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POWER LAWS, SCALE INVARIANCE AND THE GENERALIZED FROBENIUS SERIES: APPLICATIONS TO NEWTONIAN AND TOV STARS NEAR CRITICALITY

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We present a self-contained formalism for calculating the background solution, the linearized solutions and a class of generalized Frobenius-like solutions to a system of scale-invariant differential equations.

We first cast the scale-invariant model into its equidimensional and autonomous forms, find its fixed points, and then obtain power-law background solutions. After linearizing about these fixed points, we find a second linearized solution, which provides a *distinct* collection of power laws characterizing the deviations from the fixed point. We prove that generically there will be a region surrounding the fixed point in which the complete general solution can be represented as a generalized Frobenius-like power series with exponents that are integer multiples of the exponents arising in the linearized problem. While discussions of the linearized system are common, and one can often find a discussion of power-series with integer exponents, power series with irrational (indeed complex) exponents are much rarer in the extant literature. The Frobenius-like series we encounter can be viewed as a variant of the rarely-discussed Liapunov expansion theorem (not to be confused with the more commonly encountered Liapunov functions and Liapunov exponents).

As specific examples we apply these ideas to Newtonian and relativistic isothermal stars and construct two separate power series with the overlapping radius of convergence. The second of these power series solutions represents an expansion around "spatial infinity," and in realistic models it is this second power series that gives information about the stellar core, and the damped oscillations in core mass and core radius as the central pressure goes to infinity. The power-series solutions we obtain extend classical results; as exemplified for instance by the work of Lane, Emden, and Chandrasekhar in the Newtonian case, and that of Harrison, Thorne, Wakano, and Wheeler in the relativistic case.

We also indicate how to extend these ideas to situations where fixed points may not exist — either due to "monotone" flow or due to the presence of limit cycles. Monotone flow generically leads to logarithmic deviations from scaling, while limit cycles generally lead to discrete self-similar solutions.

Keywords: Power law; Frobenius; Liapunov; scale-invariant; stellar structure.

1. Introduction

The presence of power-law behavior in nature is such an extremely common phenomenon that considerable lore has now grown up concerning its genesis. One of the most common situations in which it occurs is in the presence of scale-invariant systems. Such behavior occurs, for instance, in any sort of thermodynamic system undergoing a second-order phase transition, and the behavior of physical quantities (such as susceptibilities) in terms of the distance from criticality is typically given by a power-law:

$$\chi \propto \left(\frac{T - T_{\text{critical}}}{T_{\text{critical}}} \right)^{\delta} . \tag{1}$$

Second-order phase transitions are extremely well described by statistical field theories. The associated technical machinery (the renormalization group), is now so well developed that it is sometimes difficult to remember that second-order phase transitions are not the *only* route to power-law behavior. The onset of power-law behavior actually occurs at a much more primitive level, and can be analyzed directly in terms of the underlying differential equations.

Scale invariance is commonly thought to be synonymous with power-law behavior. This is not quite correct: Scale invariance is not enough to guarantee power-law behavior, while power-law behavior implies scale invariance only in the trivial sense that it is always possible to write down *some* scale-invariant differential equation. (But such a DE does not necessarily have to describe a physical theory.) In the present paper, we examine scale-invariant theories and find one direct route to power-law behavior.

This formalism consists of transforming a scale-invariant theory into its associated autonomous form[1] and then determining the location of its fixed points. These fixed points will lead inexorably to a background solution that exhibits power-law behavior. By expanding the system about its fixed point, a linearized autonomous differential equation emerges, whose solution will exhibit *different* power-law behavior. This naturally leads to two independent power laws: one for the background fixed point and a second one for the linearized deviations from the fixed point. Combining both results, our formalism indicates that the complete general solution to the system will, over some limited region, have a Frobenius-like power-series expansion. The usual Frobenius series is one of the form

$$y(x) = x^p \cdot \sum_{i=0}^{\infty} a_i x^i \tag{2}$$

with i a natural number, and is suitable for exploring linear ODE's in the region of regular singular points. In the present context, involving nonlinear DE's, the generalized Frobenius-like series we encounter are generically of the form

$$y(x) = x^p \cdot \sum_{i=0}^{\infty} a_i x^{i\lambda} \,. \tag{3}$$

The exponent in the prefactor of this Frobenius-like series (p) is governed by the exponent associated with the fixed-point power-law background behavior, while the power series is given in terms of exponents ($i\lambda$) that are integer multiples of those arising in the linearized problem (λ). This type of expansion is essentially a modification of that given by the rarely-discussed Liapunov expansion theorem for autonomous DE's,[2] wherein we have now back-tracked to give the expansion in terms of the original scale-invariant system. While one may often find a discussions of the linearized system, and often find discussion of power-series with integer exponents, power series with irrational (indeed complex) exponents are much rarer in the extant literature.

As specific examples we apply this formalism to both Newtonian and relativistic isothermal stars, where the relevant differential equations are equivalent to the exponential [isothermal] Lane–Emden equation and its relativistic generalization. The resultant Frobenius-like power series generalize the approximate solutions found, for instance, by Chandrasekhar[3] and by Harrison *et al.*,[4] and provide significantly greater analytic information concerning their structure. In particular we derive nonlinear recursion relations for the coefficients of the power-series.

This formalism is then also applied to a more general star, where the equation of state need not be linear except in the core. Both Newtonian and relativistic stars are verified to undergo damped oscillations in the total mass, pressure and radius as the central density goes to infinity. These oscillations can accurately be described by a Frobenius-like power series with complex irrational exponents. The relevant exponents (though not the numerical values of the coefficients) drop out naturally from the linearized problem. This power law solution matches the behavior observed in numerical calculations to great accuracy.

2. Why Power Laws are Ubiquitous

In this section we lay down the basic foundations of the formalism. First, we define "scale-invariant," "equidimensional," and "autonomous," and present the appropriate substitutions to go from one form to another. A single nth-order differential equation is chosen as a model because of its simplicity, and then the formalism is extended to a system of I coupled nth-order differential equations. After determining the existence of fixed points in the equation, the background solution is obtained. The differential equation is then expanded about its fixed points to obtain a second power-series solution near the fixed point (linearized solution). Combining both results, a Frobenius-like series solution is obtained in a neighborhood close enough to the fixed point.

2.1. *Fundamental definitions*

Let us define scale-invariant differential equations in the usual manner, as those which remain unchanged under the substitutions $x \to ax$ and $y(x) \to a^p y(ax)$, where a^p is a fixed but arbitrary parameter.[1] Equations that remain unchanged under the substitutions $x \to bx$ and $y(x) \to y(x)$, where b is another fixed but arbitrary parameter,[1] will be defined as equidimensional-in-x. Thus equidimensional equations can always be viewed as scale-invariant equations corresponding to the particular value $p = 0$ for the exponent. Finally, autonomous equations will be defined as those equations which remain unchanged under the substitutions $x \to x + c$ and $y(x) \to y(x)$, where c is a third fixed but arbitrary parameter.[1] This last condition is that of translation invariance in the independent variable x.

It is a standard result that it is always possible to transform scale-invariant equations into equidimensional-in-x equations through the following substitution[1]

$$y(x) = x^p \cdot w(x)\,. \tag{4}$$

It is also a standard result that we can transform equidimensional-in-x equations into autonomous equations by substituting[1]

$$x = e^t\,, \qquad z(t) = w(x)\,, \tag{5}$$

where t is the new variable.

2.2. *From scale invariance to autonomy*

Let us first consider a nth-order scale-invariant differential equation in a single dependent variable,

$$F\left(x, y(x), y'(x), y''(x), \ldots, \frac{d^n y(x)}{dx^n}\right) = 0\,, \tag{6}$$

where the primes stand for differentiation with respect to x. We will write this as $F(x, y(x)) = 0$, but the presence of an appropriate number of derivatives should be inferred. Now assuming this equation to be scale-invariant, let us apply Eq. (4) to make Eq. (6) equidimensional-in-x. The derivatives of $y(x)$ become

$$\frac{d^n y(x)}{dx^n} = \left(\frac{d}{dx}\right)^n [x^p w(x)] = \sum_{j=0}^{n} \frac{n!\,p!}{j!(n-j)!(p-j)!}\, x^{p-j} w^{(n-j)}(x)\,. \tag{7}$$

Note that the jth derivative of $y(x)$ involves only the jth and lower-order derivatives of $w(x)$. Rewriting F, we can define a new function $\tilde{F}$ as follows,

$$\begin{aligned} &\tilde{F}\left(x, w(x), w'(x), \ldots, \frac{d^n w(x)}{dx^n}\right) \\ &\equiv F\left(x, x^p w(x), p x^{p-1} w(x) + x^p w'(x), \ldots, \sum_{j=0}^{n} \frac{n!p!}{j!(n-j)!(p-j)!} x^{p-j} w^{(n-j)}(x)\right), \end{aligned} \tag{8}$$

since F depends only on x, $w(x)$ and its derivatives. The new differential equation $\tilde{F}(x, w(x)) = 0$ is now equidimensional-in-x.

Let us now apply the substitution in Eq. (5) to transform it into an autonomous equation. The derivatives of z are

$$\frac{d^n}{dx^n} = \left(e^{-t}\frac{d}{dt}\right)^n = e^{-nt}\prod_{j=0}^{n-1}\left(\frac{d}{dt} - j\right). \tag{9}$$

As before, note here that the product in Eq. (9) involves an nth order term plus lower-order derivatives (and no higher-order derivatives). Rewriting $\tilde{F}$ we can define a new function $\bar{F}$ via

$$\begin{aligned}&\bar{F}\left(z(t), \dot{z}(t), \ldots, \frac{d^n}{dt^n}z(t)\right)f(t)\\ &\quad = \tilde{F}\left(e^t, z(t), e^{-t}\dot{z}(t), \ldots, e^{-nt}\prod_{j=0}^{n-1}\left(\frac{d}{dt} - j\right)z(t)\right),\end{aligned} \tag{10}$$

where the dots denote differentiation with respect to t and $f(t)$ is some arbitrary function of t. The function $\bar{F}(z(t))$ is now translation invariant, since the substitution $t \to t + b$ as $z(t) \to z(t + b)$ leaves the equation $\tilde{F}(x, w(x)) = 0$ (based on setting the RHS = zero) unchanged. The only way this can be true is if we can factorize $\tilde{F}(x, w(x))$ into an autonomous $\bar{F}$ times some $f(t)$, yielding the LHS.

Our original scale-invariant equation $F(x, y(x)) = 0$, that had become equivalent to the equidimensional-in-x equation $\tilde{F}(x, w(x)) = 0$, has now become equivalent to the autonomous equation $\bar{F}(z(t)) = 0$. We can of course also work backward and prove that for any autonomous-in-t equation for the dependent variable $z(t)$ the substitution $x = e^t$ (that is, $d/dt = x\,d/dx$) produces an equation that is equidimensional-in-x with $z(t) = z(\ln x) = w(x)$. Furthermore for any value of p the substitution $w(x) = x^{-p}y(p)$ results in a differential equation that is scale-invariant. However, this scale-invariant equation need not be equal to the physical differential equation that describes the system we are concerned with. Without some additional information beyond the mere existence of a power law, there is simply not much one can say about the physics.

2.3. *Scale-invariant ordinary differential equations of nth-order*

2.3.1. *Background solution*

In order to obtain the background solution of the ODE, we must look at the fixed points. Let us assume that $\bar{F}$ has a fixed point z_* such that all derivatives of $z(t)$ vanish at that fixed point, i.e.

$$\bar{F}\left(z(t) = z_*, \frac{d^i z(t)}{dt^i} = 0\right) = 0, \tag{11}$$

where $i \in [1, n]$ denotes any positive integer in that range. Hence, a solution to the differential equation $\bar{F}(z(t)) = 0$ is $z(t) = z_*$. Tracing back all the substitutions we made, we obtain

$$y(x) = x^p z_* \quad \text{which solves} \quad F(x, y(x)) = 0\,. \tag{12}$$

From this equation, the power law behavior is obvious. However, not all scale-invariant equations possess fixed points. A scale-invariant differential equation, whose associated autonomous equation *does not* have any fixed points, *will not* possess power-law solutions. Refer to App. A for an explicit counter-example. In light of this, we can formulate the following theorem:

Theorem 1. *Any first-order differential equation that is scale-invariant* and *whose associated autonomous equation possesses a fixed point will have a* background *solution in the form of a power law* [*Eq.* (12)] *with exponent given by the scale invariant condition.*

2.3.2. *Linearized solution*

Let us linearize the autonomous equation by setting $z(t) = z_* + z_1(t) + O[z_1(t)^2]$, where z_* is the fixed point and $z_1(t)$ is a small perturbation about z_*. $\bar{F}(z(t)) = 0$ reduces to

$$\bar{F}(z(t)) = \bar{F}(z_*) + \bar{F}_1(z_1(t)) + O[z_1(t)^2] = \bar{F}_1(z_1(t)) + O[z_1(t)^2] = 0\,. \tag{13}$$

So in the linear approximation, we must have

$$\bar{F}_1(z_1(t)) = 0\,. \tag{14}$$

But $\bar{F}_1(z_1(t)) = 0$ is an ODE which is both autonomous and linear, and still retains its character as a nth-order differential equation. Autonomous plus linear is a very powerful constraint, and in fact implies a differential equation with constant coefficients. That is, there is some polynomial $P(\cdot)$ such that the function $\bar{F}_1(z_1(t))$ satisfies

$$\bar{F}_1(z_1(t)) \equiv P\left(\frac{d}{dt}\right) z_1(t)\,. \tag{15}$$

The standard technique[1] for solving differential equations of this type is to insert a trial solution of the form $z_1(t) = A \exp(\lambda t)$. Then the differential equation reduces to the simple nth-order polynomial

$$P(\lambda) = 0\,, \tag{16}$$

which has at most n distinct solutions. Assuming for the time being that all roots are distinct, the general solution to the linearized autonomous equation is

$$z_1(t) = \sum_{i=1}^{n} A_i \exp(\lambda_i t)\,. \tag{17}$$

That is

$$z(t) = z_* + \sum_{i=1}^{n} A_i \exp(\lambda_i t) + O(A^2)\,. \tag{18}$$

In terms of the original variable $y(x)$

$$y(x) = x^p \left[z_* + \sum_{i=1}^{n} A_i x^{\lambda_i} + O(A^2) \right]. \tag{19}$$

Near the fixed point, we now see the presence of multiple power laws. The prefactor x^p is governed directly by the scale-invariant condition, while the subsidiary exponents arise from solving a polynomial equation based on the linearized approximation.

We are now in a position to formulate the following theorem:

Theorem 2. *Any nth-order ordinary differential equation that is scale-invariant* and *whose associated autonomous equation possesses a fixed point will have a linearized solution which is generically expressible in the form of a power law* (*Eq.* (19)) *in a neighborhood surrounding the fixed point.*

We can also enunciate a general rule of thumb: Because the exponent p ultimately arises from algebraic manipulations involving the physical dimensionalities of the various quantities present in the ODE it is likely to be a rational number. In contrast the exponents λ_i, being solutions of a polynomial equation, will generically be irrational.

In the exceptional case where the roots of the polynomial $P(\lambda)$ are not distinct, then there are additional technical complications. Let m of the roots be distinct, and let g_i with $i \in [1, m]$ be the multiplicity of the ith distinct root. Then we have $\sum_i^m g_i = n$ and

$$z_1(t) = \sum_{i=1}^{m} \sum_{j=1}^{g_i - 1} A_{ij} t^{i-1} \exp(\lambda_i t)\,. \tag{20}$$

That is

$$z(t) = z_* + \sum_{i=1}^{m} \sum_{j=1}^{g_i - 1} A_{ij} t^{i-1} \exp(\lambda_i t) + O(A^2)\,. \tag{21}$$

In terms of the original variable $y(x)$

$$y(x) = x^p \left[z_* + \sum_{i=1}^{m} \sum_{j=1}^{g_i - 1} A_{ij} (\ln x)^{i-1} x_i^{\lambda} + O(A^2) \right]. \tag{22}$$

Thus logarithmic deviations from power law behavior, while not generic, are certainly relatively easy to get — despite the lore based in statistical field theory and second-order phase transitions a logarithm is not necessarily the result of one-loop physics. Logarithms can arise from causes as mundane as a repeated root in a linearized autonomous ODE.

2.3.3. *Frobenius-like solution*

Going beyond the linearized approximation, in the vicinity of the fixed point, one can write the expansion more systematically as $z(t) = z_* + \epsilon z_1(t) + \epsilon^2 z_2(t) + O[\epsilon^3]$, where $\epsilon \ll 1$, in which case $\bar{F}(z(t)) = 0$ reduces to

$$\bar{F}(z(t)) = \bar{F}(z_*) + \epsilon \bar{F}_1((z_1(t)) + \epsilon^2[\bar{F}_1(z_2(t)) + \bar{F}_2(z_1(t), z_1(t))] + O(\epsilon^3)\,. \tag{23}$$

Here $\bar{F}_2(\cdot,\cdot)$ represents a second-order functional derivative of $\bar{F}(\cdot)$ around the fixed point, which will be some messy quadratic differential operator with constant coefficients. This implies both

$$\bar{F}_1(z_1(t)) = 0\,, \tag{24}$$

and

$$\bar{F}_1(z_2(t)) = -\bar{F}_2(z_1(t), z_1(t))\,. \tag{25}$$

That is

$$z_2(t) = -\bar{F}_1{}^{-1}[\bar{F}_2(z_1(t), z_1(t))]\,. \tag{26}$$

But we already know that $z_1 = \sum_{i=1}^{n} A_i \exp(\lambda_i t)$, so z_2 consists of pieces of the form $\exp([\lambda_i + \lambda_j]t)$. Indeed

$$z_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} B(A_i, A_j)\exp([\lambda_i + \lambda_j]t)\,. \tag{27}$$

with $B(A_i, A_j)$ being some complicated function of the A_i that is deducible from the precise form of $\bar{F}_1$ and $\bar{F}_2$.

Collecting terms

$$z(t) = z_* + \sum_{i=1}^{n} A_i \exp(\lambda_i t) + \sum_{i=1}^{n}\sum_{j=1}^{n} B(A_i, A_j)\exp([\lambda_i + \lambda_j]t) + O(\epsilon^3)\,. \tag{28}$$

In terms of the original variable $y(x)$

$$y(x) = x^p \cdot \left[z_* + \sum_{i=1}^{n} A_i x^{\lambda_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} B(A_i, A_j) x^{\lambda_i + \lambda_j} + O(\epsilon^3) \right]. \tag{29}$$

The general pattern is now clear. Let us take $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and let $\mathbf{m} = (m_1, m_2, \ldots m_n) \in \mathbb{N}^n$. Then formally

$$z(t) = \sum_{\mathbf{m}\in\mathbb{N}^n} A(\mathbf{m})\exp(\boldsymbol{\lambda}\cdot\mathbf{m}t)\,. \tag{30}$$

This series may be asymptotic, in general we make no claims about convergence, though in specific cases it often does converge in an open region surrounding the fixed point. The coefficients $A(\mathbf{m})$ are certainly not independent but are interrelated via the underlying differential equation. This is basically a variant of the rarely-discussed Liapunov expansion theorem. See for instance Lefschetz.[2] Readers interested in additional mathematical rigor are directed to that reference. Be careful not to confuse Liapunov's expansion theorem with the much more common notions of Liapunov functions and/ or Liapunov exponents.

Near the fixed point our equation becomes, in terms of the original variables $y(x)$,

$$y(x) = x^p \cdot \sum_{\mathbf{m} \in \mathbb{N}^n} A(\mathbf{m}) x^{\boldsymbol{\lambda} \cdot \mathbf{m}} \,. \tag{31}$$

This type of series is Frobenius-like in the sense that the prefactor x^p will be governed by the background solution, while the exponents of the power series will be given by the linearized problem. Hence, we can formulate the following theorem:

Theorem 3. *Any* n*th-order ordinary differential equation that is scale invariant* and *whose associated autonomous equation possesses a fixed point will generically have a formal solution in the form of a Frobenius-like power series* [*Eq.* (31)] *in a neighborhood close to the fixed point.*

As per the previous discussion, this form holds only if the roots of the linearized problem are distinct. For coincident roots one should expect the Frobenius-like series above to be modified by various logarithmic terms. Indeed based on Liapunov's expansion theorem[2] the coefficients $A(\mathbf{m})$ become polynomial in t, and so polynomial in $\ln z$ [cf. Eq. (22) above]. The resulting series is too clumsy to be of immediate use and fortunately none of the specific examples we encounter exhibit this particular behavior.

2.4. *Systems of* n*th-order scale-invariant differential equations*

2.4.1. *Background solution*

All of the previous analysis can be generalized to systems of I coupled nth-order ordinary differential equations. There are three possible routes:

(1) Develop a suitable "vector" and "matrix" notation.
(2) Use the fact that a system of I coupled nth-order differential equations can always be reduced to a system of $N = I \times n$ coupled first-order differential equations,[1] but you would still need to develop a suitable "vector" and "matrix" notation.
(3) Use the fact that a system of I coupled nth-order differential equations can generically be "decoupled" by being reduced to a single DE of order $N = I \times n$ in one of the dependent variables, though there seems to be no constructive algorithm for doing so.[1]

On balance, we have found it most useful to directly develop a suitable "vector" and "matrix" notation. Let us begin by considering the equation

$$[F] = 0 \,, \tag{32}$$

where $[F]$ is the column vector of ODE's

$$[F] = \begin{pmatrix} F_1\left(x, y_1(x), y_1'(x), \ldots, y_2(x), y_2'(x), \ldots, y_I(x), \ldots, \dfrac{d^n y_I(x)}{dx^n}\right) \\ F_2\left(x, y_1(x), y_1'(x), \ldots, y_2(x), y_2'(x), \ldots, y_I(x), \ldots, \dfrac{d^n y_I(x)}{dx^n}\right) \\ \vdots \\ F_I\left(x, y_1(x), y_1'(x), \ldots, y_2(x), y_2'(x), \ldots, y_I(x), \ldots, \dfrac{d^n y_I(x)}{dx^n}\right) \end{pmatrix}. \tag{33}$$

When we say that this system is scale-invariant, we mean that as $x \to ax$ and

$$[Y(x)] \to [A][Y(x)]\,, \tag{34}$$

where

$$[A] = \begin{bmatrix} a^{p_1} & 0 & 0 & \ldots & 0 \\ 0 & a^{p_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a^{p_I} \end{bmatrix} \tag{35}$$

and $[Y(x)]$ is a column vector of dependent variables, the equation $[F] = 0$ remains invariant. Note that this means that we have chosen our dependent variables $y_i(x) \cdots y_I(x)$ in some convenient manner to make their scale transformation properties both simple and independent of each other — this will almost always occur automatically in systems of ODE's derived from an underlying physical problem.

Previously, we also showed that any nth-order differential equation in a single dependent variable that is scale-invariant can be put into its associated autonomous form by applying the substitutions described in Eqs. (4) and (5). Therefore, it carries over that we can make the same kind of substitution in each equation of the system. After doing so, we can define a new column vector $[\bar{F}]$ in the following manner:

$$[\bar{F}] = \begin{pmatrix} \bar{F}_1\left(z_1(t), \dot{z}_1(t), \ldots, z_2(t), \dot{z}_2(t), \ldots, z_I(t), \ldots, \dfrac{d^n z_I(t)}{dt^n}\right) \\ \bar{F}_2\left(z_1(t), \dot{z}_1(t), \ldots, z_2(t), \dot{z}_2(t), \ldots, z_I(t), \ldots, \dfrac{d^n z_I(t)}{dt^n}\right) \\ \vdots \\ \bar{F}_I\left(z_1(t), \dot{z}_1(t), \ldots, z_2(t), \dot{z}_2(t), \ldots, z_I(t), \ldots, \dfrac{d^n z_I(t)}{dt^n}\right) \end{pmatrix}, \tag{36}$$

where for each $i \in [1, I]$ we have $y_i(x) = x^{p_i} w_i(x) = e^{tp_i} z_i(t)$ and furthermore $x = e^t$. From the previous analysis we know that each equation in this new vector equation $[\bar{F}] = 0$ is now autonomous, hence, the whole system is autonomous. Let's now assume that this autonomous system has a fixed point $[Z^*]$ such that all derivatives of $z(t)$ with respect to t in $[\bar{F}]$ vanish at the fixed point, where we mean

$$[Z^*] = \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_I^* \end{pmatrix} . \tag{37}$$

That is

$$[Z(t)] = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_I(t) \end{pmatrix} = [Z^*] = \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_I^* \end{pmatrix} . \tag{38}$$

is one specific solution to the system of ODE's.

Substituting back we obtain the following equations for $[Y(x)]$:

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_I(x) \end{pmatrix} = \begin{bmatrix} x^{p_1} & 0 & 0 & \cdots & 0 \\ 0 & x^{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{p_I} \end{bmatrix} \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_I^* \end{pmatrix} = \begin{pmatrix} z_1^* x^{p_1} \\ z_2^* x^{p_2} \\ \vdots \\ z_I^* x^{p_I} \end{pmatrix} . \tag{39}$$

Therefore, we can formulate the following theorem:

Theorem 4. *Any system of I nth-order differential equations that is scale invariant*, and *whose associated system of autonomous equations possesses a fixed point*, *will have solutions in the form of a collection of power laws* (*Eq.* (39)), *with exponents given by the scale-invariant condition.*

2.4.2. *Linearized solution*

If we linearize the nth order system, then for each $\bar{F}_i$ we will obtain a linear differential equation that has the capability of mixing the various z_i. Specifically let

$$z_i(t) = z_i^* + \epsilon z_{1,i}(t) + O(\epsilon^2) , \tag{40}$$

then

$$\bar{F}_i(z_j(t)) = \bar{F}_i(z_j^*) + \sum_{j=1}^{n} \{\bar{F}_1\}_{ij}(z_{1,j}) + O(\epsilon^2) . \tag{41}$$

Here the $\{\bar{F}_1\}_{ij}$ are linear differential operators that lead to a system of linear and autonomous differential equations

$$\sum_{j=1}^{n} \{\bar{F}_1\}_{ij}(z_{1,j}(t)) = 0\,. \tag{42}$$

This implies that each of the $\{\bar{F}_1\}_{ij}$ is a constant coefficient polynomial in d/dt

$$\{\bar{F}_1\}_{ij} \equiv P_{ij}\left(\frac{d}{dt}\right). \tag{43}$$

Inserting a trial solution of the form $z_{1,j}(t) = A_j \exp(\lambda t)$ now yields the vector equation

$$\sum_{j=1}^{n} P_{ij}(\lambda) A_j = 0\,. \tag{44}$$

There is a nontrivial solution if and only if the $I \times I$ matrix $P_{ij}(\lambda)$ is singular ($\det\{P_{ij}(\lambda)\} = 0$), in which case the A_i correspond to the singular eigenvector. Since each element of the matrix $P_{ij}(\lambda)$ is itself a polynomial of order n, (where n is the order of the individual differential equations in the system), the determinant is a polynomial of order $N = I \times n$ having up to N separate roots. Let us assume these roots are distinct and call them λ_α with $\alpha \in (1, N)$ and denote the corresponding eigenvector by $(A_\alpha)_i$. As long as the roots are distinct these eigenvectors, multiplied by the associated exponentials, will span the solution space and we can write the general solution to our autonomous system as

$$z_{1,i}(t) = \sum_{\alpha=1}^{N} (A_\alpha)_i \exp(\lambda_\alpha t)\,. \tag{45}$$

In terms of our original equation,

$$y_i(x) = x^{p_i}\left\{ z_i^* + \sum_{\alpha=1}^{N} (A_\alpha)_i x^{\lambda_\alpha} + O(\epsilon^2) \right\}. \tag{46}$$

This now allows us to formulate the following theorem:

Theorem 5. *Any system of I nth-order differential equations that is scale invariant* and *whose associated autonomous equation possesses a fixed point will generically have solutions in the form of a collection of power laws* (*Eq.* (46)) *in a neighborhood close to the fixed point. The prefactors* $[x^{\mathcal{P}}]$ *will be governed by the background solution, while the λ_α are given by the linearized problem.*

If the roots are not distinct then a much messier equation involving logarithms in x can be constructed, but is beyond the scope of the applications we have in mind.

2.4.3. *Frobenius-like solution*

We can again go beyond the linearized approximation in the vicinity of the fixed point and write $z_i(t) = z_i^* + \epsilon z_{1,i}(t) + \epsilon^2 z_{2,i}(t) + O[\epsilon^3]$. Then,

$$\bar{F}_i(z_j(t)) = \bar{F}_i(z_j^*) + \epsilon \sum_{j=1}^{n} \{\bar{F}_1\}_{ij}(z_{1,j})$$
$$+ \epsilon^2 \left[\sum_{j=1}^{N} \{\bar{F}_1\}_{ij}(z_{2,j}) + \sum_{j=1}^{N} \sum_{k=1}^{N} \{\bar{F}_2\}_{ijk}(z_{1,j}(t), z_{1,k}(t)) \right] + O(\epsilon^3)\,, \quad (47)$$

where, similarly to the preceding discussion for a single dependent variable, the quantity $\{\bar{F}_2\}_{ijk}(\cdot,\cdot)$ represents the second-order functional derivative of $\{\bar{F}\}_i(\cdot)$ around the fixed point $[Z^*]$. Using the same reasoning as before, we will obtain a solution of the form

$$z_{2,i}(t) = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \{B(A_\alpha, A_\beta)\}_i \exp([\lambda_\alpha + \lambda_\beta]t)\,. \quad (48)$$

Collecting terms and substituting back to our original solutions of $y_i(x)$ we obtain

$$y_i(x) = x^{p_i} \left[z_i^* + \sum_{\alpha=1}^{N} \{A_\alpha\}_i x^{\lambda_\alpha} + \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \{B(A_\alpha, A_\beta)\}_i x^{\lambda_\alpha + \lambda_\beta} + O(\epsilon^3) \right], \quad (49)$$

The general pattern is again clear

$$y_i(x) = x^{p_i} \cdot \sum_{\mathbf{m} \in \mathbb{N}^N} \{A(\mathbf{m})\}_i x^{\boldsymbol{\lambda} \cdot \mathbf{m}}\,, \quad (50)$$

where as before, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\mathbf{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{N}^N$. We can now formulate the following theorem:

Theorem 6. *Any system of I nth-order differential equations that is scale-invariant* and *whose associated autonomous equation possesses fixed points z_{i*} will generically have a solution in the form of a Frobenius-like power series* (*Eq.* (50)) *in a neighborhood close to the fixed point. The prefactors x^{p_i} will be governed by the background solution*, *while the exponents of the power series will be given by the linearized problem.*

As previously, there is an explicit requirement that the exponents arising from the linearized problem be distinct, otherwise (as per Liapunov's expansion theorem) messy expressions polynomial in $\log x$ will be encountered. (See Lefschetz[2] for technical details.)

2.5. *Discussion*

Before we turn to several specific astrophysical examples to show how this formalism works in detail, let us summarize key points:

- Scale invariance does not automatically imply power-law behavior; the existence of one or more fixed points is an additional requirement. An explicit example of a scale-invariant ODE without any power-law solution is presented in App. A.
- If we perturb slightly away from the fixed points, there will no longer be an exact power-law behavior, but we have something similar: Generically there will be a Frobenius-like expansion of the form

$$y(x) = x^p \cdot \sum_{i=0}^{\infty} a_i x^{i\lambda} \,, \tag{51}$$

  where the exponent p depends on the fixed point and the exponent λ depends on the linearized ODE around the fixed point. The coefficients a_i are of course constrained by the original exact ODE, and since it is nth-order we expect n of these coefficients to be "free" (that is, to depend on initial conditions) with the other coefficients in principle being determinable from the first n and the exact ODE.
- We feel that this formalism, attributing as it does power-law behavior to rather general features of differential equations, goes a long way towards explaining the apparent ubiquitous occurrence of power-laws in nature.

3. Application: Isothermal Newtonian Stars

In the following section, we apply the general formalism developed above to a static isothermal Newtonian star. The relevant system of ODE's is scale-invariant, and contains two fixed points, one corresponding to the center of the star, and the other corresponding to the point at infinity (where the star develops an infinitely thin halo). We formally solve the differential system associated with it and obtain power-series solutions for the total pressure, radius, compactness and mass of the star in terms of a Frobenius-like series about either fixed point. For each of these physical quantities, the radius of convergence of these series (one for each fixed point) overlap, so they constitute a complete power-series solution. These results subsume and extend previous results found by Chandrasekhar,[3] and the classic results of Lane and Emden. In particular, these series are then used to discuss the behavior of such stars near the onset of collapse.

3.1. *Background solution*

The basic equations of (static) nonrelativistic stellar structure[3] are ($G \equiv 1$)

$$\frac{dp}{dr} = -\rho(r)\,\frac{m(r)}{r^2}\,, \tag{52}$$

$$m(r) = \int_0^r 4\pi {r'}^2 \rho(r') dr' \,. \tag{53}$$

Let us now adopt the following particularly simple equation of state, which is appropriate, for instance, for an isothermal star:

$$p(r) = c_s^2 \rho \,. \tag{54}$$

Then

$$\frac{dp}{dr} = -\frac{p(r)}{c_s^2}\frac{m(r)}{r^2}\,, \tag{55}$$

$$\frac{dm(r)}{dr} = \frac{4\pi p(r) r^2}{c_s^2}\,, \tag{56}$$

where c_s stands for the speed of sound. The relevant boundary conditions are $m(0) = 0$ and $p(0) = p_0$. This system of differential equations is scale-invariant under $r \to ar$, $p \to a^{-2}p$ and $m \to a^{+1}m$. This means that from our previous analysis we have $(y_1, y_2) = (p, m)$ and $(p_1, p_2) = (-2, 1)$. As stated previously, we can make suitable transformations to convert this scale-invariant system into an autonomous system. For this purpose, let us substitute

$$\zeta(r) = \frac{2\pi p(r) r^2}{c_s^4}\,, \tag{57}$$

and

$$\chi(r) = \frac{m(r)}{(2c_s^2 r)}\,. \tag{58}$$

Then, the system becomes

$$r\,\frac{d\zeta(r)}{dr} = 2\zeta(r)[1 - \chi(r)]\,, \tag{59}$$

$$r\,\frac{d\chi(r)}{dr} = \zeta(r) - \chi(r)\,. \tag{60}$$

As expected this system is now equidimensional-in-r. Define a new variable $t = \ln(r)$ so $d/dt = r d/dr$, then

$$\frac{d\zeta(t)}{dt} = 2\zeta(t)[1 - \chi(t)]\,, \tag{61}$$

$$\frac{d\chi(t)}{dt} = \zeta(t) - \chi(t)\,. \tag{62}$$

As we can clearly see this system is now autonomous. There is a fixed point at $\chi = 1$ and $\zeta = 1$. For any initial condition this point is an attractor as $t \to \infty$. The center of the star is at $r = 0$ ($t = -\infty$). For a regular star with finite p_0 we have $\zeta = 0$ and $\chi = 0$ as $t \to -\infty$; which is the second fixed point. So solving our special dimensionless differential equation is equivalent to looking for the unique integral curve that emerges from the fixed point at $(0, 0)$ and terminates on the fixed point at $(1, 1)$. Applying Theorem 4 we can easily check that a singular solution to the system is of the form

$$p(r) = C_1 r^{-2}\,, \tag{63}$$

and

$$m(r) = C_2 r^1\,, \tag{64}$$

where C_1 and C_2 are constants. Solving for the constants we see that the $(1,1)$ fixed point corresponds to the power-law solution

$$m(r) = 2c_s^2 r\,, \tag{65}$$

and

$$p(r) = \frac{c_s^4}{2\pi r^2}\,, \tag{66}$$

while the $(0,0)$ fixed point corresponds to the more prosaic $p(r) = 0 = m(r)$.

3.2. *Power series in terms of the radius*

3.2.1. *Linearization about the fixed points*

In order to obtain the linearized solution it is necessary to linearize about its critical points. Let us first make the substitution $\chi = \chi_1 + 1$ and $\zeta = \zeta_1 + 1$. Then,

$$\frac{d\zeta_1(t)}{dt} = -2\chi_1(t)\,, \tag{67}$$

$$\frac{d\chi_1(t)}{dt} = \zeta_1(t) - \chi_1(t)\,. \tag{68}$$

Close enough to the $(1,1)$ stagnation point, solutions are of the form

$$\begin{pmatrix}\zeta(t)\\ \chi(t)\end{pmatrix} = \exp\left\{\begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}(t-t_i)\right\}\begin{pmatrix}\zeta(t_i)\\ \chi(t_i)\end{pmatrix}. \tag{69}$$

The eigenvalues of this matrix are $\delta = (-1 \pm i\sqrt{7})/2$, and the approximate linearized solutions approach the critical point exponentially in t — as $\exp(\delta t) = r^\delta$.

There is a second (unstable) stagnation point at $(0,0)$ — a repeller. If we linearize around $(0,0)$ we get

$$\frac{d\zeta_1(t)}{dt} = +2\zeta_1(t)\,, \tag{70}$$

$$\frac{d\chi_1(t)}{dt} = \zeta_1(t) - \chi_1(t)\,. \tag{71}$$

The corresponding eigenvalues are $(\gamma_1, \gamma_2) = (2, -1)$, corresponding to regular r^2 behavior for ζ and χ at the center of the star, plus an otherwise possible but irregular and unphysical behavior for χ that goes as $1/r$. This $1/r$ behavior would correspond to a finite $m(0)$ if such were possible.

3.2.2. *Power series about* $(0,0)$ *in terms of the radius*

Based on Theorem 2 and our general comments regarding Frobenius-like series it is reasonable to assume the following expansion to solve the system

$$\chi(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn} z^{m\gamma_1 + n\gamma_2} . \tag{72}$$

Here we have introduced the dimensionless variable

$$z = \sqrt{\frac{4\pi\rho_0}{c_s^2}}\, r = \left(\frac{\sqrt{4\pi p_0}}{c_s^2}\right) r , \tag{73}$$

with the particular numerical coefficients being chosen for future convenience. In this case, since the $\gamma_2 = -1$ index is unphysical, it will be excluded from the analysis. Hence, we restrict attention to an expansion of the following form:

$$\chi(z) = \sum_{n=0}^{\infty} b_n z^{n\gamma_1} \to \sum_{n=0}^{\infty} b_n z^{2n} . \tag{74}$$

In order to solve for the correct coefficients b_n we need to plug the previous equation into our differential equations. First, by uncoupling the system (eliminating ζ to obtain a single second-order differential equation for χ) we obtain

$$\frac{d^2\chi(z)}{dz^2} - \frac{2\chi(z)}{z^2} + \frac{2\chi(z)^2}{z^2} + \frac{2\chi(z)}{z}\frac{d\chi(z)}{dz} = 0 . \tag{75}$$

The relevant boundary conditions may be inferred from the fact that the pressure (and so also the density) is finite at the origin:

$$\chi(0) = \chi'(0) = 0 , \qquad \chi''(0) = \frac{1}{3} , \qquad \chi' \equiv \frac{d\chi}{dz} . \tag{76}$$

The fact that only integer exponents arise in our series solution for $\chi(z)$ is ultimately due to the fact that the second-order ODE above, though nonlinear, has what amounts to a regular fixed point at $z = 0$. So standard results in terms of ordinary Frobenius series suffice to develop an appropriate ansatz.

Now multiplying by z^2, we can write

$$\left(z\frac{d}{dz}\right)^2 \chi - z\frac{d}{dz}\chi - 2\chi = -2\chi^2 - 2\chi z\frac{d}{dz}\chi , \tag{77}$$

or in operator form,

$$\left[\left(z\frac{d}{dz}\right)^2 - z\frac{d}{dz} - 2\right]\chi = -\left[2 + z\frac{d}{dz}\right]\chi^2 . \tag{78}$$

Now we plug in our assumption of the expansion in Eq. (72) and we get a recursion relation of the form

$$[(2n)^2 - (2n) - 2]b_n = -[2 + (2n)]\sum_{i=0}^{n} b_i b_{n-i} . \tag{79}$$

Because $b_0 = 0$ we can rewrite this as

$$b_n = -\frac{(n+1)}{(n-1)(2n+1)} \sum_{i=1}^{n-1} b_i b_{n-i} \,. \tag{80}$$

This recursion relation is, to our knowledge, a new result. Together with the initial condition $b_1 = 1/6$, derived from the fact that $\chi''(0) = 1/3$, it completely specifies the power series about the $(0,0)$ critical point. It is straightforward to compute these coefficients numerically, though there does not seem to be any simple explicit closed form for the general term.

For ζ we use the relation

$$z \frac{d}{dz} \chi = \zeta - \chi \,. \tag{81}$$

So if

$$\zeta(z) = \sum_{n=0}^{\infty} a_n z^{2n} \,, \tag{82}$$

then

$$a_n = (2n+1) b_n \,. \tag{83}$$

Keeping the first 250 terms in this series, and comparing with an explicit 4th order Runge–Kutta integration, gives very high accuracy $[O(10^{-6})]$ out to $z \approx 3$.

In terms of the physical variables $m(r)$ and $p(r)$ we have

$$m(r) = 2c_s^2 r \chi(r) = 2c_s^2 r \sum_{n=1}^{\infty} b_n \left(\frac{4\pi \rho_0 r^2}{c_s^2} \right)^n = 2c_s^2 r \sum_{n=1}^{\infty} b_n \left(\frac{4\pi p_0 r^2}{c_s^4} \right)^n , \tag{84}$$

and

$$p(r) = \frac{\zeta(z) c_s^4}{2\pi r^2} = \frac{c_s^4}{2\pi r^2} \sum_{n=1}^{\infty} a_n \left(\frac{4\pi \rho_0 r^2}{c_s^2} \right)^n = \frac{c_s^4}{2\pi r^2} \sum_{n=1}^{\infty} a_n \left(\frac{4\pi p_0 r^2}{c_s^4} \right)^n . \tag{85}$$

These are the Taylor series for the solution to the differential equations that describe an isothermal star, with the coefficients being specified by the recursion relations determined above. By bounding the coefficients using the recursion relation, one can show that the series definitely converges for $|z| \leq 3$ and that it definitely diverges for $|z| \geq \sqrt{12}$. Numerical methods seem to indicate a radius of convergence of $R \approx 3.273687274$. The a_n and b_n coefficients do not have a simple explicit analytic form, but computing them numerically from the recursion relation is trivial.

Note that in more classical language this analysis is equivalent to finding a power series solution of the isothermal Lane–Emden equation. See App. C for details.

3.2.3. *Power series about* $(1,1)$ *in terms of the radius*

Turning to the other critical point, from Eq. (77) we see that $\chi(\infty) = 1$. Now making a shift of variables appropriate to the $(1,1)$ fixed point, $\chi = 1 + \chi_1$, where χ_1 is small, we obtain

$$\left(z\frac{d}{dz}z\frac{d}{dz} + z\frac{d}{dz} + 2\right)\chi_1 = -\left(2 + z\frac{d}{dz}\right)\chi_1^2 \,, \tag{86}$$

with $\chi_1(\infty) = 0$. Let us assume an expansion for χ_1 of the form given in Eq. (72) with the substitution $(\gamma_1, \gamma_2) \to (\delta_1, \delta_2)$. The real parts of $\delta_{1,2}$ must be negative so that the expansion has the right limit at infinity, where $\chi_1 \to 0$. By construction we know that $b_{00} = 0$. In order for χ to be real, the matrix of coefficients b_{mn} must be Hermitian. Now let us insert this ansatz into our shifted differential equation [Eq. (86)]. For convenience define $\Delta = m\delta_1 + n\delta_2$, since then

$$z\frac{d}{dz}z^{m\delta_1+n\delta_2} = z\frac{d}{dz}z^{\Delta} = \Delta\, z^{\Delta} \,, \tag{87}$$

and so Eq. (86) becomes

$$(\Delta^2 + \Delta + 2)b_{mn} = -(\Delta + 2)\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}b_{m-i;n-j} \,. \tag{88}$$

This implies the recursion relation

$$b_{mn} = -\frac{(\Delta + 2)}{(\Delta^2 + \Delta + 2)}\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}b_{m-i;n-j} \,, \tag{89}$$

which is again (to our knowledge) a new result. Since $b_{00} = 0$, the right hand side does not involve b_{mn} itself. Thus b_{mn} can be calculated once the lower-order b_{ij} are known. In particular once $b_{01} = b_{10}^*$ is specified, all other coefficients can be calculated recursively. Unfortunately the only known way of calculating $b_{01} = b_{10}^*$ is by performing a numerical integration (e.g. Runge–Kutta) out to large radius and then numerically fitting the data. Summing up, we obtain

$$\chi(z) = 1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}z^{m\delta_1+n\delta_2} \,. \tag{90}$$

A similar power series will exist for $\zeta(z)$. Indeed suppose

$$\zeta(z) = 1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}z^{m\delta_1+n\delta_2} \,. \tag{91}$$

Then

$$a_{mn} = (\Delta + 1)b_{mn} = (m\delta_1 + n\delta_2 + 1)b_{mn} \,. \tag{92}$$

In terms of the physical variables $m(r)$ and $p(r)$

$$m(r) = 2c_s^2 r\chi(r) = 2c_s^2 r\left\{1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}\left(\frac{4\pi p_0 r^2}{c_s^4}\right)^{(m\delta_1+n\delta_2)}\right\}, \tag{93}$$

and

$$p(r) = \frac{\zeta(z)c_s^4}{2\pi r^2} = \frac{c_s^4}{2\pi r^2}\left\{1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\left(\frac{4\pi p_0 r^2}{c_s^4}\right)^{(m\delta_1+n\delta_2)}\right\}. \tag{94}$$

These are Frobenius-like power series with irrational complex exponents for the isothermal star [recall $\delta_{1,2} = (-1 \pm i\sqrt{7})/2$], where the coefficients a_{mn} and b_{mn} are specified by the recursion relations determined above. These expansions are considerably more general than the discussion presented in Chandrasekhar,[3] and the related results for the relativistic case implicit in Harrison *et al.*[4] Those results are tantamount to just keeping the first sub-asymptotic term, which is also contained in our solution, namely

$$m(r) \approx 2c_s^2 r\left\{1 + 2\Re\left[b_{10}\left(\frac{4\pi p_0 r^2}{c_s^4}\right)^{\delta_1}\right]\right\}, \tag{95}$$

and

$$p(r) \approx \frac{c_s^4}{2\pi r^2}\left\{1 + 2\Re\left[a_{10}\left(\frac{4\pi p_0 r^2}{c_s^4}\right)^{\delta_1}\right]\right\}. \tag{96}$$

Using analytic techniques, the series can be shown to converge for $|z| \geq \frac{121}{4}|b_{01}|^2$. This is a weak bound but at least establishes a non-zero radius of convergence. Numerically fitting $|b_{01}|$ by matching to a 4th-order Runge–Kutta integration implies guaranteed convergence for $|z| > 5.246853637$. Direct numerical evidence suggests that the power series is reasonably well behaved down to $z \approx 3$ with the first three terms providing 0.5% accuracy. By numerically applying the ratio test, we find that this power series seems to converge for $|z| \geq 0.165827$. Indeed if we keep the first 100 "levels" ($n+m \leq 100$) then the power series reproduces a Runge-Kutta integration to within 6 significant figures for $z > 0.21$.

The combination of both power series, a Taylor series around $r = 0$ and a fractional power series around $r = \infty$, provides a solution everywhere to arbitrarily good accuracy, since the two expansions have overlapping radii of convergence.

3.3. *Power series in terms of the pressure*

3.3.1. *Linearization about the fixed points*

Instead of expanding χ and ζ in terms of radius r, we now want to develop an expansion in terms of pressure p. This will be useful if we want to generalize beyond a simple linear equation of state.

We can rewrite the equation of hydrostatic equilibrium in the following manner:

$$\frac{dp}{dr} = \frac{2p(r)\chi(r)}{r}\,. \tag{97}$$

Now, by applying the chain rule to $d\zeta/dp$, differentiating Eq. (59) with respect to r, and plugging in our new expression for the equation of hydrostatic equilibrium, we obtain

$$p\,\frac{d\zeta}{dp} = \zeta - \frac{\zeta}{\chi}\,. \tag{98}$$

The same can be done to Eq. (60) to obtain

$$p\,\frac{d\chi}{dp} = \frac{1}{2}\left(1 - \frac{\zeta}{\chi}\right). \tag{99}$$

This new system of differential equations in terms of the pressure still has a stagnation point at $(\zeta, \chi) = (1, 1)$. This is the same stagnation point that we encountered earlier. This stagnation point is again an attractor, which means that for any initial condition as $p \to 0$ almost all solutions of the system approach this solution. Note that the second stagnation point (the repeller) at $(0, 0)$ is still there but somewhat disguised, since the system in terms of p is highly singular there. We will not need the second stagnation point in the following discussion. Linearizing about the $(1, 1)$ stagnation point we obtain

$$p\,\frac{d\zeta_1}{dp} = \chi_1\,, \tag{100}$$

$$p\,\frac{d\chi_1}{dp} = \frac{1}{2}(\chi_1 - \zeta_1)\,. \tag{101}$$

Close to the $(1, 1)$ stagnation point, the solutions to the system are of the form

$$\begin{pmatrix}\hat{\zeta}_1(p)\\ \hat{\chi}_1(p)\end{pmatrix} = \exp\left\{\begin{bmatrix}0 & 1\\ -1/2 & 1/2\end{bmatrix}\ln\left(\frac{p}{p_i}\right)\right\}\begin{pmatrix}\hat{\zeta}_1(p_i)\\ \hat{\chi}_1(p_i)\end{pmatrix}. \tag{102}$$

The eigenvalues of this matrix are $\lambda = (1 \pm i\sqrt{7})/4$. Note that this is simply $-\delta/2$, where δ are the exponents as encountered when working with the variable r.

3.3.2. *Power series about* $(1, 1)$ *in terms of the pressure*

It is essential to decouple the system of differential equations in order to obtain a power series expression for the mass and the radius in terms of the pressure That is, we want to replace the first-order system by a single second-order equation. In order to decouple Eqs. (98) and (99), we solve for ζ in terms of χ from Eq. (98):

$$\zeta = \chi\left(1 - 2p\frac{d\chi}{dp}\right). \tag{103}$$

Now, if we insert this expression back into the first differential equation, we obtain

$$-1 + \chi + p\frac{d\chi}{dp} = 2\left\{\chi p\frac{d\chi}{dp} - \left(p\,\frac{d\chi}{dp}\right)^2 - \chi p\,\frac{d}{dp}\,p\frac{d}{dp}\,\chi\right\}. \tag{104}$$

We can see from the analysis in the previous section that $\chi(r=\infty)=\chi(p=0)=1$. It is convenient to write $\chi = 1+\hat{\chi}$ because we now have $\hat{\chi}(p=0)=0$. After this shift in variables, our decoupled differential equation becomes

$$\left\{1 - p\frac{d}{dp} + 2p\frac{d}{dp}\,p\frac{d}{dp}\right\}\hat{\chi} = \left\{p\frac{d\hat{\chi}}{dp} - p\frac{d}{dp}\,p\frac{d}{dp}\right\}\hat{\chi}^2\,. \tag{105}$$

Now, in line with Theorem 2 and our general discussion regarding Frobenius-like series, we assume an expansion of the form

$$\hat{\chi}(p) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\left(\frac{p}{p_0}\right)^{m\lambda_1+n\lambda_2}. \tag{106}$$

Previously we used the dimensionless variable $z = (\sqrt{4\pi p_0}/c_s^2)r$ to create a power series expansion about the stagnation point $(1,1)$ for χ and ζ. This time we find it more convenient to expand in powers of p/p_0. It is interesting to note that our eigenvalues, $\lambda_{1,2}$, make physical sense because we need positive real parts in order to satisfy the boundary condition $\chi(0)=0$. For $\hat{\chi}$ we have $d_{00}=0$ and the Hermiticity condition discussed previously still holds. Again, let us for simplicity define $\Lambda = m\lambda_1 + n\lambda_2$. We then have

$$p\frac{d}{dp}\left(\frac{p}{p_0}\right)^{m\lambda_1+n\lambda_2} = p\frac{d}{dp}\left(\frac{p}{p_0}\right)^{\Lambda} = \Lambda\left(\frac{p}{p_0}\right)^{\Lambda}. \tag{107}$$

And therefore Eq. (105) implies

$$d_{mn} = \frac{\Lambda(1-\Lambda)}{(1-\Lambda+2\Lambda^2)}\sum_{i=0}^{m}\sum_{j=0}^{n} d_{ij}\, d_{m-i;n-j}\,. \tag{108}$$

Note that by construction $d_{00}=0$, so once $d_{01}=d_{10}^*$ is specified all other coefficients can be calculated recursively. Now this recursion relation holds for the coefficients in $\hat{\chi}$ only. For χ itself we have

$$\chi(p/p_0) = 1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\left(\frac{p}{p_0}\right)^{m\lambda_1+n\lambda_2}. \tag{109}$$

A similar power series will exist for $\hat{\zeta}(p/p_0) = \zeta(p/p_0) - 1$. Let us assume the following expansion:

$$\hat{\zeta}(p/p_0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\left(\frac{p}{p_0}\right)^{\Lambda}. \tag{110}$$

Solving for ζ, from Eq. (98) we obtain

$$\zeta = \chi - p\frac{d\chi^2}{dp}\,, \tag{111}$$

which, upon shifting to the fixed point, is equal to

$$\hat{\zeta} = \hat{\chi} - p\frac{d}{dp}\left(2\hat{\chi} + \hat{\chi}^2\right). \tag{112}$$

Comparing exponents we obtain

$$c_{mn} = d_{mn} - \Lambda\left(2d_{mn} + \sum_{i=0}^{m}\sum_{j=0}^{n} d_{ij}\, d_{m-i;n-j}\right). \tag{113}$$

Using the recursion relation for the d_{mn}

$$c_{mn} = d_{mn} - \Lambda\left(2d_{mn} + \frac{(2\Lambda^2 - \Lambda + 1)}{\Lambda(1-\Lambda)} d_{mn}\right) = -\frac{2\Lambda}{1-\Lambda}\, d_{mn}\,. \tag{114}$$

Note that by construction $c_{00} = 0$. Now this relation holds for the coefficients in $\hat{\zeta}$ only. For ζ itself we have

$$\zeta\left(\frac{p}{p_0}\right) = 1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\left(\frac{p}{p_0}\right)^{m\lambda_1 + n\lambda_2}. \tag{115}$$

We could of course unwrap the definitions of ζ and χ to obtain Frobenius-like power series for the physical observables $r(p)$ and $m(p)$.

3.4. *Mass, pressure, and radius oscillations of a star on the verge of collapse* ($p_0 \to \infty$)

Let us now consider a more realistic equation of state:

$$\rho(p) = \begin{cases} \rho_c = \dfrac{p_c}{c_s^2} & p < p_c\,, \\ \dfrac{p}{c_s^2} & p > p_c \end{cases}\,, \tag{116}$$

where p_c and ρ_c stand for the pressure and density at the surface of the core. Most of the analysis carried until now still holds, at least for the central region of the star. But we now chose to replace the previous equation of state [$\rho(p) = p/c_s^2$, which was linear out to arbitrarily low density] with a new equation that is somewhat more realistic. We have a dense core described by a linear equation of state surrounded by an envelope of constant density in which the pressure drops rapidly to zero.

Since in the previous section we have established the existence of a power series for $\zeta(p/p_0)$, we can formally solve Eq. (57) for the radius of the core itself by setting $p \to p_c$ (at this stage making no comment about the envelope). Hence,

$$r_{\mathrm{core}}(p_c, p_0) = \sqrt{\frac{c_s^4}{2\pi p_c}}\sqrt{\zeta\left(\frac{p_c}{p_0}\right)} = \frac{c_s^2}{\sqrt{2\pi p_c}}\sqrt{1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\left(\frac{p_c}{p_0}\right)^{\Lambda}}\,. \tag{117}$$

This last equation clearly shows the oscillatory behavior of the core radius as a function of central pressure, due to the imaginary exponents in the power series.

If we now look at equation (58) and solve for the mass of the core in terms of the compactness and radius we obtain

$$\begin{aligned} m_{\text{core}}(p_c, p_0) &= 2c_s^2 r_{\text{core}}(p_c, p_0)\chi\left(\frac{p_c}{p_0}\right) \\ &= \frac{2c_s^4}{\sqrt{2\pi p_c}}\sqrt{1+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\left(\frac{p_c}{p_0}\right)^{\Lambda}}\left\{1+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\left(\frac{p_c}{p_0}\right)^{\Lambda}\right\}. \end{aligned} \tag{118}$$

As we can see from this equation the mass of the core will also oscillate as a function of central pressure due to the imaginary part of the exponent of the power series. Indeed there is a related power series such that

$$m_{\text{core}}(p_c, p_0) = \frac{2c_s^4}{\sqrt{2\pi p_c}}\left\{1+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m_{mn}\left(\frac{p_c}{p_0}\right)^{\Lambda}\right\}, \tag{119}$$

with the Hermitian coefficients m_{mn} being in principle calculable in terms of the c_{mn} and d_{mn}.

The above expression gives us the mass of the core of the star, not of the star itself. In order to calculate the total mass of the star we need to first calculate the mass of the envelope of the star:

$$m_{\text{envelope}}(r) = \int_{r_{\text{core}}}^{r} 4\pi r'^2 \rho_c\, dr'\,. \tag{120}$$

Hence,

$$m_{\text{total}}(r) = \frac{4}{3}\pi\rho_c(r^3 - r_{\text{core}}^3) + m_{\text{core}}\,. \tag{121}$$

We calculate r_{surface} from our equation of state and equation of hydrostatic equilibrium:

$$p_{\text{envelope}}(r) = p_c - \int_{r_{\text{core}}}^{r} \frac{\rho_c m_{\text{envelope}}(r')}{r'^2}\, dr'\,, \tag{122}$$

where we have used the fact that $\rho \to \rho_c$ is constant throughout the envelope. Integrating

$$p_{\text{envelope}}(r) = p_c + \left(\rho_c m_{\text{core}} - \frac{4}{3}\pi\rho_c^2 r_{\text{core}}^3\right)\left(\frac{1}{r} - \frac{1}{r_{\text{core}}}\right) + \frac{2}{3}\pi\rho_c^2\left(r_{\text{core}}^2 - r^2\right). \tag{123}$$

We can solve for r_{surface} by solving for $p_{\text{envelope}}(r = r_{\text{surface}}) = 0$. The resulting cubic equation is

$$p_{\text{envelope}}(r = r_{\text{surface}}) = 0 = C_1 r_{\text{surface}}^3 + C_2 r_{\text{surface}} - C_3\,, \tag{124}$$

where we have chosen the constants C_1, C_2 and C_3 to be

$$C_1 = \frac{2}{3}\pi\rho_c^2\,, \tag{125}$$

$$C_2 = \frac{\rho_c m_{\rm core}}{r_{\rm core}} - 2\pi\rho_c^2 r_{\rm core}^2 - p_c\,, \tag{126}$$

$$C_3 = \rho_c m_{\rm core} - \frac{4}{3}\pi\rho_c^2 r_{\rm core}^3\,. \tag{127}$$

This cubic equation can be explicitly [if tediously] solved for $r_{\rm surface}$ as a function of ρ_c, $m_{\rm core}$, and $r_{\rm core}$. Suffice it to say that we now have implicitly solved for the total mass of the star ($M_{\rm total}$), where we have taken $r_{\rm surface}$ as a parameter (which can ultimately be expressed, via $m_{\rm core}$, and $r_{\rm core}$, as an oscillating function of p_c and p_0). That is

$$M_{\rm total}(p_c, p_0) = \frac{4}{3}\pi\rho_0\big(r_{\rm surface}(p_c,p_0)^3 - r_{\rm core}(p_c,p_0)^3\big) + m_{\rm core}(p_c,p_0)\,. \tag{128}$$

Ultimately, since the individual terms above exhibit oscillatory behavior, there will be some Frobenius-like power series such that

$$M_{\rm total}(p_c, p_0) = M_{\rm total}(p_c, p_0 \to \infty)\left\{1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} M_{mn}\Big(\frac{p_c}{p_0}\Big)^{\Lambda}\right\}. \tag{129}$$

From this expression for the total mass of the star we can clearly see the oscillatory behavior mentioned earlier.

Note that this behavior is much more extensive than the scaling results discussed in Chandrasekhar[3] [see also Harrison *et al.*[4]]. Their result is equivalent to keeping only the first nontrivial term in the power series. That is:

$$M_{\rm total}(p_c, p_0) \approx M_{\rm total}(p_c, p_0 \to \infty)\left\{1 + 2\Re\left[M_{10}\left(\frac{p_c}{p_0}\right)^{\delta_1}\right]\right\}. \tag{130}$$

We now see that the scaling behavior they discussed is only the first term in an infinite series.

It may be objected that our current analysis is only valid for the special case of a linear equation of state with a constant density envelope. However, from the way we have set up the general theory it is clear that this restriction is not particularly serious. First, any reasonable equation of state will saturate to a linear at high enough densities, so our results can be applied without modification to the core of any reasonable stellar configuration as the central pressure $p_0 \to \infty$. Furthermore, suppose one has some generic equation of state. Then the exact system will not be scale-invariant. Nevertheless there will be a special critical configuration in which the central pressure is infinite, and since by hypothesis the equation of state becomes linear at sufficiently high pressure, the critical solution will exhibit approximate scale invariance in the high-pressure region — and this will correspond to a fixed point in the autonomous system. Linearization around the fixed point will generate some critical exponents (δ_1, δ_2), which depend only on the high-pressure limit of the equation of state. But deviations from the scale-invariant critical configuration in the core of the star will now feed into the exact scale noninvariant equations, still leading to the total mass being given in terms of some Frobenius-like series. Ultimately this Frobenius-like power series behavior is generic to many systems of differential equations and intrinsically does not have anything to do with gravity *per se*.

4. Application: Isothermal Relativistic Stars

4.1. *Background behavior*

In order to treat relativistic stars, we need to consider the relativistic equation of hydrostatic equilibrium (TOV equation)

$$\frac{dp}{dr} = -\left(\rho + \frac{p}{c^2}\right)\frac{(m + 4\pi r^3 p/c^2)}{r^2(1 - 2m/(rc^2))}\,. \tag{131}$$

We again have a system of two first-order differential equations, namely equations (131) and the mass equation (56), which must be supplemented by an algebraic equation of state. Note that in the TOV equation c is the speed of light, and that as $c \to \infty$ we formally recover Newtonian physics. We will consider an equation of state identical to that used in the previous chapter, i.e. $p(r) = c_s^2 w\rho(r)$, where c_s stands for the speed of sound in the fluid. Making the appropriate substitutions we obtain the closed system

$$S\begin{cases} \dfrac{dp}{dr} = -p\left(\dfrac{1}{c_s^2} + \dfrac{1}{c^2}\right)\dfrac{(m + 4\pi r^3 p/c^2)}{r^2(1 - 2m/(rc^2))}\,, \\ \dfrac{dm}{dr} = \dfrac{4\pi r^2 p}{c_s^2}\,. \end{cases} \tag{132}$$

Note that (as in the Newtonian case) this system S is invariant under the substitutions $r \to ar$, $p \to a^{-2}p$ and $m \to a\,m$. Revisiting our theoretical analysis we also discover that, $(y_1, y_2) = (p, m)$ and $(p_1, p_2) = (-2, 1)$, again as in the Newtonian case.

It is possible to systematically transform the system S into an equidimensional-in-r system S_1, and subsequently into an autonomous system S_2. Let us first substitute $\zeta = 2\pi p(r) r^2/c_s^4$ and $\chi = m/(2rc_s^2)$. Let us also define $t = \ln(r)$ so $d/dt = r d/dr$. Then, the system becomes

$$S_1\begin{cases} \dfrac{d\zeta}{dt} = 2\zeta - 2\zeta(1+\beta^2)\dfrac{(\chi + \beta^2\zeta)}{1 - 4\beta^2\chi}\,, \\ \dfrac{d\chi}{dt} = \zeta - \chi\,, \end{cases} \tag{133}$$

where $\beta = c_s/c$. As we can clearly see S_1 is now autonomous, and therefore, translation invariant. There are two fixed points, namely (α, α) and $(0, 0)$, where $\alpha = 1/(1 + 6\beta^2 + \beta^4)$. Note that $\beta \to 0$ corresponds to the Newtonian limit.

As we found previously, the first fixed point is an attractor as $t \to \infty$, which corresponds to $1/r^2$ behavior for the pressure. The second fixed point at $(0,0)$ is a repeller, as we also found in the Newtonian case. This fixed point corresponds to regular r^2 behavior for ζ and χ at the center of the star, plus an irregular and unphysical behavior for χ that goes as $1/r$. The center of the star is at $r = 0$ ($t = -\infty$). For a regular star with finite p_0 we have $\zeta = 0$ and $\chi = 0$ as $t \to -\infty$. So solving S_1 is equivalent to looking for the unique integral curve that emerges from $(0,0)$ and terminates at (α, α). Since system S is scale-invariant and its associated autonomous system S_1 has fixed points, we can apply Theorem 4 and easily check that the background solution to S is

$$p(r) = C_1 r^{-2}\,, \tag{134}$$

and

$$m(r) = C_2 r^1\,, \tag{135}$$

where C_1 and C_2 are constants. Inserting back into S and solving for the constants we see that the (α, α) fixed point corresponds to the power-law solution

$$m(r) = \frac{2c_s^2}{1 + 6\beta^2 + \beta^4}\, r\,, \tag{136}$$

and

$$p(r) = \frac{c_s^4}{2\pi r^2} \frac{1}{1 + 6\beta^2 + \beta^4}\,. \tag{137}$$

Note that if we take the limit $c \to \infty$, we recover the exact solutions obtained in the previous chapter for the Newtonian case. Also note that the background solution for the $(0,0)$ fixed point corresponds to the trivial $p(r) = 0 = m(r)$.

4.2. *Power series in terms of the radius*

4.2.1. *Linearization about the fixed points*

We suspect that, as in the Newtonian case, the pressure and the mass will have a series expansion in the form of an iterated power law. In order to determine the exponents of the power, let's analyze each fixed point in more detail. Linearizing about the (α, α) fixed point we obtain the following system

$$S_1^{\text{linear}} \begin{cases} \dfrac{d\hat{\zeta}}{dt} = A\hat{\zeta} + B\hat{\chi}\,, \\[2ex] \dfrac{d\hat{\chi}}{dt} = \hat{\zeta} - \hat{\chi}\,, \end{cases} \tag{138}$$

where

$$A = \frac{-2\beta^2}{1 + \beta^2}\,, \qquad B = \frac{-2(1 + 5\beta^2)}{(1 + \beta^2)^2}\,, \tag{139}$$

and $\hat{\zeta}$ and $\hat{\chi}$ are small perturbations. Therefore, close to the stagnation point, solutions are of the form

$$\begin{pmatrix} \zeta(t) \\ \chi(t) \end{pmatrix} = \exp\left\{ \begin{bmatrix} A & B \\ 1 & -1 \end{bmatrix} (t - t_i) \right\} \begin{pmatrix} \zeta(t_i) \\ \chi(t_i) \end{pmatrix}, \tag{140}$$

The exponents of the power law are given by the eigenvalues of this matrix, which are

$$\delta_{\pm} = \frac{-1 - 3\beta^2 \pm \sqrt{-7 - 42\beta^2 + \beta^4}}{2(1+\beta^2)}, \tag{141}$$

and the approximate linearized solutions approach the critical point exponentially in t — as $\exp(\delta t) = r^{\delta}$. These are the exact same power exponents found by Harrison *et al.*[4] Note that if we take the Newtonian limit $c \to \infty$, i.e. $\alpha \to 1$ and $\beta \to 0$, we recover the eigenvalues for the Newtonian case.

Let us now analyze the $(0,0)$ fixed point. Making the substitution $\chi = \hat{\chi}$ and $\zeta = \hat{\zeta}$, where $\hat{\chi}$ and $\hat{\zeta}$ are to be viewed as small perturbations, we obtain the following system:

$$S_2^{\text{linear}} \begin{cases} \dfrac{d\hat{\zeta}}{dt} = 2\hat{\chi}\,, \\[2mm] \dfrac{d\hat{\chi}}{dt} = \hat{\zeta} - \hat{\chi}\,. \end{cases} \tag{142}$$

Close to the stagnation point, solutions to this system have the form

$$\begin{pmatrix} \zeta(t) \\ \chi(t) \end{pmatrix} = \exp\left\{ \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} (t - t_i) \right\} \begin{pmatrix} \zeta(t_i) \\ \chi(t_i) \end{pmatrix}. \tag{143}$$

This is the second unstable stagnation point, which was mentioned in the previous section. This fixed point is a repeller with eigenvalues $(\gamma_1, \gamma_2) = (2, -1)$. Note that these are the same eigenvalues found for the fixed point $(0,0)$ in the Newtonian analysis. We have now determined the exponents of the critical power-law behavior for both fixed points.

4.2.2. *Power series about* $(0,0)$ *in terms of the radius*

Based on Theorem 2 and our general argument for the existence of Frobenius-like series let us now assume an expansion for χ and ζ of the form

$$\chi(r) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn}\, r^{m\gamma_1 + n\gamma_2}\,, \qquad \zeta(r) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\, r^{m\gamma_1 + n\gamma_2}\,. \tag{144}$$

We now need to calculate the actual coefficients b_n and a_n of the series. In order to find them, we need to first decouple the system S_1. Solving for ζ in the $d\chi/dr$ equation in S_1 and inserting this into the $d\zeta/dr$ equation we obtain

$$r\frac{d}{dr} r \frac{d}{dr}\chi - 2\chi - r\frac{d}{dr}\chi = -2\frac{(1+\beta^2)}{1 - 4\beta^2\chi}\left(\chi + r\frac{d}{dr}\chi\right)\left[\chi(1+\beta^2) + \beta^2 r \frac{d}{dr}\chi\right]. \tag{145}$$

It is now convenient to use the dimensionless variable $z = r\sqrt{4\pi p_0}/c_s^2$. Rearranging the above, we obtain

$$\left[2 + z\frac{d}{dz} - z\frac{d}{dz}z\frac{d}{dz}\right]\chi$$
$$= \left[2(1+6\beta^2+\beta^4) + (1+5\beta^2+2\beta^4)z\frac{d}{dz} - 2\beta^2 z\frac{d}{dz}z\frac{d}{dz}\right]\chi^2$$
$$+ 2\beta^2(3+\beta^2)\left(z\frac{d}{dz}\chi\right)^2 . \tag{146}$$

Note here that $\chi(\infty) = \alpha$ and that if we take $\beta \to 0$ we recover the Newtonian result $\chi(\infty) = 1$. We now have a second order decoupled differential equation for χ. Before solving for the coefficients of the series, we remind ourselves that for this case the $1/r$ behavior is unphysical. Hence, let us suppress all occurrences of $\gamma_2 = -1$. and restrict our analysis to $\gamma_1 = 2$. Instead of Eq. (144) we now have

$$\chi(z) = \sum_{n=1}^{\infty} b_n\, z^{2n} = \sum_{n=1}^{\infty} b_n \left(\frac{4\pi p_0 r^2}{c_s^4}\right)^n , \tag{147}$$

$$\zeta(z) = \sum_{n=1}^{\infty} a_n\, z^{2n} = \sum_{n=1}^{\infty} a_n \left(\frac{4\pi p_0 r^2}{c_s^4}\right)^n . \tag{148}$$

Plugging this ansatz into our decoupled differential equation, we obtain the following recursion relation for the coefficients b_n:

$$b_n = -\frac{(1+6\beta^2+\beta^4) + (1+5\beta^2+2\beta^4)n - 4\beta^2 n^2}{(n-1)(2n+1)} \sum_{i=1}^{n-1} b_i b_{n-i}$$
$$- \frac{4\beta^2(3+\beta^2)}{(n-1)(2n+1)} \sum_{i=1}^{n-1} i(n-i) b_i b_{n-i} . \tag{149}$$

With this recursion relation, we have an expression for $\chi(z)$ equation (147) in the form of a power series close to the fixed point $(0,0)$.

A similar power series will exist for ζ. Using the dm/dr equation from S_1, we observe that (as in the Newtonian case)

$$a_n = (2n+1)b_n . \tag{150}$$

In order to completely specify the power series, we need to have boundary conditions. The boundary conditions to begin the recursion relation are $b_0 = 0$ and $b_1 = 1/6$, which are the same conditions used previously in the Newtonian analysis of the system. Choosing $b_1 = 1/6$ is equivalent to the specific choice of z made above. Hence, inserting these conditions into the a_n equation we obtain $a_0 = 0$ and $a_1 = 1/2$. We have now a complete expression for χ and ζ close to the $(0,0)$ stagnation point given by Eqs. (147) and (148).

The radius of convergence $R(\beta)$ of the χ series will be the same as that for the ζ series, and is now a function of the parameter β. We performed numerical calculations and discovered that there is a finite radius of convergence for almost all values of β. In order to better understand the convergence of the series, let us define two regions: region I, ranging from $0 < \beta < 0.0722$ and region II, defined from $0.0722 < \beta < 1$.

The radius of convergence is well-behaved as a function of β in region I, allowing us to recover the Newtonian upper bound at $\beta = 0$ ($R(0) \approx 3.2$.) Our results also showed that the radius of convergence strictly increases monotonically until it reaches $\beta \approx 0.0722$ at which $R(0.0722) \approx 3.6736$.

In region II, the global behavior of β is given by a power law (approximately $R(\beta) \propto \beta^{-1}$), decreasing from $R \approx 3.6736$ at $\beta \approx 0.0722$ to $R \approx 1.1218$ at $\beta = 1$. The local behavior as a function of β in region II, however, is more complex, since there are values of β for which the radius of convergence drops to zero. The initial value at which our series diverges is $\beta_0 = 0.0724$, and after that it will diverge at $\beta = \beta_0 + \tau n$, where n is an integer ($n = 0, 1, 2, 3, \ldots$) and $\tau \approx 0.00026$. It seems possible to approximate the behavior of the radius of convergence in region II with a damped oscillating function, e.g. $R(\beta) \propto \cos(2\pi\beta/\tau + \phi_0)/\beta$.

4.2.3. *Power series about (α, α) in terms of the radius*

In order to determine the coefficients b_{mn} and a_{mn} for the power series about the fixed point (α, α) we need to make a shift to $\hat{\chi} = \chi - \alpha$ in our decoupled differential equation (145). By doing so, Eq. (145) can be simplified and put in operator form to obtain

$$\begin{aligned}\left[(1 - 2\alpha[1 + 5\beta^2 + 2\beta^4])z\frac{d}{dz} - (1 - 4\alpha\beta^2)z\frac{d}{dz}z\frac{d}{dz} - 2\right]\hat{\chi} \\ = \left[2(1 + 6\beta^2 + \beta^4) + (1 + 5\beta^2 + 2\beta^4)z\frac{d}{dz} - 2\beta^2 z\frac{d}{dz}z\frac{d}{dz}\right]\hat{\chi}^2 \\ + 2\beta^2(3 + \beta^2)\left(z\frac{d}{dz}\hat{\chi}\right)^2 .\end{aligned} \tag{151}$$

Again, we have a second order decoupled differential equation for $\hat{\chi}$. Note that for our series expansion we have the expression given in equation (144), where we must take $\lambda_{1,2} \to \delta_{1,2}$. Plugging this ansatz into our decoupled differential equation, we obtain a recursion relation for the coefficients b_{mn}

$$\begin{aligned} b_{mn} = -\frac{2(1 + 6\beta^2 + \beta^2) + (1 + 5\beta^2 + 2\beta^4)\Delta - 2\beta^2\Delta^2}{(1 - 4\beta^2\alpha)\Delta^2 - (1 - 2\alpha[1 + 5\beta^2 + 2\beta^4])\Delta + 2}\sum_{j=0}^{m}\sum_{i=0}^{n} b_{ij}b_{m-i;n-j} \\ - \frac{2\beta^2(3 + \beta^2)}{(1 - 4\beta^2\alpha)\Delta^2 - (1 - 2\alpha[1 + 5\beta^2 + 2\beta^4])\Delta + 2} \\ \times \sum_{j=0}^{m}\sum_{i=0}^{n} ij(m - j)(n - i)b_{ij}b_{m-i;n-j} ,\end{aligned} \tag{152}$$

where $\Delta = m\delta_1 + n\delta_2$. With this recursion relation, we again have a complete expression for χ in the form of a power series close to the fixed point (α, α). ζ will also have a power series expansion, with coefficients given (as in the Newtonian case) by

$$a_{mn} = (\Delta + 1)b_{mn} \,. \tag{153}$$

Once more, we need boundary conditions to begin the recursion relation. We know analytically that $b_{00} = 0$ and $a_{00} = 0$. If we knew $b_{01} = b_{10}^*$ [or equivalently $a_{01} = a_{10}^* = (1 + \delta_2)b_{01}$] then we could recursively calculate the entire series. As in the Newtonian case, there is no analytic theory for these coefficients.

Though we do not have an analytic way of specifying a_{01} or b_{01}, we do have the powerful result that ζ and χ are [close to the fixed point (α, α)] given by the power series of the form

$$\chi(r) = \alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \left(\frac{4\pi p_0 r}{c_s^2} \right)^{m\delta_1 + n\delta_2} , \tag{154}$$

and

$$\zeta(r) = \alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left(\frac{4\pi p_0 r}{c_s^2} \right)^{m\delta_1 + n\delta_2} . \tag{155}$$

In terms of the dimensional physical variables $p(r)$ and $m(r)$ this can explicitly be put in Frobenius-like form

$$m(r) = 2c_s^2 r \left[\alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \left(\frac{4\pi p_0 r}{c_s^2} \right)^{m\delta_1 + n\delta_2} \right] , \tag{156}$$

and

$$p(r) = \frac{c_s^4}{4\pi r^2} \left[\alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left(\frac{4\pi p_0 r}{c_s^2} \right)^{m\delta_1 + n\delta_2} \right] . \tag{157}$$

4.3. *Power series in terms of the pressure*

Though we will not go into any of the details, it is clear from the existence of the power series $p(r)$, that the series can be "reverted" to provide a power series for $r(p)$. The key points are that $p(r)$ is monotonic, so the inverse $r(p)$ exists. Then from the power series above we deduce the existence of a reverted Frobenius-like power series

$$r(p, p_0) = \frac{c_s^2}{\sqrt{4\pi p}} \left[\alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_{mn}(\beta) \left(\frac{p}{p_0} \right)^{m\delta_1 + n\delta_2} \right] . \tag{158}$$

From this we deduce that for a cutoff equation of state (linear in the core, constant density in the envelope) even in the relativistic case the core mass and core radius will exhibit damped oscillations as a function of central pressure p_0.

$$r_{\rm core}(p_c, p_0) = r(p = p_c, p_0) = \frac{c_s^2}{\sqrt{4\pi p_c}} \left[\alpha + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_{mn}(\beta) \left(\frac{p_c}{p_0} \right)^{m\delta_1 + n\delta_2} \right]. \quad (159)$$

Similarly the total mass and total radius will exhibit damped oscillations.

It should also be clear that the details of the equation of state at low density do not matter. As long as the equation of state at high densities is linear, then the analysis above implies that as the central pressure goes to infinity the core radius and core mass will undergo damped oscillations as a function of central pressure. Once the core undergoes these oscillations, the envelope serves only to communicate this internal behavior out to the stellar surface. Thus the total mass and total radius will similarly undergo damped oscillations. These damped oscillations will involve an infinite power series in terms of $p_0^{\triangle}$, which will contain the first-order term discussed in Harrison *et al.*,[4] plus an infinite collection of higher-order terms.

5. Conclusions

We have presented a unified formalism for calculating the solution to scale-invariant theories in the form of power laws, and deviations from pure power-law behavior. We have proved that some of the solutions to a scale-invariant differential system, whose associated autonomous equation possesses fixed points, will exhibit pure power-law behavior. We have also proved that after linearizing about these fixed points, the solution to the scale-invariant linearized system can also be written in the form of an independent collection of power laws. Generically, these collections of power-law solutions can be combined to develop a Frobenius-like series in a neighborhood surrounding the fixed point.

We applied these ideas to several problems taken from astrophysics, such as static isothermal stars both in the Newtonian and TOV frameworks. We showed, both numerically and analytically, that in all the static cases the solution to the scale-invariant differential system associated with each example exhibits power-law behavior as the star approaches the point of collapse. We also showed analytically that the damped oscillations in the pressure, mass and radius of a star, as the central density goes to infinity, can be represented as a Frobenius-like power series, which is easily obtained by applying our formalism. In these examples the coefficients of the power series are calculable by novel and relatively simple nonlinear recursion relations. The existence of a power series expansion around the center of the star, with integer exponents, is standard. The existence of a second convergent power-series (around "spatial infinity") with irrational complex exponents, appears to be a new result.

More generally, it would be advantageous to consider truly dynamical systems involving PDE's in both space and time. Work along these lines is progressing.

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Appendix A. Autonomous Equation Without Fixed Points

Caution must be exercised since not all scale-invariant equations possess fixed points. A scale-invariant differential equation, whose associated autonomous equation *does not* have any fixed points, will *not* possess power-law solutions. As an example, let's analyze what happens to the following relatively simple differential equation:

$$F(x,y(x)) \equiv \frac{dy}{dx} - \frac{py(x)}{x} - fx^{p-1} = 0\,, \tag{A.1}$$

where f is a nonzero constant. This is easily verified to be a scale-invariant equation and the associated equidimensional-in-x equation is

$$\tilde{F}(x,w(x)) \equiv x^p\left(\frac{dw(x)}{dx} - \frac{f}{x}\right) = 0\,. \tag{A.2}$$

This factorizes to give

$$f(t)\bar{F}(z(t)) = e^{(p-1)t}\left(\frac{dz(t)}{dt} - f\right) = 0\,. \tag{A.3}$$

This last equation is autonomous and it does not possess a fixed point since we are assuming that $f \neq 0$. The general solution for this differential equation is easily verified to be

$$y(x) = x^p(f\ln x + A)\,. \tag{A.4}$$

Note that the solution to our example will be a power law if and only if $f = 0$, regardless of the fact that our example is explicitly scale-invariant for all values of f. In conclusion, scale invariance by itself does not guarantee that the solution to a scale-invariant differential equation will exhibit power law behavior. More complicated examples of this behavior can easily be constructed, but this simple equation is already enough to make the point that scale invariance does not necessarily lead to power-law solutions.

However, if we impose *both* scale invariance and the constraint that the associated autonomous equation has fixed points, then the solution will certainly exhibit power law behavior.

Appendix B. Limit Cycles and Discrete Self-Similarity

In the theory of autonomous differential equations the existence of a fixed point is one of the simplest things one could wish for; the next most complicated structure one might encounter is a limit cycle.

Suppose we start with a scale-invariant differential equation and that the associated autonomous equation has a limit cycle rather than a fixed point. This means there is a special solution $z_*(t)$ with a period T such that

$$z_*(t+nT) = z_*(t) \quad \forall n \in Z\,. \tag{B.1}$$

In terms of the equidimensional variables this implies

$$w_*(\exp(nT)x) = w_*(x)\,. \tag{B.2}$$

That is, a limit cycle in the variable t becomes a discrete self-similarity in the variable x. Similarly in terms of the original variable $y(x)$ we obtain discrete power-law behavior

$$y(x) = x^p w_*(x)\,, \quad \forall n \in Z : y(\exp(nT)x) = \exp(pnT)y(x)\,. \tag{B.3}$$

This is not the classic power-law behavior discussed previously, but is, in a sense, the next least complicated thing. As an example, let us consider the second-order differential equation

$$F(x, y(x)) \equiv x^2 y''(x) - (2p-1)xy'(x) + p^2 y + \Omega^2 y(x) = 0\,. \tag{B.4}$$

This is easily verified to be scale-invariant with index p. The associated equidimensional-in-x equation is

$$\tilde{F}(x, w(x)) \equiv x^2 w''(x) + xw'(x) + \Omega^2 w(x) = 0\,. \tag{B.5}$$

The associated autonomous equation is

$$\bar{F}(z(t)) \equiv \ddot{z}(t) + \Omega^2 z(t) = 0\,. \tag{B.6}$$

Now there is a trivial fixed point at $z = 0$ but the general solution to the autonomous equation is

$$z(t) = A\cos(\Omega t) + B\sin(\Omega t)\,. \tag{B.7}$$

In terms of the equidimensional variables

$$w(x) = \Re[(A+iB)x^{i\Omega}]\,. \tag{B.8}$$

Finally, in terms of $y(x)$

$$y(x) = x^p \Re[(A+iB)x^{i\Omega}]\,. \tag{B.9}$$

This now is an explicit example of a discrete power law. The solution is not scale-invariant under arbitrary rescalings $x \to a\,x$, but it is invariant under the specific rescaling $x \to x\exp(2\pi/\Omega)$.

Note the implication: Discrete self-similarity and discrete power laws are not as peculiar as one might at first imagine; instead they are simply the next most complicated thing (after simple power-law behavior) that can happen in scale-invariant systems. (Viewed in this light the occurrence of discrete self-similarity in Choptuik's critical solution[5,6] should not at all be considered surprising.)

Appendix C. Isothermal Lane–Emden Equation

The traditional [isothermal] Lane–Emden equation is

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d\phi(z)}{dz}\right) = \exp(-\phi(z))\,, \tag{C.1}$$

$$\phi(0) = 0\,, \qquad \left.\frac{d\phi}{dz}\right|_0 = 0\,. \tag{C.2}$$

This second-order DE is completely equivalent to the first-order system we derived for a Newtonian relativistic star. Indeed

$$\phi(z) = -\ln[2\zeta(z)/z^2]\,, \tag{C.3}$$

and the series we previously determined for $\zeta(z)$ in the Newtonian case then carry over into series for the Lane-Emden function $\phi(z)$. It is then straightforward to compute

$$\phi(z) = \frac{1}{6}\,z^2 - \frac{1}{120}\,z^4 + \frac{1}{1890}\,z^6 - \frac{61}{1632960}\,z^8 + \frac{629}{224532000}\,z^{10} + O(z^{12})\,. \tag{C.4}$$

You can also derive this in a more direct way by putting a trial power series directly into the Lane–Emden DE and equating coefficients.

Perhaps more surprising is the implication that at large z the Lane–Emden function should have an expansion of the form

$$\phi(z) = \ln[z^2/2] + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_{mn}\, z^{(m\delta_1+n\delta_2)}\,. \tag{C.5}$$

[Recall $\delta_{1,2} = (-1 \pm i\sqrt{7})/2$.] The coefficients q_{mn} are in turn determined by the recursion relations for a_{mn} and b_{mn} previously discussed. An alternative more direct route is to write

$$\phi(x) = \ln[z^2/2] + q(z)\,,$$

and substitute into the Lane–Emden equation yielding

$$\left\{z\frac{d}{dz}z\frac{d}{dz} + z\frac{d}{dz}\right\}q(z) = 2\{\exp[-q(z)] - 1\}\,. \tag{C.6}$$

Linearizing around the obvious solution $q(z) = 0$ recovers the critical indices $\delta_{1,2} = (-1 \pm i\sqrt{7})/2$, while inserting a trial series consisting of integer powers of these critical exponents will determine all the higher coefficients q_{mn} in terms of $q_{01} = q_{10}^*$. Of course, as is usual, there is no analytic theory for this first coefficient since it corresponds to finding the unique boundary condition at infinity that leads to a regular star at the center — $q_{01} = q_{10}^*$ has to be determined numerically from an outward integration of the Lane–Emden equation starting from the origin where we do know the physical boundary conditions.

When it comes to writing down explicit recursion relations for the coefficients, it is more convenient to work with the compactness $\chi(z)$ in terms of which the [isothermal] Lane–Emden equation is equivalent to Eq. (78) and explicit recursion relations are given in Eqs. (80) and (89).

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