Analog models of General Relativity: Introduction and Survey

Matt Visser

Physics Department
Washington University
Saint Louis
USA

Presented at the workshop:

Analog models for General Relativity

LAFEX/CBPF, Rio de Janeiro

Oct 2000

Overview:

What? In this survey I will talk about various "analog models" for general relativity.

Why? Laboratory experiments with general relativity black holes are basically impossible (and also highly inadvisable).
There is now a lot of interest in simulating black holes by using condensed matter analogues.

Where? Here on Earth.

When? Five to ten years.

Who? Collaborative effort:
Relativity, Optics, Condensed matter, ...

Acoustic black holes:

Basic Idea:

Consider sound waves in a flowing fluid.

If the fluid is moving faster than sound, then the sound waves are swept along with the flow, and cannot escape from that region.

This sounds awfully similar to a black hole in general relativity — is there any connection?

— # # # YES! # # # —

The models:

There are analog models based on:

1. Acoustics in flowing fluids;
2. Slow light in flowing fluid dielectrics;
3. Flowing Bose-Einstein condensates;
4. Quasi-particles in superfluids;
5. Nonlinear electrodynamics;
6. The Scharnhorst effect;
7. and more.

Common themes:

An effective Lorentzian metric that governs perturbative fluctuations.

Fluctuations exhibit many of the kinematic features of general relativity.

Dynamic features [those specifically based on the Einstein-Hilbert action] typically do not carry over.

It seems plausible that we might be able to construct analog horizons in the laboratory in the not too distant future.

Such analog horizons are expected to exhibit Hawking radiation, but possibly/probably without any analog of Bekenstein entropy.

Hawking radiation:

Analog models of general relativity are useful probes of Hawking radiation.

Because the short-distance physics is explicitly known (atomic physics), the cutoff is physically understood.

This helps clarify the role of trans-Planckian frequencies in general relativity black holes, which in these condensed-matter analogs are replaced by "trans-Bohrian" physics.

I will give an overview of some of these proposals and indicate what the hopes are for laboratory tests...

Focus — Most promising proposals:

1. Acoustic black holes (dumb holes; where supersonic fluid flow traps sound),

2. Optical black holes (where a combination of extremely high refractive index and dielectric fluid flow traps light),

3. BEC holes (where phase oscillations in Bose-Einstein condensates, which travel extremely slowly, are trapped by the flowing condensate),

4. Quasi-particle holes (where quasi-particles in superfluids have a position-dependent dispersion relation governed by the background fluid flow).

Geometrical acoustics:

Acoustic propagation in fluids can be described in terms of an acoustic metric which depends algebraically on the fluid flow.

In a flowing fluid, if sound moves a distance $\mathrm{d}\vec{x}$ in time $\mathrm{d}t$ then

$$||\mathrm{d}\vec{x} - \vec{v}\;\mathrm{d}t|| = c_s\;\mathrm{d}t.$$

Write this as

$$(\mathrm{d}\vec{x} - \vec{v}\;\mathrm{d}t)\cdot(\mathrm{d}\vec{x} - \vec{v}\;\mathrm{d}t) = c_s^2\mathrm{d}t^2.$$

Now rearrange a little:

$$-(c_s^2 - v^2)\;\mathrm{d}t^2 - 2\;\vec{v}\cdot\mathrm{d}\vec{x}\;\;\mathrm{d}t + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = 0.$$

Notation — four-dimensional coordinates:

$$x^\mu = (x^0; x^i) = (t; \vec{x}).$$

Then you can write this as

$$g_{\mu\nu}\;\mathrm{d}x^\mu\;\mathrm{d}x^\nu = 0.$$

Geometrical acoustics:

Pick off the coefficients: you get an effective acoustic metric

$$g_{\mu\nu}(t,\vec{x}) \propto \begin{bmatrix} -(c_s^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{bmatrix}.$$

Acoustic geometry shares kinematic aspects of general relativity, but not the dynamics:

Euler equations versus Einstein equations.

The sound rays of geometrical acoustics are the null geodesics of this effective metric.

Geometrical acoustics, by itself, does not give you enough information to fix an overall multiplicative factor (conformal factor).

Eikonals:

This analysis also works for geometrical optics in a flowing fluid, with $c_s \to c/n$; replace the speed of sound by the speed of light in the medium (speed of light divided by refractive index).

In fact it works whenever you can make an eikonal approximation to some wave equation — replacing "waves" by "rays".

This is already enough to give you some very powerful results:

Fermat's principle is now a special case of geodesic propagation.

Ray focussing can be described by the Riemann tensor of this effective metric.

But there is a lot more hiding in the woodwork, beyond the eikonal approximation.

Physical Acoustics:

Suppose you have a non-relativistic flowing fluid, governed by the Euler equation plus the continuity equation.

Suppose the fluid flow is barotropic, irrotational, and inviscid.

Suppose we look at linearized fluctuations.

Theorem: linearized fluctuations (*aka* sound waves, *aka* phonons) are described by a scalar field (massless minimally coupled) propagating in a (3+1)-dimensional acoustic metric

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho}{c} \begin{bmatrix} -(c^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{bmatrix}.$$

(Proof: Unruh81, Visser93, Unruh94, Visser97.)

Other representations:

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{\rho c}\left[\begin{array}{ccc} -1 & \vdots & -v^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v^i & \vdots & (c^2\delta^{ij} - v^i v^j) \end{array}\right].$$

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

$$ds^2 = \frac{\rho}{c}\left[-c^2 dt^2 + ||d\vec{x} - \vec{v}\, dt||^2\right].$$

If you move with the fluid, null cones spread out at the speed of sound.

The conformal factor is required to get a nice minimally coupled d'Alembertian equation of motion for the velocity potential.

Key points:

— The signature is $(-,+,+,+)$.

— There are two distinct metrics:
the *physical spacetime metric*, and
the *acoustic metric* .

— A completely general $(3+1)$–dimensional Lorentzian geometry has 6 degrees of freedom per point in spacetime. (4×4 symmetric matrix $\Rightarrow$ 10 independent components; then subtract 4 coordinate conditions).

—The acoustic metric is specified completely by the three scalars: velocity potential, density, speed of sound. It has at most 3 degrees of freedom per point in spacetime. Continuity reduces this to 2 degrees of freedom.

Event horizons and ergo-regions:

Event horizon: the boundary of the region from which null geodesics (phonons; sound rays) cannot escape.

At the event horizon, the inward normal component of fluid velocity equals the speed of sound.

Ergo surface: the boundary of the region of supersonic flow.

In general relativity this is important for spinning black holes.

Example: Draining bathtub

A $(2+1)$ dimensional flow with a sink.

Use constant density, continuity, conservation of angular momentum: (which automatically implies that the pressure p and speed of sound c are also constant throughout the fluid flow).

The velocity of the fluid flow is

$$\vec{v} = \frac{(A\,\hat{r} + B\,\hat{\theta})}{r}.$$

The acoustic metric is

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r} dt\right)^2 + \left(r\,d\theta - \frac{B}{r} dt\right)^2.$$

Example: Draining bathtub

The acoustic metric is

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r} dt\right)^2 + \left(r\, d\theta - \frac{B}{r} dt\right)^2.$$

The acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, that is at

$$r_{horizon} = \frac{|A|}{c}.$$

Supersonic flow sets in outside the event horizon, when the magnitude of the velocity equals the speed of sound.

$$r_{ergo-surface} = \frac{\sqrt{A^2 + B^2}}{c}.$$

Example: Schwarzschild

Schwarzschild geometry in Painlevé–Gullstrand coordinates:

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2GM}{r}} dt \right)^2$$

$$+ r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2 \right).$$

Equivalently

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \sqrt{\frac{2GM}{r}} \, dr \, dt$$

$$+ dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2 \right).$$

This representation of the Schwarzschild geometry is not particularly well-known and has been rediscovered several times this century.

Surface gravity:

If we restrict attention to a static geometry, we can apply all of the standard tricks for calculating the "surface gravity" developed in general relativity.

The surface gravity is a useful characterization of general properties of the event horizon and is given in terms of a normal derivative by

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v_\perp^2)}{\partial n} = c\,\frac{\partial(c - v_\perp)}{\partial n}.$$

The surface gravity is essentially the acceleration of the fluid as it crosses the horizon.

Non static geometries are not too bad.

(Thanks to the second metric: It gives you an unambiguous background for making comparisons.)

Hawking radiation:

As discussed by Unruh81, (and many others) an acoustic event horizon will emit Hawking radiation in the form of a thermal bath of phonons at a temperature

$$k\, T_H = \frac{\hbar\, g_H}{2\pi\, c_s}.$$

(Yes, this really is the speed of sound, and g_H is really normalized to have the dimensions of a physical acceleration.)

$$T_H = (1.2{\times}10^{-6} K\, mm)\; \left[\frac{c}{km\, s^{-1}}\right] \; \left[\frac{1}{c}\frac{\partial(c - v_\perp)}{\partial n}\right].$$

Experimental verification of this acoustic Hawking effect will be rather difficult.

Surface gravity: Naive Estimate

Estimate from dimensional analysis:

$$g_H \approx \frac{c_s^2}{R}.$$

So for a 1 millimeter nozzle,
and a 1 kilometer/sec speed of sound:

$$T_H = 1.2 \times 10^{-6} K.$$

And remember, this is a thermal bath of phonons, not photons.

Difficult to detect.

Difficulties and Opportunities

Plus: You need supersonic flow without turbulence or shocks.

Difficult to achieve.

But: The dimensional analysis estimate is grossly misleading.

If you actually solve the fluid dynamics equations for supersonic flow, you tend to get infinite acceleration at the ergo-surface.

Once you add viscosity, this regulates things: You find unexpectedly large but finite accelerations at the ergo-surface — this looks like it will improve experimental prospects.

Viscosity also smears out the event horizon: It's no longer a sharp boundary.

Surface gravity: Improved estimate

With viscosity present, the typical scale for the surface gravity turns out to be

$$g_H \approx Re\, \frac{c_s^2}{R} \approx \frac{c_s R}{\nu}\, \frac{c_s^2}{R} \approx \frac{c_s^3}{\nu}.$$

(Though you can "fine tune" g_H to arbitrary values.)

Viscosity is related to molecular dynamics: There is an automatic cutoff at interatomic distances.

(Liberati, Sonego, Visser, CQG)

Slow light

Qualitatively similar phenomena happen for "slow light"

World record refractive index (1999):

$$n \approx 3 \times 10^7!$$

This is experiment, not theory!

That is:

$$c_{slow-light} = \frac{c}{n} \approx 10 \; metres/sec.$$

Reasonable hope of soon achieving:

$$n \approx 3 \times 10^{10}!$$

$$c_{slow-light} = \frac{c}{n} \approx 1 \; centimetre/sec.$$

Problem with slow light:

The refractive index in these systems is extremely frequency dependent — the high refractive index persists only over a very narrow frequency range.

(This is because you are mucking around with an atomic resonance.)

You want to sit right next to the resonance to get anomalous dispersion.

As you get close to the event horizon, the motion of the fluid speeds up, and tends to Doppler shift the light out of the "slow" regime.

The physics is rather non-trivial; several theoretical attacks in progress.

(Leonhardt, Piwnicki, PRL; PRA.)

Bose-Einstein condensates:

Almost the same mathematical steps, but now based on linearizing the nonlinear Schrodinger equation (Gross–Pitaevskii equation).

$$-i\hbar\,\partial_t\psi(t,\vec{x}) \;=\; -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{x}) + V(t,\vec{x})\,\psi(t,\vec{x}) + \lambda\,(\psi^*\psi)\,\psi(t,\vec{x}).$$

Use the Madelung representation to put the Schrodinger equation in "hydrodynamic" form:

$$\psi = \sqrt{\rho}\;\mathrm{exp}(-i\theta/\hbar).$$

Take real and imaginary parts: You get a continuity equation and something that looks like the Euler equation.

The quantum metric:

After linearizing and appropriate approximations:

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{\lambda}{c^3} \left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\,\delta^{ij} - v_0^i\, v_0^j) \end{array} \right],$$

with

$$c^2 \equiv \frac{\lambda\ \ \rho_0}{m},$$

and

$$(\vec{v}_0)^i = \delta^{ij}\ \ \nabla_j \theta_0.$$

The conformal factor is different, the eikonal approximation is the same.

(Garay, Cirac, Anglin, Zoller;
Barceló, Liberati, Visser.)

BEC: Estimates

Present day technology:

$$c_{condensate} \approx 1\ mm/sec.$$

Less hopeful: Typical size of Y2K BECs:

$$R \approx 1\ micron.$$

Crossing time:

$$T \approx 1\ millisecond.$$

The experimental issues boil down to getting a decently large BEC and manipulating it.

(Garay, Cirac, Anglin, Zoller.)

Quasiparticles in superfluids:

Basic structure the same:
different experimental issues.

— Two-fluid models (normal and superfluid).

— First, second, and third sound.

— Can also look at fermionic quasiparticles, not just "phonons".

— Relevant speeds still tend to be high:

$$c_{quasiparticle} \approx 100\; metres/sec.$$

— Cryogenics needed.

(Grigori Volovik.)

General Lessons:

Going from fluid dynamics to the acoustic metric is relatively easy; ditto slow light to the "optical metric"; BEC to the "quantum metric", etc.

Working backwards is downright impossible.

It's probably best to first worry about things like lensing, geodesic propagation, and "proof of principle" experiments before trying for event horizons.

Eventually: ergo–surfaces, and then horizons, would be nice...

So would naked singularities...

General Lessons: A Warning

Even if you find Hawking radiation, the notion of black hole entropy may not even be meaningful.

Lesson for any theory of quantum gravity:
(1) Finding Hawking radiation in your theory does not imply that you have discovered quantum gravity.
(2) If you find Hawking radiation, and you have a theory that approximates classical Einstein gravity, then you must get black hole entropy approximately proportional to area.

Hawking radiation is kinematics — it occurs for any test field on any Lorentzian geometry with event horizon independent of whether or not the Lorentzian geometry is dynamical.

Black hole entropy is dynamics — to even define black hole entropy requires a geometrical Lagrangian.

Conclusions:

Analog models for GR are very good toy models that guide us in logically separating the kinematics of gravity from the dynamics.

Analog models should allow us to experimentally test some features of GR that would be completely unattainable with physical gravity.

Analog models for GR and for black holes can teach us a lot.