Essential and inessential features of Hawking radiation

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Abstract:

There are numerous different derivations of the Hawking radiation effect.

They emphasise different features of the process, and make markedly different physical assumptions.

I will present an argument that is "minimalist" and strips the derivation of as much excess baggage as possible.

I will argue that all that is really necessary is quantum physics plus a slowly evolving future apparent horizon.

In particular, neither the Einstein equations nor black hole entropy are necessary (nor even useful) in deriving Hawking radiation.

Basic Idea:

Do as much as you can with the eikonal approximation.

(Even WKB is mild overkill.)

Look for generic features in the modes at/near the apparent horizon.

Specifically, look for a Boltzmann factor.

Historical derivations:

Collapsar — Hawking — Nature 74.

Bogolubov — Hawking — CMP 75.

Kruskal — Hartle–Hawking — PRD 76.

Horizon-chasing — Boulware — PRD 76.

Euclidean — Gibbons– Hawking — PRD 77.

Thermo-field theory — Israel — PLA 77.

. . . .

Shell-vacuum — Parentani — PRD 00.

Tunnelling — Parikh–Wilczek — PRL 00.

Complex paths — Padmanabhan et al – 00/01.

Irrelevant:

There are many things a good derivation should *not* depend on:

— Bekenstein entropy;

— Grey-body factors;

— Past horizon;

— Einstein equations;

— Specific features of the Schwarzschild geometry;

— Event horizon (absolute horizon);

— Gravity.

Relevant:

A good derivation should be:

— Universal;

Depend only on very general features of the problem:

— existence of apparent horizon;

— "surface gravity".

Mantra:

Hawking radiation is kinematics;

Bekenstein entropy is geometrodynamics.

PG metric:

Exercise: Any spherically symmetric geometry, static or not, can locally be put in the form

$$ds^2 = -c(r,t)^2\, dt^2 + (dr - v(r,t)\, dt)^2$$
$$+r^2[d\theta^2 + \sin^2\theta\, d\phi^2].$$

Equivalently

$$ds^2 = -[c(r,t)^2 - v(r,t)^2]\, dt^2 - 2v(r,t)\, dr\, dt$$
$$+dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\phi^2].$$

In matrix form (quasi–ADM)

$$g_{\mu\nu}(t,\vec{x}) \equiv \begin{bmatrix} -(c^2 - v^2) & \vdots & -v\, \hat{r}_j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v\, \hat{r}_i & \vdots & \delta_{ij} \end{bmatrix}.$$

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{c^2} \begin{bmatrix} -1 & \vdots & -v\, \hat{r}^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v\, \hat{r}^i & \vdots & (c^2\, \delta^{ij} - v^2\, \hat{r}^i\, \hat{r}^j) \end{bmatrix}.$$

PG Horizon:

Apparent horizon located at $c(r,t) = |v(r,t)|$.

Metric nonsingular at the apparent horizon.

To get a future apparent horizon, corresponding to an astrophysical black hole, and an "infalling aether", we need $v < 0$.

Define a quantity:

$$g_H(t) = \frac{1}{2} \frac{\mathrm{d}[c(r,t)^2 - v(r,t)^2]}{\mathrm{d}r}\bigg|_H$$

$$= c_H \frac{\mathrm{d}[c(r,t) - |v(r,t)|]}{\mathrm{d}r}\bigg|_H.$$

If the geometry is static, this reduces to the ordinary definition of surface gravity:

$$\kappa = g_H / c_H.$$

Eikonal approximation: S-wave

$$\phi(r,t) = \mathcal{A}(r,t)\ \exp[-i\varphi(r,t)]$$
$$= \mathcal{A}(r,t)\ \exp\left[-i\left(\omega\, t - \int^r k(r')\, \mathrm{d}r'\right)\right].$$

Then

$$g^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi = 0.$$

Equivalently

$$-\omega^2 + 2v(r,t)\,\omega k + [c(r,t)^2 - v(r,t)^2]k^2 = 0.$$

So that

$$(\omega - vk)^2 = c^2 k^2.$$

$$\omega - vk = \sigma\ ck; \qquad \sigma = \pm 1.$$

$$k = \frac{\sigma\ \omega}{c + \sigma v}.$$

Need $\omega \gg \max\{\dot{c}/c, \dot{v}/v\}$.

WKB approximation: S-wave

Conserved current

$$J_\mu = |\mathcal{A}(r,t)|^2 \, (\omega, k, 0, 0).$$

Then

$$\nabla_\mu \, J^\mu = 0 \qquad \Rightarrow$$

$$|\mathcal{A}(r,t)| \propto \frac{1}{r}.$$

Normalizing

$$\phi(r,t) = \frac{1}{\sqrt{2\omega} \; r} \exp\left[-i\left(\omega \, t - \int^r k(r') \, \mathrm{d}r'\right)\right].$$

$$k = \frac{\sigma \, \omega}{c + \sigma v} = \frac{\sigma \, c - v}{c^2 - v^2} \, \omega.$$

$\sigma = +1 \Rightarrow$ outgoing mode.

$\sigma = -1 \Rightarrow$ ingoing mode.

Near horizon modes: ingoing

In the vicinity of the future horizon $r \approx r_H$ (with $v \approx -c$) the ingoing modes $\sigma = -1$ are approximately

$$\phi(r,t)_{\text{in}} \approx \frac{1}{\sqrt{2\omega}\, r_H} \exp\left[\mp i|\omega| \left\{ t + \frac{r - r_H}{2c_H} \right\}\right].$$

This means the phase velocity of the ingoing mode as it crosses the horizon (in coordinate distance per coordinate time) is $2c_H$.

Phase velocity equals group velocity because there is no dispersion.

Near horizon modes: outgoing

Consider $\sigma = +1$

$$\int k = \int^{r} \frac{dr'}{c(r') - |v(r')|} \approx \int^{r} \frac{dr'\, c_H}{g_H (r' - r_H)}$$

$$= \frac{c_H}{g_H} \ln[r - r_H].$$

Therefore (for $r > r_H$)

$$\phi(r,t)_{\text{out}} \approx \frac{[r - r_H]^{\mp i|\omega| c_H/g_H}}{\sqrt{2\omega}\, r_H} \exp\{\mp i|\omega| t\}\,.$$

The fact that these outgoing modes have the "surface gravity" show up in such a fundamental and characteristic way is already strongly suggestive; and this is really all there is to Hawking radiation.

The phase pile-up at the horizon is characteristic of many derivations of Hawking radiation.

Near horizon modes: crossing the horizon

Continue the outgoing mode backwards to just inside the horizon. The phase picks up an imaginary contribution from the logarithm

$$\int^r \frac{dr'}{c(r') - |v(r')|} \approx \int^r \frac{dr'\, c_H}{g_H (r' - r_H)}$$

$$= \frac{c_H}{g_H} \ln |r - r_H| + i\pi\, \Theta(r - r_H).$$

So just inside the horizon

$$\phi(r,t)_{\text{out}} \approx \frac{|r - r_H|^{i\omega c_H / g_H}}{\sqrt{2\omega}\, r_H}$$

$$\times \exp\left\{ \frac{\pm \pi \omega c_H}{g_H} \right\} \exp\left[\mp i \omega t\right].$$

That is

$$|\phi(r,t)_{\text{out}(r<r_H)}|^2 \approx \exp\left\{ \frac{\pm 2\pi \omega c_H}{g_H} \right\}$$

$$|\phi(r,t)_{\text{out}(r>r_H)}|^2.$$

Boltzmann factor!

Hartle–Hawking:

(Cf: Parikh–Wilczek, Padmanabhan et al.)

The Boltzmann factor

$$\text{Prob(emit)} = \exp\left\{\frac{-2\pi\omega c_H}{g_H}\right\} \text{Prob(absorb)},$$

implies thermal spectrum with

$$k\, T_H = \frac{\hbar\, g_H}{2\pi\, c_H}.$$

. . . .

(If you don't like thermodynamic arguments you can alternatively do a Bogolubov coefficient calculation.

In the current approach, the phase pile-up at the apparent horizon is a truly elementary result.)

Beyond S-wave:

What happens if we go beyond S-wave?

$$\partial_\mu \varphi = (\omega, -k, -k_\perp).$$

In terms of partial waves

$$k_\perp^2 = \frac{\ell(\ell+1)}{r^2}.$$

Then in the eikonal approximation

$$-\omega^2 + 2v(r,t)\,\omega k + [c(r,t)^2 - v(r,t)^2]\,k^2$$
$$+c(r,t)^2\,k_\perp^2 = 0.$$

That is

$$(\omega - vk)^2 = c^2k^2 + c^2k_\perp^2.$$

Quadratic for k as a function of ω and $k_\perp$:

$$k = \frac{\sigma\sqrt{c^2\omega^2 - (c^2 - v^2)c^2k_\perp^2} - v\,\omega}{c^2 - v^2}.$$

Beyond S-wave: ingoing

Evaluate using L'Hopital's rule:

$$k_{\text{in}} \to -\frac{\omega^2 - c^2 k_\perp^2}{2\, c_H\, \omega}.$$

So the ingoing modes depend on $k_\perp$.

$$\phi(r,t)_{\text{in}} \approx \frac{1}{\sqrt{2\omega}\, r_H} \times \exp\left[\mp i|\omega| \left\{ t + \frac{(r - r_H)[\omega^2 - c^2 k_\perp^2]}{2\, c_H\, \omega} \right\}\right].$$

But that does not matter:

The ingoing modes are not the relevant ones.

Beyond S-wave: outgoing

Near the horizon

$$k_{\text{out}} \to \frac{c_H \ \omega}{g_H(r - r_H)}.$$

Asymptotic behaviour *independent* of $k_\perp$.

Phase pile-up *independent* of $k_\perp$.

Continuation across horizon *independent* of $k_\perp$.

Hawking temperature *independent* of $k_\perp$.

Behaviour universal for all partial waves.

Adding a mass term:

$$c^2 k_\perp^2 \to c^2 k_\perp^2 + (m \ c^2/\hbar)^2.$$

Essential features:

— Apparent horizon.

— Non-zero g_H.

— Slow evolution:

$$\frac{kT_H}{\hbar} \approx \omega_{\text{peak}} \gg \max\{\dot{c}/c, \dot{v}/v\}.$$

That is

$$\left.\frac{\mathrm{d}[c(r,t) - |v(r,t)|]}{\mathrm{d}r}\right|_H \gg \frac{\dot{c}_H}{c_H}.$$

Near the horizon spatial gradients should dominate over temporal gradients.

That's it.

General Lessons:

Mantra:

Hawking radiation is kinematics;

Bekenstein entropy is geometrodynamics.

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Some people still have the strange idea that Hawking radiation has something to do with gravity...

Disabuse yourselves of this notion...

. . . .

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