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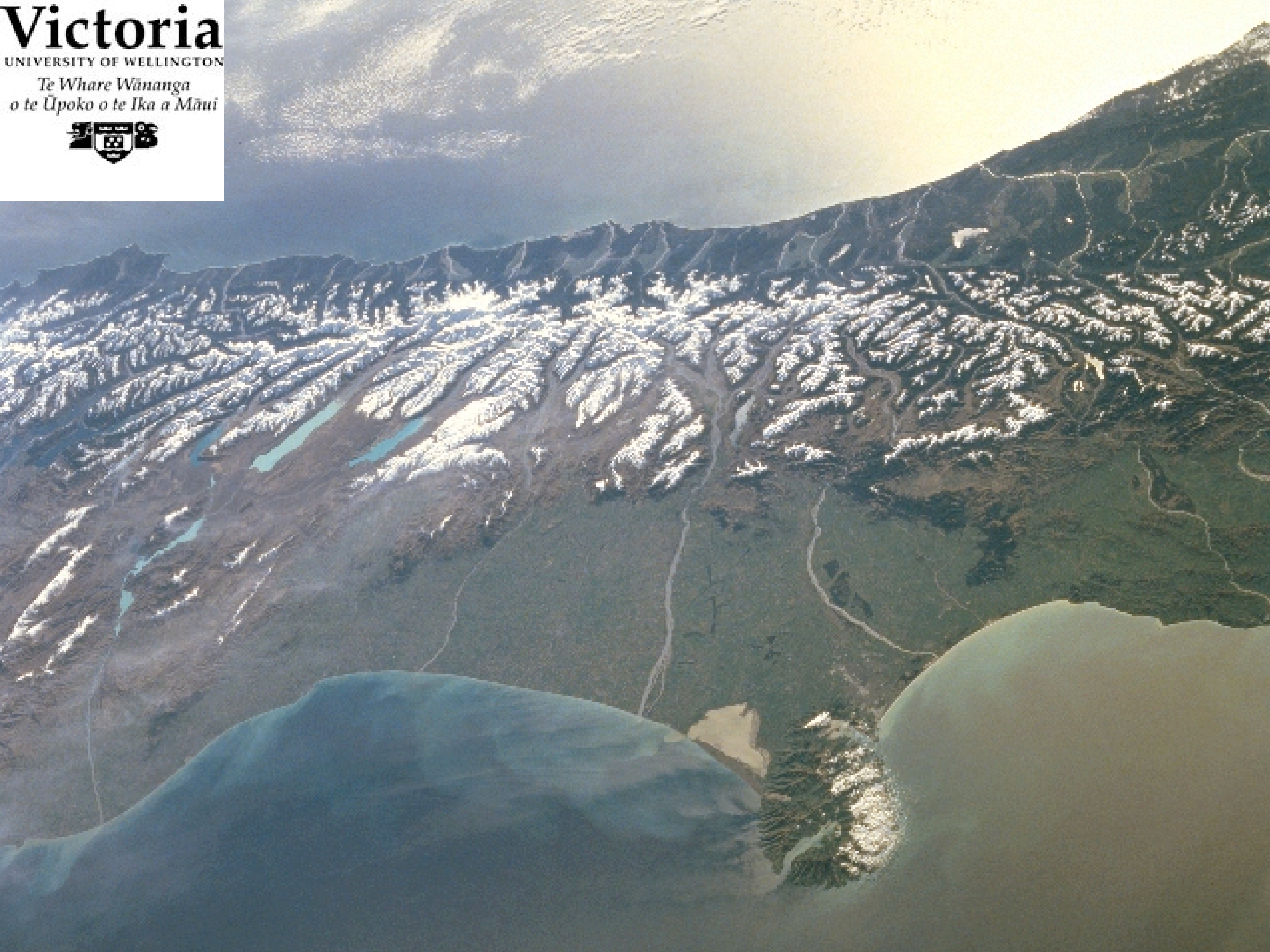

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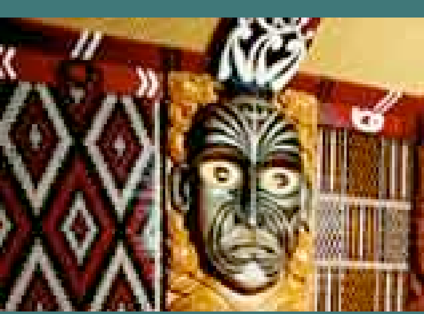
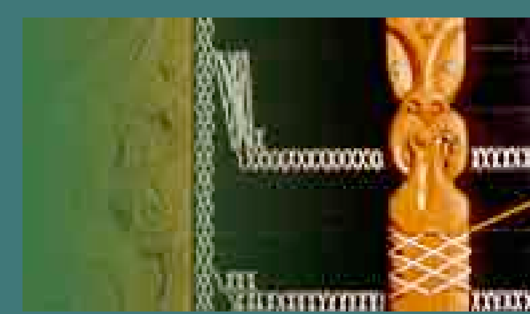
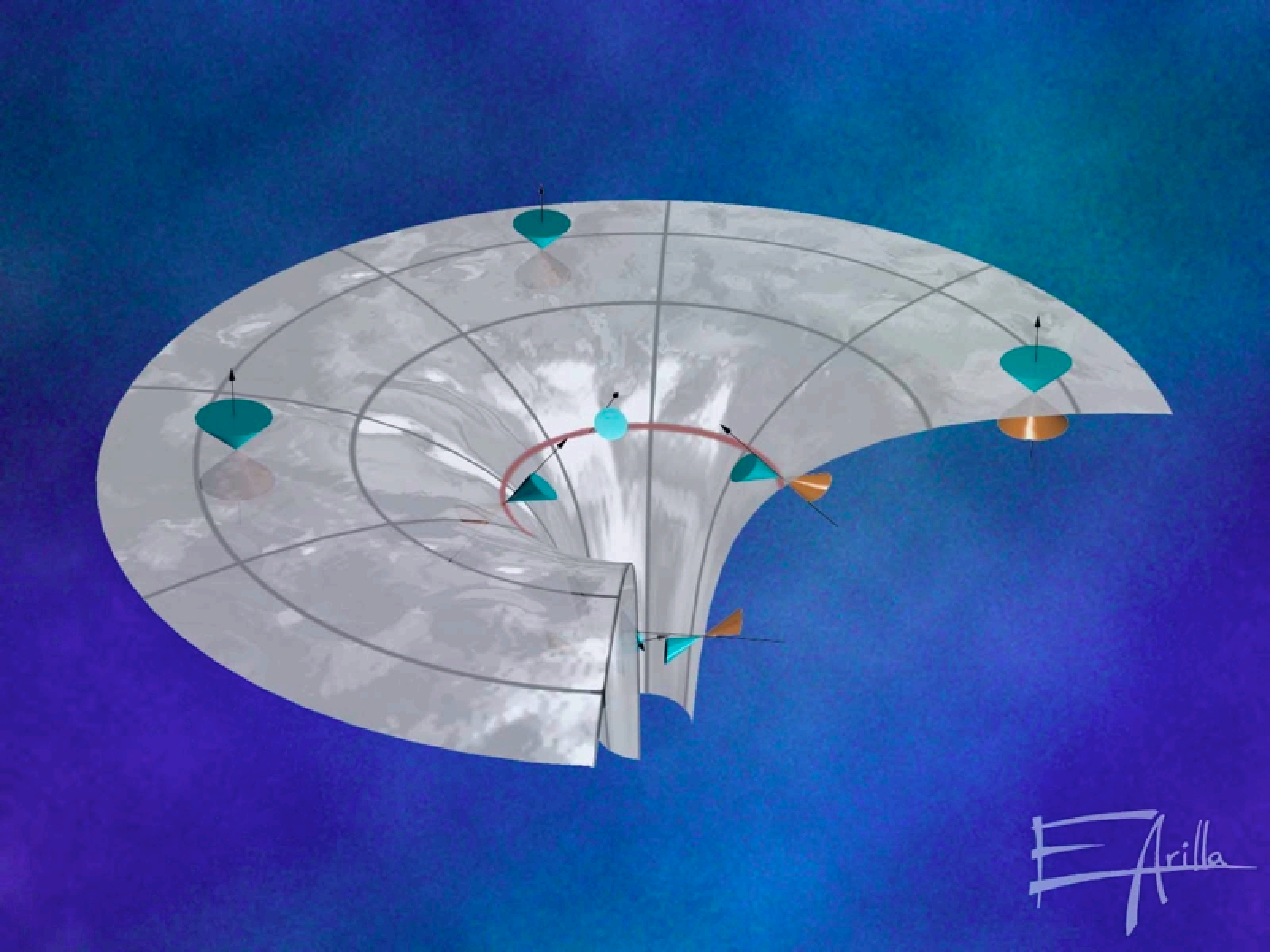
Vortex geometry for the equatorial slice of the Kerr geometry

Matt Visser & Silke Weinfurtner



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Abstract:

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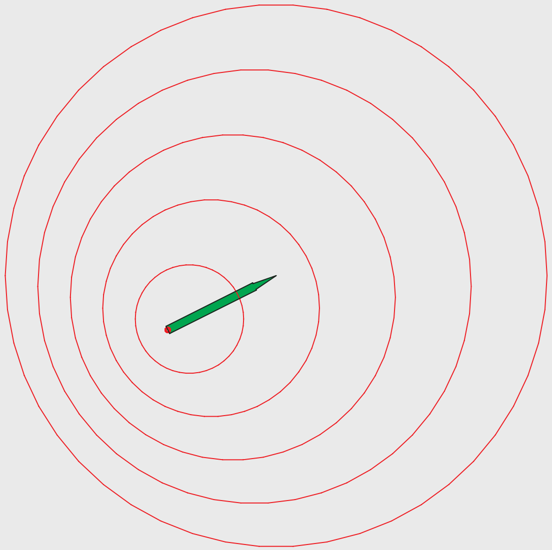
"Analogue models" for curved spacetime can be very useful for guiding physical intuition in general relativity.

The "acoustic metric" describing sound in a flowing fluid is perhaps the simplest of the "analogue models".

A "draining bathtub" vortex can be set up to exhibit both a horizon and an ergo-surface.

How close can we get to modelling the actual geometry of the Kerr spacetime using a fluid vortex?

Geometrical acoustics:



In a flowing fluid, if sound moves a distance $\mathrm{d}\vec{x}$ in time $\mathrm{d}t$ then

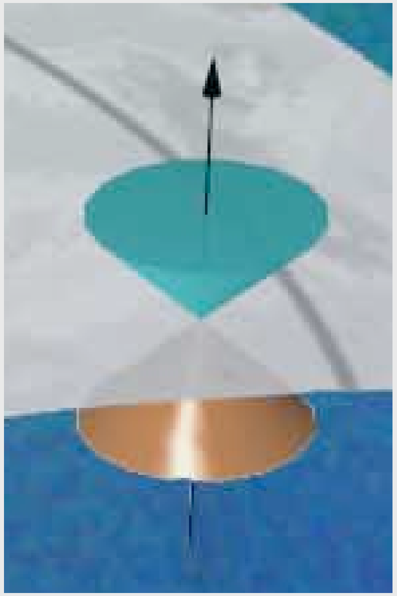
$$||\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t|| = c_s\,\mathrm{d}t.$$

Write this as

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t)\cdot(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) = c_s^2\mathrm{d}t^2.$$

Now rearrange a little: (Quadratic!)

$$-(c_s^2 - v^2)\,\mathrm{d}t^2 - 2\,\vec{v}\cdot\mathrm{d}\vec{x}\;\;\mathrm{d}t + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = 0.$$



Geometrical acoustics:

Notation — four-dimensional coordinates:

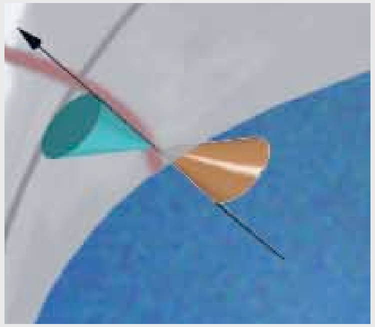
$$x^\mu = (x^0; x^i) = (t; \vec{x}).$$

Then you can write this as

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = 0.$$

With an effective acoustic metric

$$g_{\mu\nu}(t, \vec{x}) \propto \begin{bmatrix} -(c_s^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{bmatrix}.$$



Sound cones!

Physical acoustics:

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Theorem: For an irrotational barotropic inviscid fluid, linearized perturbations (sound waves, phonons) obey the wave equation:

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\; g^{\mu\nu}\;\partial_\nu\psi\right) = 0.$$

This is a massless minimally coupled scalar that "sees" the "acoustic metric":

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)\left[\begin{array}{c|c} -\{c^2 - h^{mn}\, v_n\, v_n\} & -v_j \\ \hline -v_i & h_{ij} \end{array}\right].$$

Example: Draining bathtub geometry

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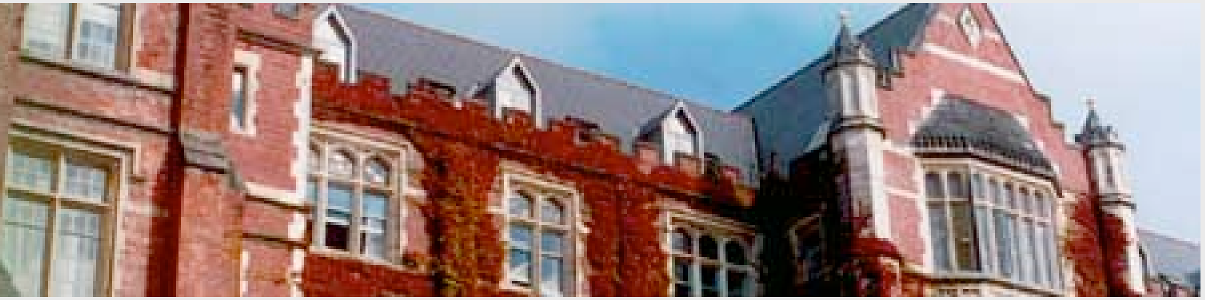
A $(2+1)$ dimensional flow with a sink.

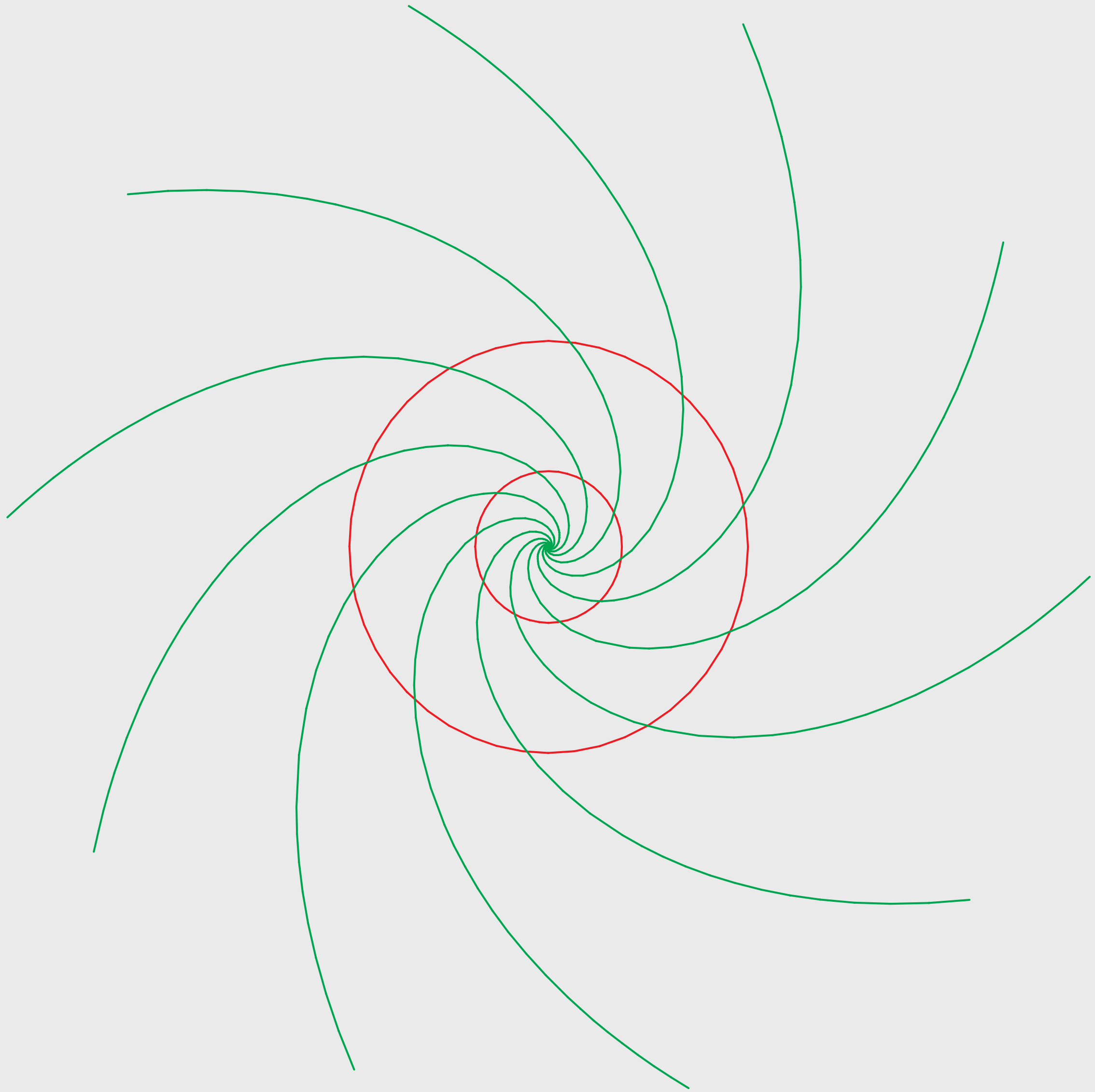
Use: constant density, constant sound speed, zero torque. (You will need an external force.)

The velocity of the fluid flow is

$$\vec{v} = \frac{(A\,\hat{r} + B\,\hat{\theta})}{r}.$$

Streamlines are equiangular spirals.





Physical acoustics: Draining bathtub

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The acoustic metric is

$$\mathrm{d}s^2 = -c_s^2 \mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\mathrm{d}t\right)^2.$$

The acoustic horizon forms once the radial velocity exceeds the speed of sound.

An ergo-surface forms once the speed exceeds the speed of sound.

$$r_{horizon} = \frac{|A|}{c}. \qquad r_{ergo-surface} = \frac{\sqrt{A^2+B^2}}{c}.$$

Mimicking Kerr spacetime:

Can we construct an acoustic geometry that mimics Kerr spacetime in detail?

There is a fundamental geometrical obstruction:

For simple fluids the spatial slices of the acoustic geometry are always conformally flat.

The spatial slices of Kerr are never conformally flat.

The best you can hope for is to consider the equatorial slice of the Kerr spacetime.

Most general line vortex:

The background flow, which determines the "acoustic metric" is governed by the continuity, Euler, and barotropic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \vec{v}) = 0.$$

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] = -\nabla p + \vec{f}.$$

$$p = p(\rho).$$

Engineering perspective:

$$\vec{f} = \rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] + \nabla p,$$

Zero radial flow:

$$\vec{v}(r) = v_{\hat{\theta}}(r)\ \hat{\theta}.$$

$$\vec{a} = (\vec{v}\cdot\nabla)\vec{v} = -\frac{v_{\hat{\theta}}(r)^2}{r}\ \hat{r}.$$

$$\vec{f} = f_{\hat{r}}\ \hat{r} = \left\{ -\rho(r)\ \frac{v_{\hat{\theta}}(r)^2}{r} + c^2\ \partial_r \rho(r) \right\} \hat{r}\ .$$

This is the external force required to maintain the vortex.

For zero external force: $$\frac{v_\theta(r)^2}{c(r)^2} = -\ r\ \partial_r \ln \rho(r)$$

General radial flow:

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$$\vec{v} = v_{\hat{r}}(r)\ \hat{r} + v_{\hat{\theta}}(r)\ \hat{\theta}.$$

Continuity implies:

$$\oint \rho(r)\ \vec{v}(r) \cdot \hat{r}\ \mathrm{d}s = 2\pi\ \rho(r)\ v_{\hat{r}}(r)\ r = 2\pi\ k_1.$$

$$\rho(r) = \frac{k_1}{r\ v_{\hat{r}}(r)}.$$

$$\vec{f} = \frac{k_1}{r v_{\hat{r}}}\ (\vec{v} \cdot \nabla)\vec{v} + c_s^2\ \partial_r \left(\frac{k_1}{r v_{\hat{r}}} \right)\ \hat{r},$$

General radial flow:

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Calculating external force and decomposing into angular and radial pieces:

$$f_{\hat{r}} = \vec{f} \cdot \hat{r} = k_1 \left\{ \frac{1}{r v_{\hat{r}}} \; \left[\frac{1}{2} \partial_r [v_{\hat{r}}(r)^2] - \frac{v_{\hat{\theta}}(r)^2}{r} \right] + c_s^2 \; \partial_r \left(\frac{1}{r v_{\hat{r}}} \right) \right\},$$

$$f_{\hat{\theta}} = \vec{f} \cdot \hat{\theta} = k_1 \left\{ \frac{1}{r^2} \partial_r [r \; v_{\hat{\theta}}(r)] \right\}.$$

With enough effort you can mimic any velocity profile.

The Kerr equator:

In Boyer--Lindquist coordinates:

$$(\mathrm{d}s^2)_{(2+1)} = -\mathrm{d}t^2 + \frac{2m}{r}(\mathrm{d}t - a\,\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{1-2m/r+a^2/r^2} + (r^2+a^2)\,\mathrm{d}\phi^2.$$

In the r-ϕ plane:

$$(\mathrm{d}s^2)_{(2)} = \frac{\mathrm{d}r^2}{1-2m/r+a^2/r^2} + \left(r^2+a^2+\frac{2ma^2}{r}\right)\mathrm{d}\phi^2.$$

This is conformally flat, but not obviously so.

Adopt new coordinates:

$$\frac{\mathrm{d}r^2}{1-2m/r+a^2/r^2} + \left(r^2+a^2+\frac{2ma^2}{r}\right)\mathrm{d}\phi^2 = \Omega(\tilde{r})^2\,[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\phi^2].$$

The Kerr equator:

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This gives two equations:

$$\left(r^2 + a^2 + \frac{2ma^2}{r}\right) = \Omega(\tilde{r})^2 \; \tilde{r}^2,$$

$$\frac{\mathrm{d}r^2}{1 - 2m/r + a^2/r^2} = \Omega(\tilde{r})^2 \; \mathrm{d}\tilde{r}^2$$

} Eliminate $\Omega(\tilde{r})$

Differential equation:

$$\frac{1}{\tilde{r}(r)} \, \frac{\mathrm{d}\tilde{r}(r)}{\mathrm{d}r} = \frac{1}{\sqrt{1 - 2m/r + a^2/r^2}\sqrt{r^2 + a^2 + 2ma^2/r}},$$

$$\tilde{r}(r) = \exp\left\{ \int \frac{\mathrm{d}r}{\sqrt{1 - 2m/r + a^2/r^2}\sqrt{r^2 + a^2 + 2ma^2/r}} \right\}.$$

The Kerr equator:

Fix boundary conditions:

$$\tilde{r}(r) = r \exp\left[-\int_r^\infty \left\{ \frac{1}{\sqrt{1 - 2m/\bar{r} + a^2/\bar{r}^2}\sqrt{\bar{r}^2 + a^2 + 2ma^2/\bar{r}}} - \frac{1}{\bar{r}} \right\} \mathrm{d}\bar{r} \right]$$

Define:

$$\begin{cases} \tilde{r} = r \; F(r), \text{ with } \lim_{r\to\infty} F(r) = 1, \\ r = \tilde{r} \; H(\tilde{r}) \text{ with } \lim_{\tilde{r}\to\infty} H(\tilde{r}) = 1. \end{cases}$$

$$\Omega(\tilde{r})^2 = \frac{r^2 + a^2 + 2ma^2/r}{\tilde{r}^2} = H(\tilde{r})^2 \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right),$$

$$(\mathrm{d}s^2)_{(2+1)} = -\mathrm{d}t^2 + \frac{2m}{r}(\mathrm{d}t^2 - 2a\,\mathrm{d}\phi\,\mathrm{d}t) + \Omega(\tilde{r})^2\,[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\phi^2].$$

The Kerr equator:

This is now in "acoustic form":

$$(\mathrm{d}s^2)_{(2+1)} = \Omega(\tilde{r})^2 \left\{ -\Omega(\tilde{r})^{-2} \left[1 - \frac{2m}{r} \right] \mathrm{d}t^2 - \Omega(\tilde{r})^{-2} \frac{4am}{r}\, \mathrm{d}\phi\, \mathrm{d}t + [\mathrm{d}\tilde{r}^2 + \tilde{r}^2\, \mathrm{d}\phi^2] \right\}.$$

Pick off the coefficients:

$$\frac{\rho}{c} = \Omega(\tilde{r})^2 = H^2(\tilde{r}) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3} \right).$$

$$v_\phi = \Omega(\tilde{r})^{-2}\, \frac{2am}{r} = -\frac{2am}{r}\, H^{-2}(\tilde{r}) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3} \right)^{-1}.$$

$$c^2 = \Omega(\tilde{r})^{-4}\, H^2(\tilde{r}) \left\{ 1 - \frac{2m}{r} + \frac{a^2}{r^2} \right\}$$

$$\rho = H(\tilde{r}) \sqrt{1 - \frac{2m}{r} + \frac{a^2}{r^2}}$$

This is the "equivalent vortex"!

The Kerr equator:

$$\frac{\rho}{c} = \Omega^2(r) = F^{-2}(r) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right).$$

$$v_\phi = -\frac{2am}{r} F^2(r) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)^{-1}.$$

$$c^2 = F^2(r) \left\{1 - \frac{2m}{r} + \frac{a^2}{r^2}\right\} \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)^{-2}.$$

$$\rho(r) = F^{-1}(r) \sqrt{1 - \frac{2m}{r} + \frac{a^2}{r^2}}$$

$$F(r) = \exp\left[-\int_r^\infty \left\{\frac{1}{\sqrt{1 - 2m/\bar{r} + a^2/\bar{r}^2}\sqrt{\bar{r}^2 + a^2 + 2ma^2/\bar{r}}} - \frac{1}{\bar{r}}\right\} \mathrm{d}\bar{r}\right].$$

This has the advantage of being completely explicit, albeit a trifle messy!

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The Kerr equator:

Ergo-surface: $$r_E = 2m.$$

Horizon: $$r_H = m + \sqrt{m^2 - a^2} < r_E.$$

The Kerr equator can [in principle] be exactly simulated by a very specific vortex.

This needs a very specific external force, and a very specific equation of state.

This is not likely to be experimentally feasible.

Somewhat disappointing!

The Kerr equator:

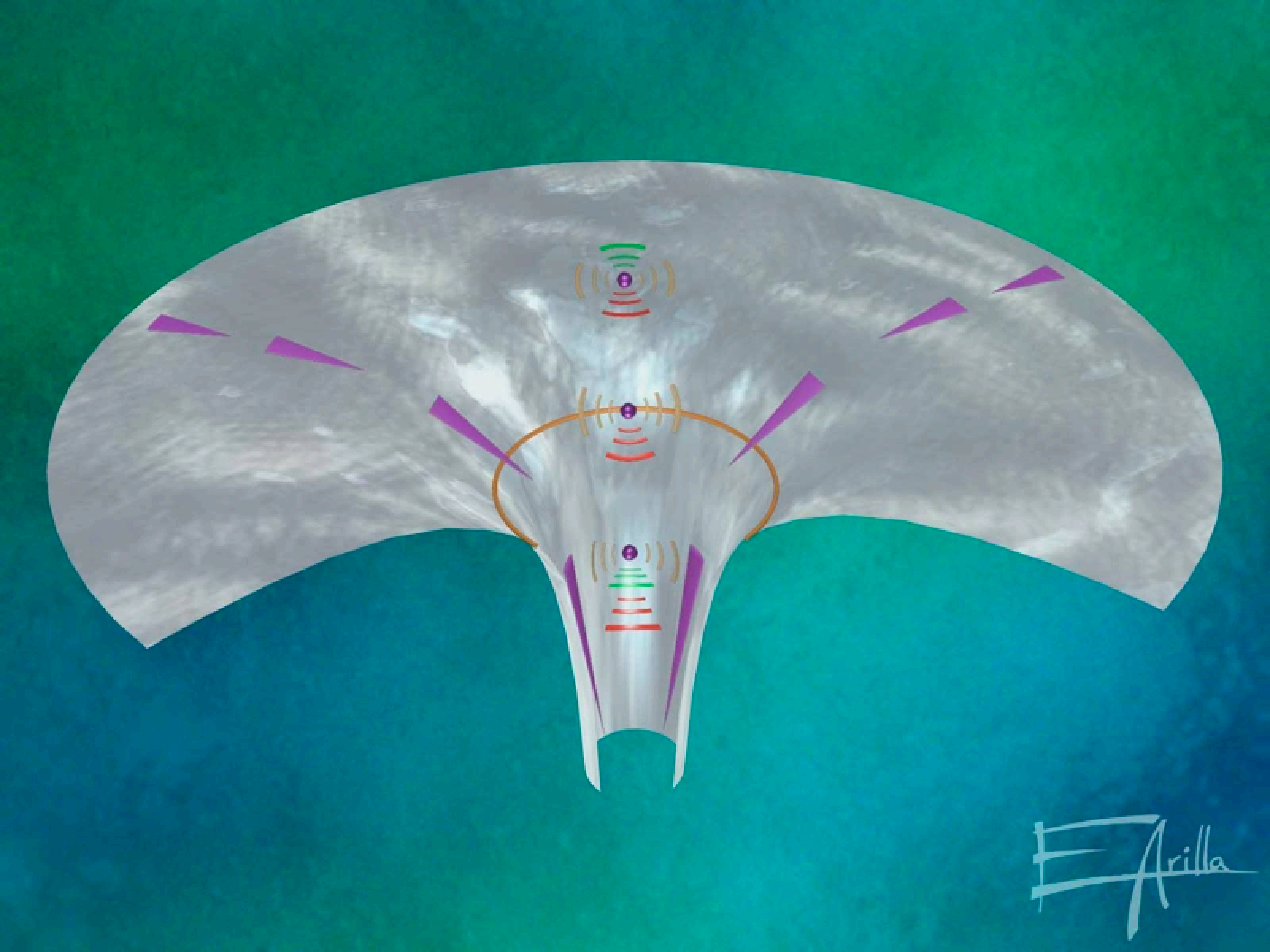
Technical surprise:

The Doran coordinates were not useful?

(Doran coordinates are the natural extension of Painleve--Gullstrand coordinates, which are very useful for the "acoustic Schwarzschild" geometry.

The problem lies with the off-diagonal parts of the space metric...

For the future: simple "analogue models" that generate fully general geometries...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