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Kerr-fest:

Celebration of Roy Kerr's 70th birthday
University of Canterbury, Christchurch, New Zealand. 26-28 August 2004.

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Near-horizon structure of generic rotating black holes

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Abstract:

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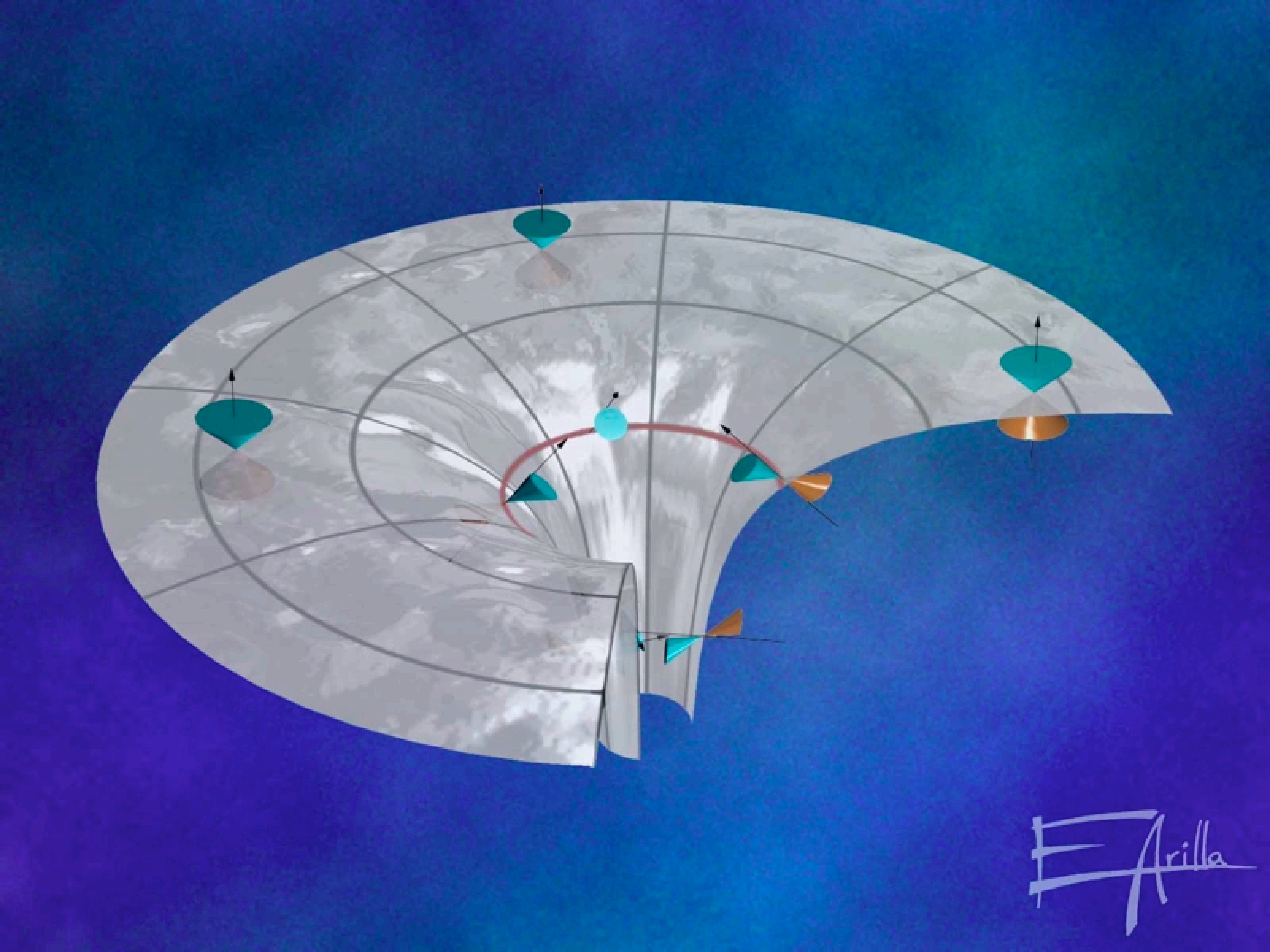
A popular explanation for the microscopic origin of Bekenstein's black hole entropy is the conjecture that this entropy can be ascribed to a collection of (1+1) dimensional conformal field theories that reside at the horizon, and defined on the two-plane perpendicular to the horizon.

If this is to be the case, then the Einstein equations must force the Ricci curvature to possess a high degree of symmetry at the horizon.

We test this hypothesis by working directly with the spacetime geometry for a generic rotating black hole - constrained only by the existence of a stationary non-static Killing horizon, and with otherwise arbitrary matter content - to show that the Einstein tensor block diagonalizes on the horizon.

This is a specific example of an "enhanced symmetry" that manifests only at the horizon itself.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Introduction:

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Bekenstein's concept of black hole entropy makes the horizon a somewhat special place.

When combined with the idea of QFT holography, it becomes plausible to think of the horizon as a "membrane" on which to define a set of surface degrees of freedom which are a "dual representation" of the internal degrees of freedom of the black hole.

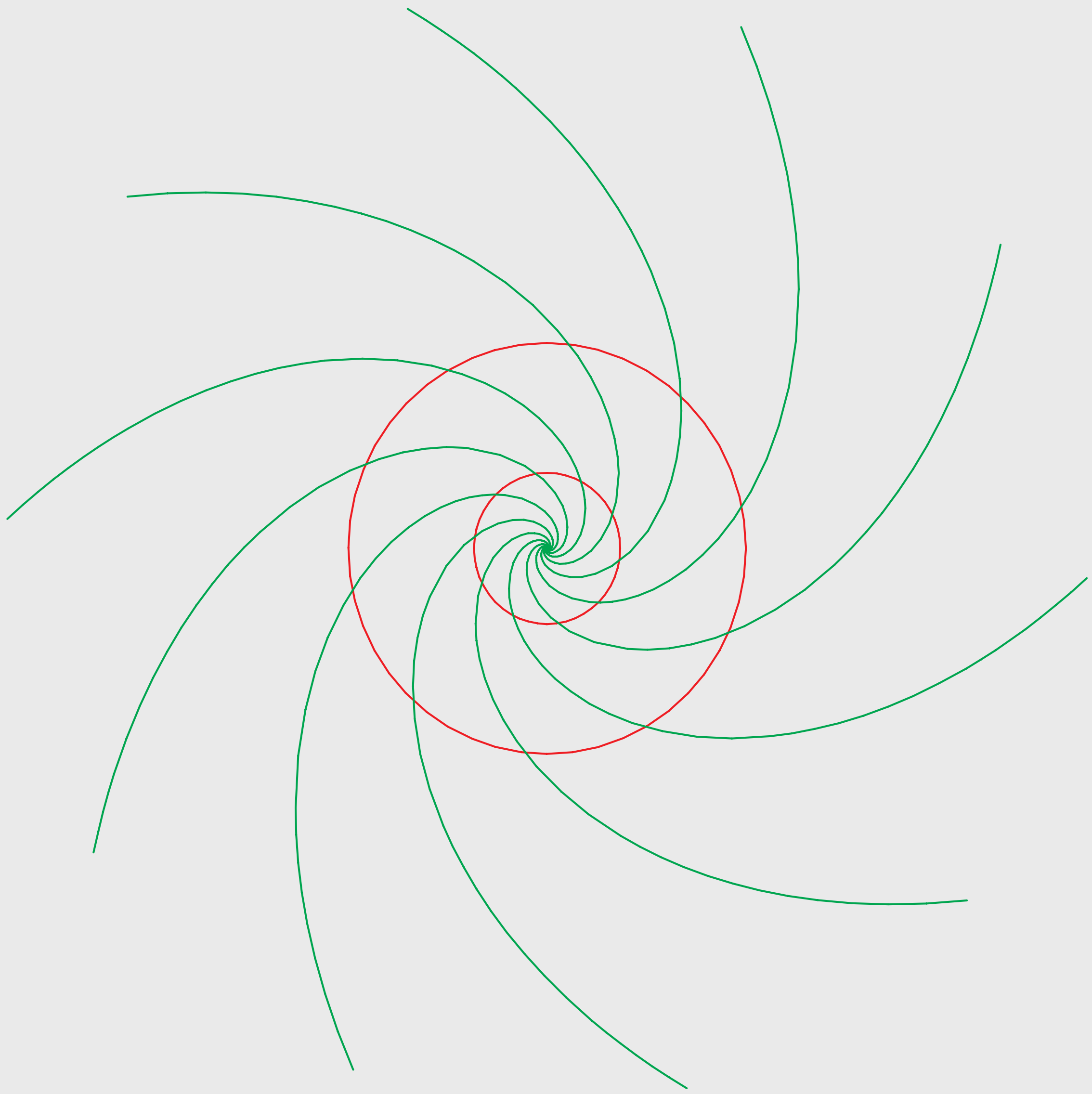
If, as many people suspect, these surface degrees of freedom are some sort of 2-dimensional CFT, this would tightly constrain the form of the stress-energy at the horizon.

Introduction:

But then the Einstein equations would constrain the form of the spacetime curvature at the horizon.

So a consistency check on this whole scenario is that the spacetime geometry must exhibit an "enhanced symmetry" at the horizon.

Since this "enhanced symmetry" must be there for arbitrary black holes, it must be possible to deduce such an enhanced symmetry on purely geometrical grounds, working solely from the geometrical definition of a black hole.



Key result:

At the horizon of any stationary (or static) black hole the Einstein tensor satisfies:

$$G_{ab}|_H = X\, g_{ab} + Y_{ab}$$

Where Y_{ab} is a tensor that lies entirely within the horizon.

This is equivalent to the statement that on any Killing horizon the Einstein tensor block diagonalizes:

$$G_{\hat{a}\hat{b}}|_H = \left[\begin{array}{cc|c} -X & 0 & 0 \\ 0 & X & 0 \\ \hline 0 & 0 & X\,\delta_{\hat{\imath}\hat{\jmath}} + Y_{\hat{\imath}\hat{\jmath}} \end{array}\right]$$

Plausibility argument:

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I first became aware of a related result in the context of spherical symmetry --- where there is a "boundary condition" which at the horizon enforces:

$$R_{\hat{t}\hat{t}} = -R_{\hat{r}\hat{r}}$$

or equivalently:

$$G_{\hat{t}\hat{t}} = -G_{\hat{r}\hat{r}}$$

The easiest way to see this is to start from the general static spherically symmetric geometry and explicitly calculate the Einstein tensor.

Static spherical symmetry:

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Write the metric in the form:

$$\mathrm{d}s^2 = -e^{-2\phi(r)}\left\{1-\frac{b(r)}{r}\right\}\mathrm{d}t^2+\frac{\mathrm{d}r^2}{1-b(r)/r}+r^2\,\mathrm{d}\Omega^2$$

Calculate:

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}$$

$$G_{\hat{r}\hat{r}} = -\frac{b'}{r^2} - \frac{2\phi'(r)}{r}\left\{1-\frac{b(r)}{r}\right\}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = -\frac{1}{2}\frac{b''}{r}+\left\{1-\frac{b(r)}{r}\right\}\left[-\phi''+\phi'\left(\phi'-\frac{1}{r}\right)\right]-\frac{3}{2}\phi'\left[\frac{b}{r^2}-\frac{b'}{r}\right]$$

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Static spherical symmetry:

The horizon occurs at: $r_H : b(r) = r$

If the geometry is to be regular at the horizon, then $\phi(r)$ and its derivatives must be finite at the horizon.

The surface gravity is then:

$$\kappa = \frac{1}{2\, r_H} \exp[-\phi(r_H)] \left\{ 1 - b'(r_H) \right\}$$

The Einstein tensor at the horizon reduces to:

$$G_{\hat{t}\hat{t}}|_H = -G_{\hat{r}\hat{r}}|_H = \frac{b'(r_H)}{r_H^2}$$

$$G_{\hat{\theta}\hat{\theta}}|_H = G_{\hat{\varphi}\hat{\varphi}}|_H = -\frac{1}{2}\frac{b''(r_H)}{r_H} - \frac{3}{2}\frac{\phi'(r_H)}{r_H}\left\{1 - b'(r_H)\right\}$$

Static spherical symmetry:

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Up to this point I have nowhere used the Einstein equations.
If I now do so, I find that (independent of the nature and type of matter in the spacetime) regularity of the horizon enforces:

$$\rho_H = -(p_r)_H$$

That is --- the energy density at the horizon is intimately related to the radial pressure.
(Indeed there must be a radial tension at the horizon.)

Question: To what extent does this result generalize away from spherical symmetry?

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Generalizations:

Obvious generalizations to look at are:

--- Static but not spherically symmetric.
(Visser, Martin, Medved, gr-qc/0402069.)

--- Stationary but not static.
(Visser, Martin, Medved, gr-qc/0403026.)

The same result continues to hold (generic matter).
At any Killing horizon:

$$G_{ab}|_H = X\ g_{ab} + Y_{ab}$$

with the tensor Y_{ab} lying entirely within the horizon.

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Bifurcate Killing horizons:

The slickest proof we have been able to find works in the case of bifurcate Killing horizons.

That proof applies equally well to generic rotating Kerr-like black holes with bifurcate Killing horizons and surrounded by arbitrary matter fields, but is so slick you may not notice the black hole is rotating!

Even if an astrophysically relevant black hole spacetime does not contain a bifurcation surface, as long as it approaches a stationary limit, that stationary limit will under mild conditions (Racz, Wald) have an extension that does contain a bifurcation surface...

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Bifurcate Killing horizons:

For all practical purposes, bifurcation surfaces are not serious restrictions.

Pick an arbitrary spacelike section of the horizon.
Set up a null basis on that section:

$$\chi^a, \quad N^a, \quad m_1^a, \quad m_2^a.$$

$$g(\chi,\chi) = 0; \quad g(N,N) = 0; \quad g(\chi,N) = -1;$$

$$g(m_i,m_j) = \delta_{ij}; \quad g(m_i,\chi) = 0; \quad g(m_i,N) = 0.$$

Here χ^a is the Killing vector,
while N^a is the "other" null normal.

Now decompose the Einstein tensor in this null basis.

Bifurcate Killing horizons:

$$
\begin{aligned}
G_{ab} \;=\;& G_{++}\,\chi_a\chi_b + G_{--}\,N_aN_b + G_{+-}\,\{\chi_aN_b + N_a\chi_b\} \\
&+G_{+1}\,\{\chi_a[m_1]_b + [m_1]_a\chi_b\} + G_{+2}\,\{\chi_a[m_2]_b + [m_2]_a\chi_b\} \\
&+G_{-1}\,\{N_a[m_1]_b + [m_1]_aN_b\} + G_{-2}\,\{N_a[m_2]_b + [m_2]_aN_b\} \\
&+G_{11}\,[m_1]_a[m_1]_b + G_{22}\,[m_2]_a[m_2]_b \\
&+G_{12}\,\{[m_1]_a[m_2]_b + [m_2]_a[m_1]_b\}\;.
\end{aligned}
$$

But anywhere on any Killing horizon it is a standard result that:

$$G^b{}_a\,\chi_b \propto \chi_a$$

The Killing vector is a null eigenvector of the Einstein tensor! This implies:

$$G_{--} = G_{-1} = G_{-2} = 0$$

Bifurcate Killing horizons:

So anywhere on any Killing horizon:

$$\begin{aligned} G_{ab} &= G_{++}\, \chi_a \chi_b + G_{+-}\, \{\chi_a N_b + N_a \chi_b\} \\ &\quad + G_{+1}\, \{\chi_a [m_1]_b + [m_1]_a \chi_b\} + G_{+2}\, \{\chi_a [m_2]_b + [m_2]_a \chi_b\} \\ &\quad + G_{11}\, [m_1]_a [m_1]_b + G_{22}\, [m_2]_a [m_2]_b \\ &\quad + G_{12}\, \{[m_1]_a [m_2]_b + [m_2]_a [m_1]_b\} \; . \end{aligned}$$

But at the bifurcation surface:

$$\chi_a \to 0$$

$$\chi_a\, N_a + N_a\, \chi_b + [m_1]_a\, [m_1]_b + [m_2]_a\, [m_2]_b \to g_{ab}$$

while the coefficients in the expansion are constants.

Bifurcate Killing horizons:

So on the bifurcation surface:

$$
\begin{aligned}
G_{ab} \;=\; & G_{+-}\, g_{ab} \\
& +(G_{11} - G_{+-})\, [m_1]_a [m_1]_b \\
& +(G_{22} - G_{+-})\, [m_2]_a [m_2]_b \\
& +G_{12}\, \{[m_1]_a [m_2]_b + [m_2]_a [m_1]_b\}
\end{aligned}
$$

That is:

$$G_{ab}|_H = X\, g_{ab} + Y_{ab}$$

Where Y_{ab} is a tensor that lies entirely within the bifurcation surface.

Bifurcate Killing horizons:

Although the present argument is limited to bifurcate Killing horizons, it is truly generic.

The form of the stress-energy at the horizon is very tightly constrained by the geometry.

The argument is easily seen to be dimension independent, and the symmetries are appropriate to a collection of CFTs living on the bifurcation surface, with SETs transverse to the bifurcation surface --- plus in-surface interactions.

Can we make all of this more explicit?


Mt. Taranaki
Egmont National Park Boundary

Stationary non-static Killing horizons:

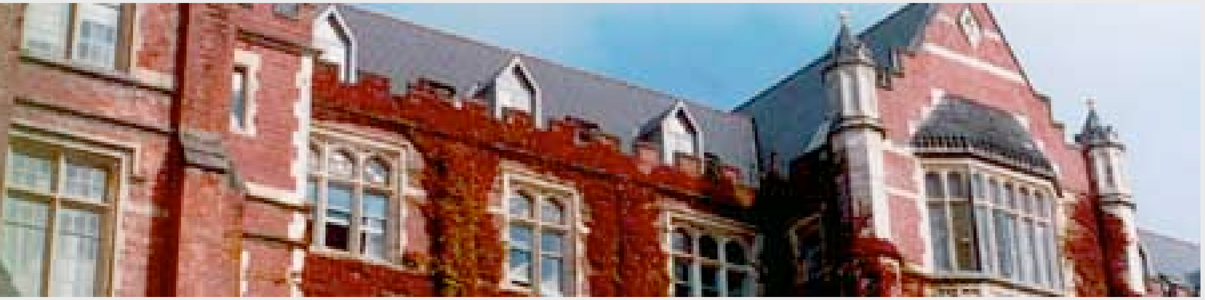
These are appropriate specifically for rotating black holes.

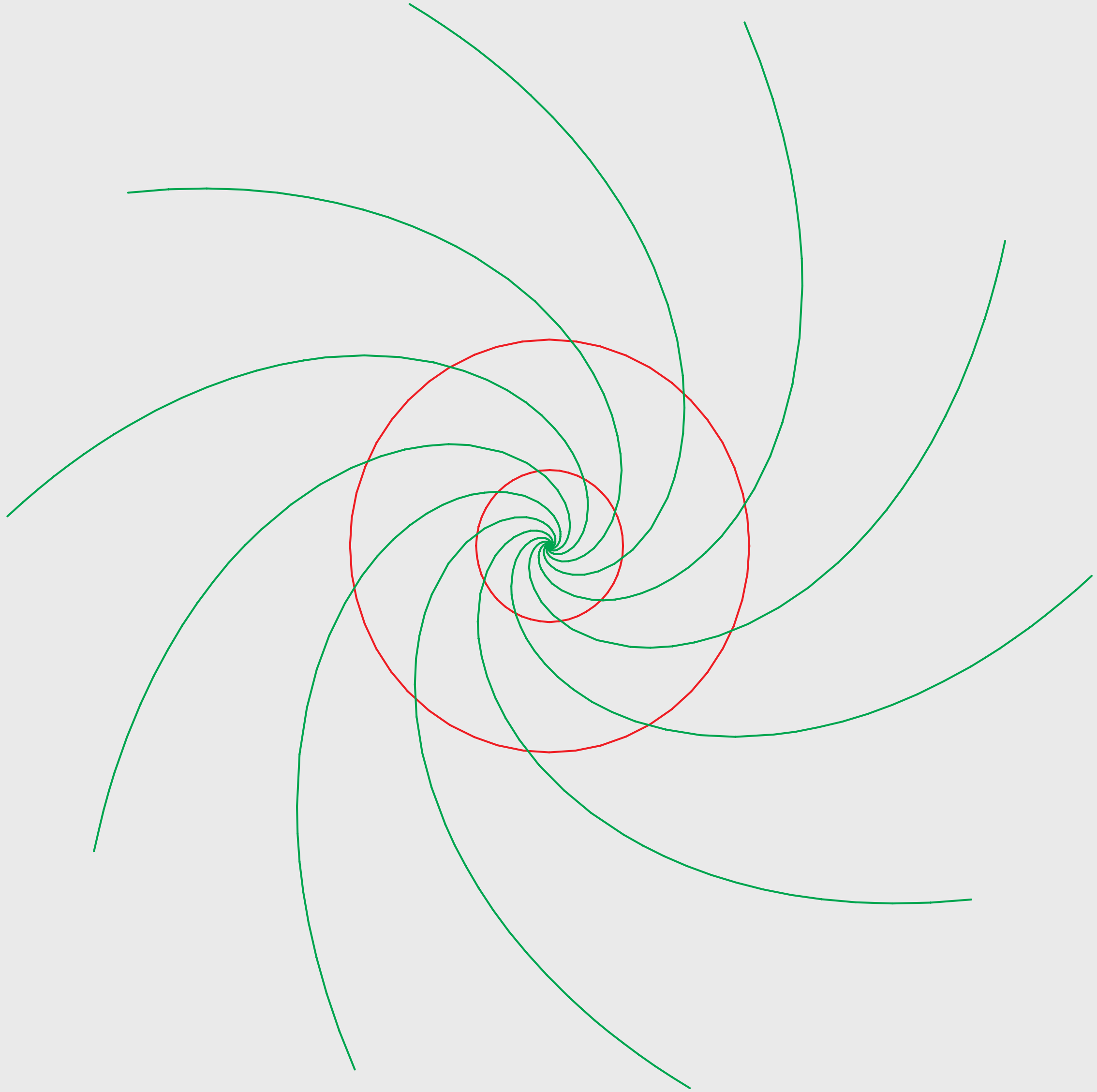
Under mild technical conditions:

$$g_{\mu\nu} = \begin{bmatrix} g_{tt} & g_{\phi t} & 0 & 0 \\ g_{\phi t} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{bmatrix}.$$

Convenience: adopt ADM-like form:

$$g_{\mu\nu} = \begin{bmatrix} -[N^2 - g_{\phi t}^2/g_{\phi\phi}] & g_{\phi t} & 0 & 0 \\ g_{\phi t} & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{bmatrix},$$





Stationary non-static Killing horizons:

Horizon: Located at N=0.

Convenience: Define omega, normal coordinate:

$$g_{\mu\nu} = \begin{bmatrix} -[N^2 - g_{\phi\phi}\,\omega^2] & -g_{\phi\phi}\,\omega & 0 & 0 \\ -g_{\phi\phi}\,\omega & g_{\phi\phi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}$$

Metric:

$$\mathrm{d}s^2 = -N(n,z)^2\,\mathrm{d}t^2 + g_{\phi\phi}(n,z)\{\mathrm{d}\phi - \omega(n,z)\,\mathrm{d}t\}^2 + \mathrm{d}n^2 + g_{zz}(n,z)\,\mathrm{d}z^2\ .$$

Surface gravity:

$$\kappa_H \equiv \lim_{n\to 0} \partial_n N$$

Stationary non-static Killing horizons:

Consider a Taylor series expansion:

$$N(n,z) = \kappa_H(z)\, n + \frac{1}{2}\kappa_1(z)\, n^2 + \frac{1}{3!}\kappa_2(z)\, n^3 + O(n^4) \,,$$

$$\omega(n,z) = \omega_H(z) + \omega_1(z) n \; + \frac{1}{2}\omega_2(z)\, n^2 + \frac{1}{3!}\omega_3(z)\, n^3 + O(n^4) \,.$$

Ditto for other metric components.

Demand curvature invariants finite at the horizon:

$$\frac{\mathrm{d}\omega_H(z)}{\mathrm{d}z} = 0 \quad \Rightarrow \quad \omega_H(z) = \omega_H = \text{constant} \,,$$

$$\omega_1(z) = 0 \,.$$

Rigidity theorem!

Stationary non-static Killing horizons:

$$\frac{\mathrm{d}\kappa_H(z)}{\mathrm{d}z} = 0 \quad \Rightarrow \quad \kappa_H(z) = \kappa_H = \text{constant}\,,$$

$$\kappa_1(z) = 0\,,$$

Zero'th law!

$$\frac{\mathrm{d}g_{zz}}{\mathrm{d}z}\Big|_{n=0} = 0 \quad \Rightarrow \quad g_{zz} = [g_{zz}(z)]_H + o(n^2)\,,$$

$$\frac{\mathrm{d}g_{\phi\phi}}{\mathrm{d}z}\Big|_{n=0} = 0 \quad \Rightarrow \quad g_{\phi\phi} = [g_{\phi\phi}(z)]_H + o(n^2)\,.$$

For non-extremal black holes this is enough to keep all curvature invariants finite at the horizon.

(Similar results in extremal case.)

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Stationary non-static Killing horizons:

Now calculate the on-horizon Einstein tensor.

For convenience define:

$$ds_{\|}^2 = g_{\phi\phi}(z)\, \mathrm{d}\phi^2 + g_{zz}(z)\, \mathrm{d}z^2 \ ,$$

$$R_{\|} = \frac{1}{2}\left\{ \frac{(\partial_z [g_H]_{\phi\phi})^2}{[g_H]_{zz}\ [g_H]_{\phi\phi}^2} + \frac{\partial_z [g_H]_{\phi\phi}\ \partial_z [g_H]_{zz}}{[g_H]_{zz}^2\ [g_H]_{\phi\phi}} - 2\frac{\partial_z^2 [g_H]_{\phi\phi}}{[g_H]_{zz}\ [g_H]_{\phi\phi}} \right\} \ .$$

Transverse to the horizon set:

$$\chi^a = \xi^a + \Omega_H\ \psi^a \ .$$

$$ds_{\perp}^2 = -\left[N^2 - g_{\phi\phi}\,(\omega - \Omega_H)^2\right] \mathrm{d}\tilde{\chi}^2 + \mathrm{d}n^2 \ ,$$

Stationary non-static Killing horizons:

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Compute:

$$R_\perp = -2\left\{\frac{\kappa_2}{\kappa_H}\right\} + \frac{3}{2}\,\frac{[g_H]_{\phi\phi}(z)\;\omega_2(z)^2}{\kappa_H^2}\,.$$

Now pick up the pieces (Maple) :

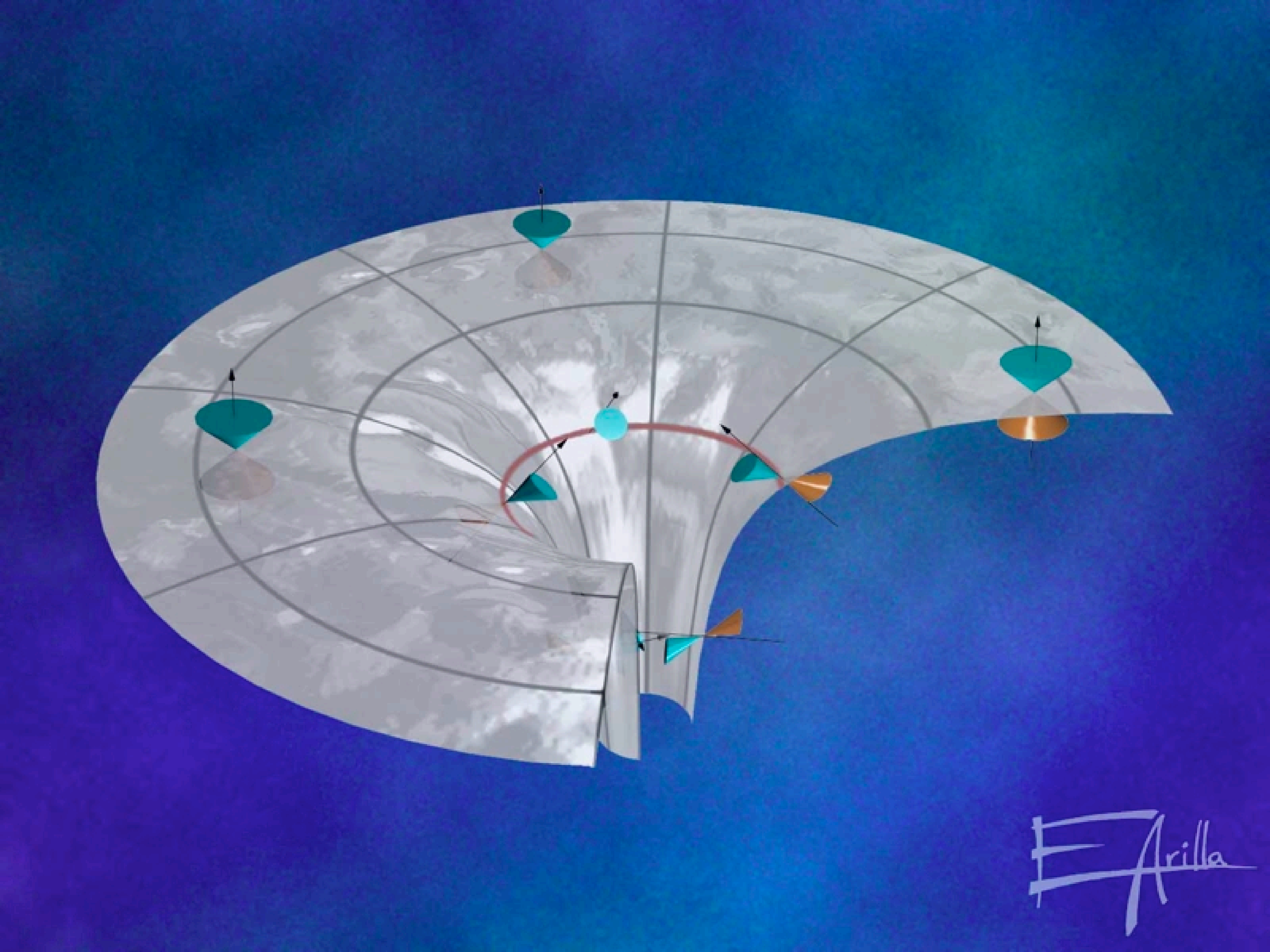
$$\begin{aligned}
[G_{\hat\chi\hat\chi}]_H &= -\frac{1}{2}R_\parallel - \frac{1}{2}\mathrm{tr}[g_2] - \frac{1}{4}[g_H]_{\phi\phi}\,\frac{\omega_2^2}{\kappa_H^2}\,,\\
[G_{nn}]_H &= -[G_{\hat\chi\hat\chi}]_H\,,\\
[G_{\hat z\hat z}]_H &= -\frac{1}{2}R_\perp + \frac{[g_2]_{\phi\phi}}{[g_H]_{\phi\phi}} + \frac{1}{2}\,[g_H]_{\phi\phi}\,\frac{\omega_2^2}{\kappa_H^2}\,,\\
[G_{\hat\phi\hat\phi}]_H &= -\frac{1}{2}R_\perp + \frac{[g_2]_{zz}}{[g_H]_{zz}}\,,
\end{aligned}$$

Stationary non-static Killing horizons:

This is a lot more explicit than before, and relates the on-horizon Einstein tensor to the on-horizon and transverse-horizon curvatures, plus surface gravity, plus a few derivatives...

This verifies, by explicit computation, the result of our general argument, and provides additional and specific information regarding the coefficients...

It also provides, almost for free, a new viewpoint regarding the zero'th law and the rigidity theorem.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Conclusion:

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At any Killing horizon:

$$G_{ab}|_H = X\ g_{ab} + Y_{ab}$$

where the tensor Y_{ab} lies entirely within the horizon.

This purely geometrical statement strongly constrains the near-horizon stress tensor in a manner compatible with the existence of on-horizon CFT-like microstates, supporting the idea of a generic low-energy basis for the Bekenstein black hole entropy.

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