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Nuclear Physics B 584 (2000) 415–435

NUCLEAR PHYSICS B

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Brane surgery: energy conditions, traversable wormholes, and voids

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Received 5 April 2000; revised 10 May 2000; accepted 6 June 2000

Abstract

Branes are ubiquitous elements of any low-energy limit of string theory. We point out that negative tension branes violate all the standard energy conditions of the higher-dimensional spacetime they are embedded in; this opens the door to very peculiar solutions of the higher-dimensional Einstein equations. Building upon the $(3+1)$-dimensional implementation of fundamental string theory, we illustrate the possibilities by considering a toy model consisting of a $(2+1)$-dimensional brane propagating through our observable $(3+1)$-dimensional universe. Developing a notion of "brane surgery", based on the Israel–Lanczos–Sen "thin shell" formalism of general relativity, we analyze the dynamics and find traversable wormholes, closed baby universes, voids (holes in the spacetime manifold), and an evasion (not a violation) of both the singularity theorems and the positive mass theorem. These features appear generic to any brane model that permits negative tension branes: This includes the Randall–Sundrum models and their variants. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 04.60.Ds; 04.62.+v; 98.80.Hw
Keywords: Branes; Brane surgery; Energy conditions; Wormholes; Voids

1. Introduction

Branes, ubiquitous elements of any low-energy limit of string theory, have recently attracted much attention as essential ingredients of the semi-phenomenological Randall–Sundrum models [1,2]. These models have been used to both ameliorate the "hierarchy problem" [1] and to explore the possibility of "exotic" Kaluza–Klein theories with their infinitely large extra dimensions [2]. Essential ingredients in these RS models are the existence of both *positive* and *negative* tension branes.

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0550-3213/00/$ – see front matter
PII: S0550-3213(00)00379-5

Now a brane tension is normally thought of as being completely equivalent to an internal cosmological constant, and from the point of view of physics constrained to the brane this is certainly correct. However, from the higher-dimensional point of view (that is, as seen from the embedding space) this is not correct: For a $(p+1)$-brane embedded in $(n+1)$ dimensions a brane tension leads to the stress energy

$$T^{\mu\nu} = -\Lambda_D\, g^{\mu\nu}_{\text{induced}}\,\delta^{n-p}(\eta^a) = -\Lambda_D\left(g^{\mu\nu} - \sum_{a=1}^{n-p} n^\mu_a n^\nu_a\right)\delta^{n-p}(\eta^a), \tag{1.1}$$

where the sum runs over the $n-p$ normals to the brane, and the η^a are suitable Gaussian normal coordinates. Contracting with a higher-dimensional null vector, k_μ, we see

$$T^{\mu\nu}k_\mu k_\nu = -\Lambda_D\, g^{\mu\nu}_{\text{induced}}\,k_\mu k_\nu \delta^{n-p}(\eta^a) = \Lambda_D\left[\sum_{a=1}^{n-p}\left(n^\mu_a k_\mu\right)^2\right]\delta^{n-p}(\eta^a). \tag{1.2}$$

If the brane tension is negative, $\Lambda_D < 0$, and the null vector is even slightly orthogonal to the brane, then on the brane

$$T^{\mu\nu}k_\mu k_\nu < 0. \tag{1.3}$$

That is, the embedding-space null energy condition (NEC) is violated. In fact, integrating across the brane, even the averaged null energy condition (ANEC) is violated. (*Ipso facto*, all the energy conditions are violated.) This is a classical violation of the energy conditions, which we shall soon see is even more profound than the classical violations due to non-minimally coupled scalar fields [3].

In a recent series of papers [4–6] we have made a critical assessment of the current status of the energy conditions, finding a variety of both classical and quantum violations of the energy conditions. We now see that uncontrolled violations of the energy conditions are also a fundamental and intrinsic part of any brane-based low-energy approximation to fundamental string theory. Among the possible consequences of these energy condition violations we mention the occurrence of traversable wormholes (violations of topological censorship), possible violations of the singularity theorems (more properly, evasions of the singularity theorems), and even the possibility of negative asymptotic mass.

A particular example of this sort of phenomenon occurs in the (finite size) Randall–Sundrum models, where one has two parallel branes (our universe plus a hidden brane) of equal but opposite brane tension. One or the other of these branes (depending on whether one is considering the RS1 or RS2 model) violates the $(4+1)$-dimensional energy conditions and exhibits the "flare out" behaviour reminiscent of a traversable wormhole [7]. That these branes do not quite represent traversable wormholes in the usual sense [8,9] follows from the fact that the "throat" is an entire flat $(3+1)$ Minkowski space, instead of the more usual $R^1 \times S^{d-1}$. Furthermore, in the infinite-size version of the Randall–Sundrum (RS2) model, where the hidden sector has been pushed out to hyperspatial infinity, our universe is itself represented by a positive-tension $(3+1)$-brane, which does not violate any $(4+1)$-dimensional energy conditions. The energy-condition violating brane has in this particular model been pushed out to hyperspatial infinity and discarded. Be that as it may, the occurrence of negative tension branes in modern semi-phenomenological models is generic, and a feel for some of the peculiar geometries they can engender is essential to developing any deep understanding of the physics.

In this particular paper we shall for illustrative purposes choose a particularly simple model: We work with a $(3+1)$-dimensional bulk, which contains a $(2+1)$-dimensional brane (of either positive or negative brane tension). We choose this particular model because it is sufficiently close to reality to make the points we wish to make as forcefully as possible, and because it arises naturally in certain types of fundamental string theory. While it is most often the case that fundamental string theories (or their various offspring: membrane models, M-theory, etc.) are formulated in either $(9+1)$ or $(10+1)$ dimensions, [2] this is not absolutely necessary: There is an entire industry based on formulating string theories directly in $(3+1)$ dimensions, with the price that has to be paid being the inclusion of extra $(1+1)$-dimensional quantum fields propagating on the world-sheet [10–14]. [3] Now even in such a $(3+1)$-dimensional incarnation of string theory, open strings will terminate on D-branes (Dirichlet branes), and an effective theory involving the $(3+1)$-dimensional bulk plus $(2+1)$-, $(1+1)$-, and $(0+1)$-dimensional D-branes ("domain walls", "cosmic strings", and "soliton-like particles") can be contemplated as a low-energy approximation. [4] While D-branes are perhaps the most straightforward examples of membrane-like solitons in string theory, they do come with additional technical baggage: the most elementary implementation of D-branes occurs in bosonic string theories [15], but often D-branes are associated with specific implementations of supersymmetric string theories [16] and carry various types of Ramond–Ramond or Neveu–Schwarz charge. There are in addition other types of brane-like configurations that sometimes arise in fundamental string theory such as non-dynamical "orientifold planes" [16], which generate gravitational fields corresponding to negative tensions, but which do not themselves exhibit

[2] In many specific cases the actual implementation is directly in terms of a Euclidean-signature 10- or 11-dimensional spacetime; with the underlying Lorentzian-signature reality hidden under several layers of scaffolding.

[3] Consider for example the bosonic string, which is most often viewed as a $(1+1)$-dimensional world sheet propagating in $(25+1)$ dimensions: There is a trivial re-interpretation in which the bosonic string propagates in $(3+1)$ dimensions and there are 22 free scalar fields propagating on the world-sheet. These 22 scalar fields are there just to soak up the conformal anomaly and make the theory manageable. If these scalar fields are now constrained by appropriate identifications the re-interpretation is less trivial — it is an example of the fact that compactifications of *some* of the dimensions of the higher-dimensional embedding spacetime that the world sheet propagates through can be traded off for a lower-dimensional uncompactified embedding spacetime plus interacting fields on the world-sheet. When this procedure is applied to superstrings the technical details are considerably more complex, but the basic result still holds.

[4] More traditional string theorists who absolutely insist on working directly in the higher-dimensional embedding space can view the current calculations as a particular toy model in which only selected sub-sectors of the grand total degrees of freedom are excited. Additionally, it should be borne in mind that many of the generic features of the analysis presented in this paper will extend *mutatis mutandis* to embedding spaces and branes of higher dimensionality. You do not want the bulk to have fewer than $(3+1)$ dimensions since then bulk gravity is either completely or almost trivial. You do not want the bulk to have more than $(10+1)$ dimensions since the model is then difficult to interpret in terms of fundamental string theory. For technical reasons (to be able to use the thin-shell formalism) you want the brane to be of co-dimension 1, so if the bulk is $(n+1)$-dimensional the brane should be $([n-1]+1)$-dimensional. Within these dimensional limitations, the qualitative features of this paper are generic.

internal dynamics. We will not delve further into this bestiary, but will instead content ourselves with the observation that the low-energy limit of fundamental string theory (of whatever persuasion) generically leads to an effective theory containing brane-like excitations.

This overall picture is actually very similar to the notion of extended topological defects arising from symmetry breaking in point-particle field theories: There are many semi-phenomenological GUT-based point particle field theories that naturally contain domain walls, cosmic strings, and/or solitons. The key difference here is that point particle field theories inevitably lead to positive brane tensions, with negative brane tensions being energetically disfavoured (they correspond to an unnatural form of symmetry breaking that forces one to the *top* of the potential). The key difference in brane-based models is that there is no longer any particular barrier to negative brane tension — in fact negative brane tensions are ubiquitous, now being so commonly used as to almost not require explicit mention. An exhaustive list of papers using negative tension branes would by now be impractical. Among many instances of their use (apart from [1] and [2]) we mention: [17–26].

Within the model we have chosen, we demonstrate that negative tension branes lead to traversable wormholes — in some cases to stable traversable wormholes. (Positive tension branes quite naturally lead to closed baby universes; these are *not* FLRW universes, and are not suitable for cosmology, but are perhaps of interest in their own right.) We also explore the possibility of viewing the brane as an actual physical boundary of spacetime, with the region on the "other side" of the brane being null and void.

The basic tools used are the idea of "Schwarzschild surgery" as developed in [27] (see also the more detailed presentation in [9]), which we first extend to "brane surgery", specialize to "Reissner–Nordström–de Sitter" surgery, and then use to present an analysis of both static and dynamic spherically-symmetric $(2+1)$-dimensional branes in a $(3+1)$-dimensional Reissner–Nordström–de Sitter background geometry.[5] We find both stable and unstable traversable wormhole solutions, stable and unstable baby universes, and stable and unstable voids.

2. Brane surgery

We start by considering a rather general static spherically symmetric geometry (not the most general, but quite sufficient for our purposes)

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2\,\mathrm{d}\Omega_2^2. \tag{2.4}$$

To build the class of geometries we are interested in, we start by taking two copies of this geometry, truncating them at some time-dependent radius $a(t)$, and sewing the resulting geometries together along the boundary $a(t)$. The result is a manifold without boundary

[5] As we shall soon see, brane surgery is essentially a specific implementation of the Israel–Lanczos–Sen junction conditions of general relativity; as such it has been used implicitly in many brane-related papers (see, for example, [1,2,28–30]); the key difference in the present paper is in the details and in the questions we address.

that has a "kink" in the geometry at $a(t)$. If we sew together the two external regions $r \in (a(t), \infty)$, then the result is a wormhole spacetime with two asymptotic regions. On the other hand, if we sew together the two internal regions $r \in (0, a(t))$, then the result is a closed baby universe.

At the "kink" $a(t)$ the spacetime geometry is continuous, but the radial derivative (and hence the affine connexion) has a step-function discontinuity. The Riemann tensor in this situation has a delta-function contribution at $a(t)$, and this geometry can be analyzed using the Israel–Lanczos–Sen "thin shell" formalism of general relativity [32–35]. The relevant specific implementation of the thin-shell formalism can be developed by extending the formalism of [27] and [9]. Because of its relative simplicity we shall start with the static case $a =$ constant.

2.1. Brane statics

The unit normal vector to the sphere $a =$ constant is (depending on whether one is considering inward or outward normals)

$$n^\mu = \pm\Big(0, \sqrt{F(a)}, 0, 0\Big), \qquad n_\mu = \pm\left(0, \frac{1}{\sqrt{F(a)}}, 0, 0\right). \tag{2.5}$$

The extrinsic curvature (second fundamental form) can be written in terms of the normal derivative

$$K_{\mu\nu} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial\eta} = \frac{1}{2}n^\sigma\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \pm\frac{1}{2}\sqrt{F(a)}\,\frac{\partial g_{\mu\nu}}{\partial r}. \tag{2.6}$$

If we go to an orthonormal basis, the relevant components are [6]

$$\begin{aligned} K_{\hat{t}\hat{t}} &= \mp\frac{1}{2}\sqrt{F(r)}\,\frac{\partial g_{tt}}{\partial r}g^{tt} = \mp\frac{1}{2}\sqrt{F(r)}\,\frac{\partial F(r)}{\partial r}\frac{1}{F(r)} \\ &= \mp\frac{1}{2}F(r)^{-1/2}\frac{\partial F(r)}{\partial r}\bigg|_{r=a}, \end{aligned} \tag{2.7}$$

$$K_{\hat{\theta}\hat{\theta}} = \pm\frac{1}{2}\sqrt{F(r)}\,\frac{\partial g_{\theta\theta}}{\partial r}g^{\theta\theta} = \pm\frac{1}{2}\sqrt{F(r)}\,\frac{\partial r^2}{\partial r}\frac{1}{r^2} = \pm\frac{\sqrt{F(r)}}{r}\bigg|_{r=a}. \tag{2.8}$$

The discontinuity in the extrinsic curvature is related to the jump in the normal derivative of the metric as one crosses the brane

$$\kappa_{\mu\nu} = K^+_{\mu\nu} - K^-_{\mu\nu}. \tag{2.9}$$

In general, one could take the geometry on the two sides of the brane to be different $[F^+(r) \neq F^-(r)]$, but in the interests of clarity the present models will all be taken to have a Z_2 symmetry under interchange of the two bulk regions. [7] Under these conditions

[6] The use of an orthonormal basis makes it particularly easy to phase the calculation in terms of the physical density and physical pressure.

[7] Remember that we have already decided to take the range of the r coordinate to be *either* two copies of $(a(t), \infty)$, corresponding to a wormhole; *or* two copies of $(0, a(t))$, corresponding to a baby universe. Then Z_2 symmetry corresponds to $F^+(r) = F^-(r)$, with a kink in the geometry at $r = a(t)$. Our normal vectors do not flip sign as we cross the brane.

$$\kappa_{\hat{t}\hat{t}} = \mp F(r)^{-1/2} \left. \frac{F(r)}{\partial r} \right|_{r=a}, \tag{2.10}$$

$$\kappa_{\hat{\theta}\hat{\theta}} = \pm 2 \left. \frac{\sqrt{F(r)}}{r} \right|_{r=a}. \tag{2.11}$$

The upper sign refers to a wormhole geometry where the two exterior regions have been sewn together (discarding the two interior regions), while the lower sign is relevant if one has kept the two interior bulk regions.

The thin-shell formalism of general relativity [32–35] relates the discontinuity in extrinsic curvature to the energy density and tension localized on the junction: [8]

$$\sigma = -\frac{1}{4\pi}\kappa_{\hat{\theta}\hat{\theta}} = \mp \frac{1}{2\pi r} \left. \sqrt{F(r)} \right|_{r=a}, \tag{2.12}$$

$$\theta = -\frac{1}{8\pi}[\kappa_{\hat{\theta}\hat{\theta}} - \kappa_{\hat{t}\hat{t}}] = \mp \frac{1}{4\pi r} \frac{\partial}{\partial r} \left. \left(r\sqrt{F(r)} \right) \right|_{r=a}. \tag{2.13}$$

If the material located in the junction is a "clean" brane (a brane in its ground state, without extra trapped matter in the form of stringy excitations), then its equation of state is $\sigma = \theta$ and the condition for a static brane configuration (either a wormhole or baby universe geometry) is simply

$$\begin{aligned} \sigma = \theta \quad &\Rightarrow \quad 2\sqrt{F(r)}\Big|_{r=a} = \frac{\partial}{\partial r}\left(r\sqrt{F(r)} \right)\Big|_{r=a} \\ &\Rightarrow \quad \frac{\partial}{\partial r}\left(\frac{F(r)}{r^2} \right)\Big|_{r=a} = 0. \end{aligned} \tag{2.14}$$

Thus we have a very simple result: static wormholes (baby universes) correspond to extrema of the function $F(r)/r^2$, though at this stage we have not yet made any assertions about stability or dynamics. The only difference between wormholes and baby universes is that for wormholes the brane tension must be negative, whereas for baby universes it is positive.

It is instructive to note that the locations of these static brane solutions correspond to *circular photon orbits* in the original spacetime (and this is true for arbitrary $F(r)$). That is: at these static brane solutions any "particle" that is emitted form the brane, which then follows null geodesics (of the bulk spacetime), and which initially has no radial momentum, will just skim along the brane; never moving off into the bulk. (Note that this is a purely kinematic effect that occurs over and above any "trapping" due to stringy interactions between the brane and excited string states.)

This may easily be verified by considering the photon orbits for arbitrary $F(r)$. The time-translation and rotational Killing vectors lead to conserved quantities

[8] The numerical coefficients appearing herein are dimension-dependent (because of the implicit trace over the Ricci tensor and extrinsic curvature hidden in the Einstein equations).

$$\left(\frac{\partial}{\partial t}, k\right) = -\epsilon \quad \Rightarrow \quad g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = -\epsilon \quad \Rightarrow \quad F\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \epsilon. \tag{2.15}$$

$$\left(\frac{\partial}{\partial \phi}, k\right) = -\ell \quad \Rightarrow \quad g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \ell \quad \Rightarrow \quad r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \ell. \tag{2.16}$$

Inserting this back into the condition that the photon momentum be a null vector, $(k, k) = 0$, we see

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{F(r)\ell^2}{r^2} = \epsilon^2. \tag{2.17}$$

Now λ is an arbitrary affine parameter, so we can reparameterize $\lambda \to \ell\lambda$ and define $b = \epsilon/\ell$ to see that photon orbits are described by the equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{F(r)}{r^2} = b^2. \tag{2.18}$$

The circular photon orbits (and at this stage we make no claims about stable versus unstable circular photon orbits) are, as claimed, at extrema of the function $F(r)/r^2$ (which coincide with the location of the static brane configurations).

2.2. *Brane dynamics*

If now the brane is allowed to move radially $a \to a(t)$, we start the analysis by first parameterizing the motion in terms of proper time along a curve of fixed θ and ϕ. That is: the brane sweeps out a world-volume

$$X^\mu(\tau, \theta, \phi) = \big(t(\tau), a(\tau), \theta, \phi\big). \tag{2.19}$$

The 4-velocity of the (θ, ϕ) element of the brane can then be defined as

$$V^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}a}{\mathrm{d}\tau}, 0, 0\right). \tag{2.20}$$

Using the normalization condition and the assumed form of the metric, and defining $\dot{a} = \mathrm{d}a/\mathrm{d}\tau$,

$$V^\mu = \left(\frac{\sqrt{F(a) + \dot{a}^2}}{F(a)}, \dot{a}, 0, 0\right), \qquad V_\mu = \left(-\sqrt{F(a) + \dot{a}^2}, \frac{\dot{a}}{F(a)}, 0, 0\right). \tag{2.21}$$

The unit normal vector to the sphere $a(\tau)$ is

$$n^\mu = \pm\left(\frac{\dot{a}}{F(a)}, \sqrt{F(a) + \dot{a}^2}, 0, 0\right), \qquad n_\mu = \pm\left(-\dot{a}, \frac{\sqrt{F(a) + \dot{a}^2}}{F(a)}, 0, 0\right). \tag{2.22}$$

The extrinsic curvature can still be written in terms of the normal derivative

$$K_{\mu\nu} = \frac{1}{2}n^\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma}. \tag{2.23}$$

If we go to an orthonormal basis, the $\hat{\theta}\hat{\theta}$ component is easily evaluated [9,27]

$$K_{\hat{\theta}\hat{\theta}} = \pm\frac{1}{2}\sqrt{F(a)+\dot{a}^2}\,\frac{\partial g_{\theta\theta}}{\partial r} g^{\theta\theta} = \pm\frac{\sqrt{F(a)+\dot{a}^2}}{a}. \tag{2.24}$$

The $\tau\tau$ component is a little messier, but generalizing the calculation of [27] or [9] (which amounts to calculating the four-acceleration of the brane) quickly leads to [9]

$$K_{\hat{\tau}\hat{\tau}} = \mp\frac{1}{2}\frac{1}{\sqrt{F(a)+\dot{a}^2}}\left(\frac{\mathrm{d}F(r)}{\mathrm{d}a} + 2\ddot{a}\right) = \mp\frac{\mathrm{d}}{\mathrm{d}a}\sqrt{F(a)+\dot{a}^2}. \tag{2.25}$$

Applying the thin-shell formalism now gives:

$$\sigma = \mp\frac{1}{2\pi a}\sqrt{F(a)+\dot{a}^2}, \tag{2.26}$$

$$\theta = \mp\frac{1}{4\pi a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a\sqrt{F(r)+\dot{a}^2}\right). \tag{2.27}$$

These equations can easily be seen to be compatible with the conservation of the stress energy localized on the brane

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sigma a^2\right) = \theta\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a^2\right). \tag{2.28}$$

So as usual, *two* of these three equations are independent, and the third is redundant.

From the above we see that traversable wormhole solutions, corresponding to the minus sign above, require negative brane tension (and so positive internal pressure and negative internal energy density). This is in complete agreement with [31] where it was demonstrated that even for dynamical wormholes there must be violations of the null energy condition at (or near) the throat.

If the material located in the junction is again assumed to be a "clean" brane ($\sigma = \theta$) then all the dynamics can be reduced to a *single* equation [10]

$$\dot{a}^2 + F(a) = (2\pi\sigma)^2 a^2. \tag{2.29}$$

This single dynamical equation applies equally well to both wormholes and baby universes, the $\mp$ that shows up in the brane Einstein equations quietly goes away upon squaring — thus for questions of dynamics and stability these surgically constructed baby universes and wormholes can be dealt with simultaneously — the *only* difference lies in question of whether the brane tension is positive or negative.

Note that we could re-write this dynamical equation as

$$\left(\frac{\mathrm{d}\ln(a)}{\mathrm{d}\tau}\right)^2 + \frac{F(a)}{a^2} = (2\pi\sigma)^2. \tag{2.30}$$

From this it is clear that static solutions must be located at extrema of the function $F(a)/a^2$, in agreement with the static analysis.

[9] We do not repeat the details here since this calculation is now standard textbook fare [9], pp. 182–183. If one wishes to avoid the need for this particular calculation one can instead work backwards from the conservation of stress-energy, together with the already-calculated expression for $K_{\hat{\theta}\hat{\theta}}$, to *deduce* an expression for $K_{\hat{\tau}\hat{\tau}}$. But if you choose this route you lose the opportunity to make a consistency check.

[10] And if the brane is not "clean" in this sense one only needs to keep track of one additional piece of information — the on-brane conservation equation (2.28).

In the next section we shall make use of this general formalism by specializing $F(r)$ to the Reissner–Nordström–de Sitter form. We shall then exhibit some explicit solutions to these brane equations of motion, and perform the relevant stability analysis.

3. Reissner–Nordström–de Sitter surgery

For the Reissner–Nordström–de Sitter geometry

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2. \tag{3.31}$$

It is then most instructive to write the dynamical equation in the form

$$\left(\frac{\mathrm{d}\ln(a)}{\mathrm{d}\tau}\right)^2 + V(a) = E, \tag{3.32}$$

with a “potential”

$$V(a) = \frac{F(a)}{a^2} = \frac{1}{a^2} - \frac{2M}{a^3} + \frac{Q^2}{a^4} - \frac{\Lambda}{3}, \tag{3.33}$$

and an “energy”

$$E = +(2\pi\sigma)^2. \tag{3.34}$$

The extrema of this potential are easily located, their positions are independent of Λ and occur at

$$r_\pm = \frac{3M}{2} \pm \sqrt{\left(\frac{3M}{2}\right)^2 - 2Q^2}. \tag{3.35}$$

(As promised, these are indeed the locations of the circular photon orbits of the Reissner–Nordström–de Sitter geometry; note that the cosmological constant does *not* affect the location of these circular photon orbits.) The *value* of this potential at these extrema is somewhat tedious to calculate, we find

$$\begin{aligned} V_\pm(M, Q, \Lambda) &\equiv V\big(r_\pm(M, Q)\big) \\ &= -\frac{1}{4Q^2}\left(1 - \frac{9}{2}\frac{M^2}{Q^2} + \frac{27}{8}\frac{M^4}{Q^4}\right) \pm \frac{M}{4Q^6}\left[\left(\frac{3M}{2}\right)^2 - 2Q^2\right]^{3/2} - \frac{\Lambda}{3}. \end{aligned} \tag{3.36}$$

Though it may not be obvious, the $Q \to 0$ limit formally exists and is given by

$$V_-(M, Q\to 0, \Lambda) \to -\infty, \qquad V_+(M, Q \to 0, \Lambda) \to \frac{1}{27M^2} - \frac{\Lambda}{3}. \tag{3.37}$$

The behaviour of the potential $V(a)$ is qualitatively:

- $V(a) \to +\infty$ as $a \to 0$ ($Q \neq 0$);
- $V(a) \to -\Lambda/3$ as $a \to +\infty$;
- There is at most one local minimum (V_- located at r_-) and one local maximum (V_+ located at r_+).

Two figures, where we have plotted $V(a)$ for two special cases, are provided in the discussion below. When looking for a stable brane solution we want to satisfy the following:

1. We want the local minimum to exist, and the brane to be located in its basin of attraction.
2. The energy must be at least equal to V_- (to even get a solution), and should not exceed V_+ (to avoid having the solution escape from the local well located around r_-).
3. We also do not want (at least for now) the brane to fall inside (or even touch) any horizon the original Reissner–Nordström–de Sitter geometry might have — for two reasons:
   (a) Because if it does fall inside (or even touch) an event horizon the wormhole geometry is operationally indistinguishable from a Reissner–Nordström–de Sitter black hole and therefore not particularly interesting (but see the discussion regarding singularity avoidance later in this paper) whereas the baby universe geometry is for $Q = 0$ doomed to a brief and unhappy life, and for $Q \neq 0$ is just plain weird.
   (b) For technical reasons (r is now timelike and t spacelike) a few key minus signs flip at intermediate steps of the calculation, more on this later.

These physical constraints now imply:

1. To get a local minimum we need $M > \sqrt{8/9}Q$.
2. To then trap the solution, to make it one of bounded excursion, we need

$$V_-(M, Q, \Lambda) \leqslant +(2\pi\sigma)^2 \leqslant V_+(M, Q, \Lambda). \tag{3.38}$$

3. Horizon avoidance requires $F(a) > 0$ over the entire range of motion; this implies

$$V(a) = \frac{F(a)}{a^2} > 0 \quad \Rightarrow \quad V_-(M, Q, \Lambda) > 0. \tag{3.39}$$

   In view of this the horizon avoidance condition might more properly be called horizon elimination — horizons can be avoided if and only if the inner and outer horizons are actually eliminated. (We could however still have a cosmological horizon at very large distances, this cosmological horizon is never reached if the bounded excursion constraint is satisfied.) We can also explicitly separate out the cosmological constant to write the horizon elimination condition as

$$\Lambda < 3V_-(M, Q, \Lambda \to 0), \tag{3.40}$$

   which makes it clear that a powerful enough negative (bulk) cosmological constant is guaranteed to eliminate all the event horizons from the geometry.

That these constraints can simultaneously be satisfied (at least in certain parameter regimes) can now be verified by inspection. The best way to proceed is to sub-divide the discussion into several special cases.

3.1. $M > |Q| = 0$

There is one maxima (at $a = 3M$) and no minimum. There are no stable solutions, though the "arbitrarily advanced civilization" posited by Morris and Thorne [8] might like to try to artificially maintain the unstable static solution at $a = 3M$. (This solution is unstable to both collapse and explosion.)

Adding $Q \neq 0$ provides a hard core to the potential so that collapse is avoided (modulo the horizon crossing issue which must be dealt with separately).

3.2. $M > |Q| \neq 0$

There are now both a local maximum (at $r_+ < 3M$) and a global minimum (at $r_- > 0$). The potential is plotted in Fig. 1. Stable solutions exist (both static stable solutions and stable solutions of bounded excursion), but since $V_- < 0$ ($\Lambda = 0$) at the global minimum horizon avoidance requires

$$\Lambda < 3V_-(M, Q, \Lambda \to 0) < 0. \tag{3.41}$$

That is, stable traversable wormhole or baby universe solutions exist only if the bulk is anti-de Sitter space ($adS_{(3+1)}$) with a strong enough negative cosmological constant.

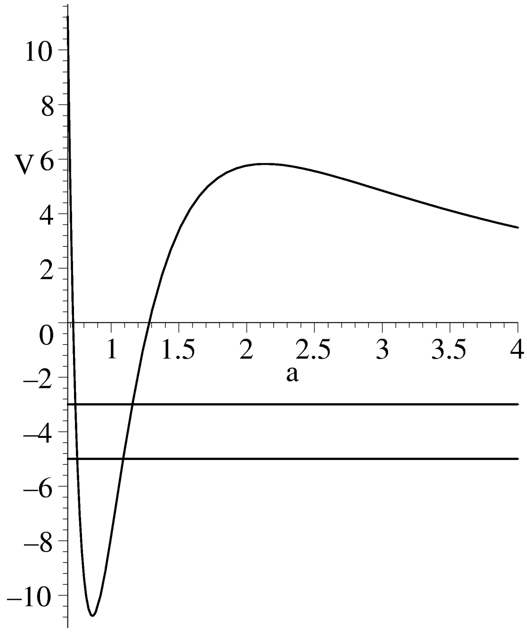

Fig. 1. Sketch of the potential $V(a)$ for $M > |Q|$ and $\Lambda = 0$. Adding a cosmological constant merely moves the entire curve up or down: the lower horizontal line represents $\Lambda/3$, and for Λ sufficiently large and negative the inner and outer horizons (which are given by the intersection of this horizontal line with the $\Lambda = 0$ potential) are guaranteed to be eliminated. The upper horizontal line represents $\Lambda/3 + (2\pi\sigma)^2$, and its intersection with the $\Lambda = 0$ potential gives the turning points of the motion. If inner and outer horizons exist they lie between the inner and outer turning points.

Indeed, if you consider the original geometry prior to brane surgery and extend it down to $r = 0$ then for this choice of parameters (because of the large negative cosmological constant) you encounter a naked singularity. For the stable wormhole geometries based on this brane prescription this is *not* a problem since the naked singularity was in the part of the spacetime that you threw away in setting up the brane construction. (The baby universe models on the other hand, while stable, explicitly do contain naked singularities.) [11]

A particularly simple sub-class of these solutions occurs when the bulk cosmological constant is tuned to a special value in terms of the brane tension. This is the analog of the Randall–Sundrum fine tuning [1,2] and corresponds to a zero "effective" cosmological constant, in the sense that the brane equation of motion can be rearranged and reinterpreted as being governed by $E = 0$ and

$$\Lambda_{\text{effective}} = \Lambda + 3(2\pi\sigma)^2. \tag{3.42}$$

If this $\Lambda_{\text{effective}}$ is now tuned to zero

$$\Lambda = -3(2\pi\sigma)^2 < 3V_-(M, Q, \Lambda \to 0) < 0. \tag{3.43}$$

3.3. $M = |Q|$

There are still both a local maximum (at $r_+ = 2M$) and a global minimum (at $r_- = M$). Stable solutions exist. Since now $V_-(\Lambda \to 0) = 0$ at the global minimum horizon avoidance requires anti de Sitter space with an arbitrarily weak cosmological constant. (And again this is an example of horizon elimination.)

3.4. $M \in \left(\sqrt{8/9}\,|Q|, |Q|\right)$

There are still both a local maximum (at $r_+ < 2M$) and a global minimum (at $r_- > M$). The potential is plotted in Fig. 2. Stable solutions exist. Since now $V_-(\Lambda \to 0) > 0$ at the global minimum, horizon avoidance can be achieved with zero cosmological constant in the bulk. For instance, picking

$$\Lambda = 0, \qquad (2\pi\sigma)^2 = V_-(M, Q, \Lambda \to 0), \tag{3.44}$$

yields the stable static solution at r_-. This is perhaps the most "physical" of these traversable wormholes in that it resides in an asymptotically flat spacetime.

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[11] This is part of a general pattern: The stable (or even merely static) brane configurations that do not possess naked singularities in the bulk region are the wormhole configurations with negative brane tension. This observation also applies to the other sub-cases discussed below. This is compatible with the discussion of Chamblin, Perry, and Reall [36] who discovered qualitatively similar behaviour for $(8+1)$-dimensional branes in a $(9+1)$-dimensional bulk. Specifically, they found that static $(8+1)$-dimensional brane configurations with positive brane tension led to naked singularities in the bulk, and that eliminating the naked singularities forced the adoption of negative brane tension (and implicitly a wormhole configuration). This observation also serves to buttress our previous comments to the effect that the qualitative features of the calculations presented in this paper are generic, and are not just limited to $(2+1)$ branes in $(3+1)$ dimensions.

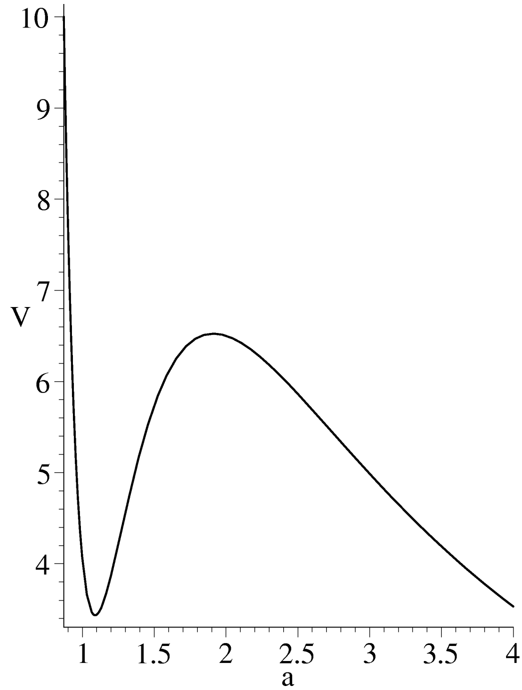

Fig. 2. Sketch of the potential $V(a)$ for M in the critical range $(\sqrt{8/9}\,|Q|, |Q|)$, and $\Lambda = 0$. Adding a cosmological constant merely moves the entire curve up or down. In this case, even for $\Lambda = 0$, we see a stable minimum at r_- with no event horizons. For small positive Λ a cosmological horizon will form at very large radius, but this is of no immediate concern because of the barrier at r_+. If Λ becomes too large however, $\Lambda > V_-(M, Q, \Lambda \to 0)$, inner and outer horizons will reappear between the inner and outer turning points.

3.5. $M = \sqrt{8/9}\,|Q|$

The maximum and minimum merge into a single point of inflection (at $r_\pm = 3M/2$). There are no stable solutions. All the solutions exhibit runaway to large radius.

3.6. $M \in \left(\sqrt{8/9}\,|Q|, 0\right)$

There is not even a point of inflection: the potential is monotonic decreasing. There are no stable solutions.

3.7. $M = 0$, $Q \neq 0$

There is not even a point of inflection: the potential is monotonic decreasing. There are no stable solutions.

3.8. $M < 0$

Letting the central mass M go negative is not helpful — $M < 0$ helps stabilize against collapse, but actually destroys the possibility of stable solutions because the location of "extrema" $r_\pm$ is pushed to unphysical nominally negative values of the radius.

3.9. *Baby bangs?*

The fact that so many of these baby universe models are unstable to explosion is intriguing, and potentially of phenomenological interest. While these particular baby-universe models are not suitable cosmologies for our own universe, we believe that more realistic scenarios can be developed.

3.10. *Singularity avoidance?*

We have so far sought to implement horizon avoidance in our models: we have sought conditions that would prevent the brane from falling through or even touching any horizon that might be present in the underlying pre-surgery spacetime. Suppose we now relax that constraint. The best way to analyze the situation is to note that inside the horizon (more precisely between the outer horizon and the inner horizon) the pre-surgery metric can be written in the form

$$\mathrm{d}s^2 = +|F(r)|\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{|F(r)|} + r^2\,\mathrm{d}\Omega_2^2. \tag{3.45}$$

The calculation of the four-velocity, normal, extrinsic curvatures, and their discontinuities can be repeated, with the result that in this region $[F(r) < 0]$

$$\begin{aligned} V^\mu &= \left(-\frac{\sqrt{\dot{a}^2 - |F(a)|}}{|F(a)|}, \dot{a}, 0, 0\right), \\ n^\mu &= \pm\left(-\frac{\dot{a}}{|F(a)|}, \sqrt{\dot{a}^2 - |F(a)|}, 0, 0\right), \end{aligned} \tag{3.46}$$

and

$$\sigma = \mp\frac{1}{2\pi a}\sqrt{\dot{a}^2 - |F(a)|}. \tag{3.47}$$

After rearrangement this leads to the *same* dynamical equation as before

$$\left(\frac{\mathrm{d}\ln(a)}{\mathrm{d}\tau}\right)^2 + \frac{F(a)}{a^2} = (2\pi\sigma)^2. \tag{3.48}$$

So that all of our previous arguments can be extended inside the event horizon.
A few key observations:

- The two turning points occur at $F(a)/a^2 = (2\pi\sigma)^2 > 0$. Thus $F(a) > 0$ at the turning points. So if there are horizons present (that is, if $F(a) = 0$ has solutions $r^\pm_{\text{horizon}} \neq r_\pm$), and one is in the potential well near r_-, then one turning point will be outside the outer horizon, and the second turning point will be inside the inner horizon.

- Even though the brane oscillation will take finite proper time τ this corresponds to infinite t-parameter time — when the brane re-emerges from the outer horizon it will emerge from a past outer horizon of a "future incarnation" of the universe; the brane will not re-emerge into our own universe. (For simplicity you may wish to set $\Lambda = 0$ and consider the Penrose diagram of the maximally extended Reissner–Nordström geometry as presented, for instance, on page 158 of Hawking and Ellis [37]. A partial Penrose diagram for Reissner–Nordström–de Sitter may be found in [38,39]. See also Fig. 3.)
- Operationally, from "our" asymptotically flat region, once the brane passes the horizon the geometry will be indistinguishable from an ordinary Reissner–Nordström–de Sitter black hole.
- The original pre-surgery spacetime has two asymptotic regions, two outer horizons, and two inner horizons, which are then repeated an infinite number of times in the maximal analytic extension. If the brane starts out in the rightmost asymptotic region and falls through the right (future) outer horizon, then you can quickly convince yourself that it must pass through the *left* inner horizon (twice, once on the way in,

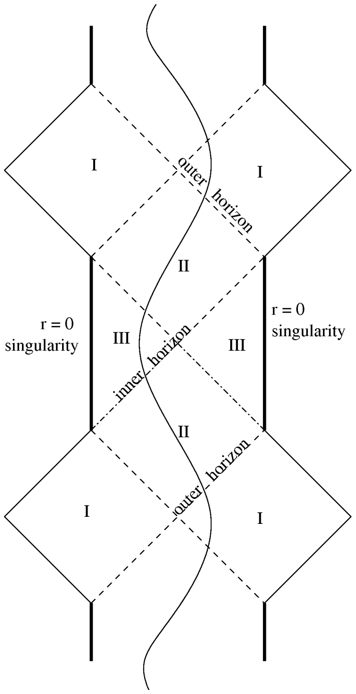

Fig. 3. Sketch of the Penrose diagram for the maximally extended Reissner–Nordström geometry when $M > |Q|$ ($\Lambda = 0$). A timelike (2 + 1)-brane [spacelike normal] will oscillate between the turning points r_+ and r_-, but each oscillation will take infinite coordinate time even if it takes finite proper time. For wormhole solutions keep the right half of the diagram, make two copies, and sew them together along the brane. For baby universe geometries keep the left half of the diagram, make two copies and sew them up along the brane.

and once again on the rebound) before moving back out through the right (past) outer horizon back into the (next incarnation of) the right asymptotic region. (See Fig. 3.)

- The wormhole geometry based on this brane surgery is an explicit example of *partial* evasion of the usual singularity theorems [37]. (We say evasion, not violation, because the presence of the negative tension brane vitiates the usual hypotheses used in proving the singularity theorems.) The wormhole geometry certainly has trapped surfaces once the brane falls inside the horizon, but by construction there is no *left* curvature singularity. (The *right* curvature singularity is still there, and the right inner horizon is still a Cauchy horizon.)[12] Note that this is a idealized statement appropriate to "clean" wormhole universes containing only a few test particles of matter: in any more realistic model where the universe contains a finite amount of radiation, inner event horizons are typically unstable to a violent blue shift instability, and are typically converted by back-reaction effects to some sort of curvature singularity [38,39]. This process however, lies far beyond the scope of the usual singularity theorems.

If you wish to eliminate *both* left and right singularities a more drastic fix is called for: You will need to use a $(3+0)$-dimensional brane, something you might call an instanton-brane because it represents a spacelike hypersurface through the spacetime — at early times there's nothing there, the brane "switches on" for an instant, and then it's gone again. The simplest example of such a instanton-brane is to place one at r_-, the static minimum of the potential $V(a)$.[13] If there are event horizons then this minima will be inside the event horizon (between inner and outer horizons) and a hypersurface placed at r_- will be spacelike. Placing the instanton-brane at this location will eliminate *both* singularities and *both* inner horizons — you are left with two asymptotic regions and two (outer) event horizons, infinitely repeated.

More generally one could think of an instanton-brane described by a location $a(\ell)$, where ℓ is now proper length along the brane (and the notion of *dynamics* is somewhat obscure). The spacelike tangent and timelike normal are now (outside the horizon)

$$V^\mu = \left(\frac{\sqrt{(\mathrm{d}a/\mathrm{d}\ell)^2 - F(a)}}{F(a)}, \frac{\mathrm{d}a}{\mathrm{d}\ell}, 0, 0 \right),$$

$$n^\mu = \pm \left(\frac{1}{F(a)} \frac{\mathrm{d}a}{\mathrm{d}\ell}, \sqrt{(\mathrm{d}a/\mathrm{d}\ell)^2 - F(a)}, 0, 0 \right), \tag{3.49}$$

and a brief computation yields

$$\sigma = \mp \frac{1}{2\pi a} \sqrt{(\mathrm{d}a/\mathrm{d}\ell)^2 - F(a)}. \tag{3.50}$$

[12] If you think of the Reissner–Nordström–de Sitter geometry as arising from gravitational collapse of an electrically charged star, then it is the left curvature singularity (which is eliminated by the present construction) that would arise from the central density of the star growing to infinity. The right curvature singularity (which is unaffected by the present construction) has a totally different genesis as it arises in a matter-free region due to gravitational focussing of the electromagnetic field.

[13] Although this is a static minimum of the usual $V(a)$ it is in the present context not *stable*. This arises because for a spacelike shell the overall *sign* of the potential flips.

This can be rearranged to give

$$\left(\frac{\mathrm{d}\ln(a)}{\mathrm{d}\ell}\right)^2 - \frac{F(a)}{a^2} = (2\pi\sigma)^2. \tag{3.51}$$

So the net result is that for an instanton-brane the *sign* of the potential has flipped, but that of the brane contribution to the energy has not. (And exactly the same result continues to hold inside the horizon, a few intermediate signs flip, but that's all.)

In summary: certain varieties of brane wormhole provide explicit evasions (either partial or complete) of the usual singularity theorems.

4. Voids: the brane as a spacetime boundary

A somewhat unusual feature of brane physics is that the brane could also be viewed as an actual physical boundary to spacetime, with the "other side" of the brane being null and void. In general relativity as it is normally formulated the notion of an actual physical boundary to spacetime (that is, an accessible boundary reachable at finite distance) is anathema. The reason that spacetime boundaries are so thoroughly deprecated in general relativity is that they become highly artificial special places in the manifold where some sort of boundary condition has to be placed on the physics by an act of black magic. Without such a postulated boundary condition all predictability is lost, and the theory is not physically acceptable. Since there is no physically justifiable reason for picking any one particular type of boundary condition (Dirichlet, Neumann, Robin, or something more complicated), the attitude in standard general relativity has been to exclude boundaries, by appealing to the cosmic censor whenever possible and by hand if necessary.

The key difference when a brane is used as a boundary is that now there is a specific and well-defined boundary condition for the physics: D-branes (D for *Dirichlet*) are defined as the loci on which the fundamental open strings end (and satisfy Dirichlet-type boundary conditions). D-branes are therefore capable (at least in *principle*) of providing both a physical boundary *and* a plausible boundary condition for spacetime. For Neveu–Schwarz branes the boundary conditions imposed on the fundamental string states are more complicated, but they still (at least in principle) provide physical boundary conditions on the spacetime.

When it comes to specific calculations, this may however not be the best mental picture to have in mind — after all, how would you try to calculate the Riemann tensor for the edge of spacetime? And what would happen to the Einstein equations at the edge? There is a specific trick that clarifies the situation: Take the manifold with brane boundary and make a second copy, then sew the two manifolds together along their respective brane boundaries, creating a single manifold without boundary that contains a brane, and exhibits a Z_2 symmetry on reflection around the brane. Because this new manifold is a perfectly reasonable no-boundary manifold containing a brane, the gravitational field can be analyzed using the usual thin-shell formalism of general relativity [32–35]: The metric is continuous, the connection exhibits a step-function discontinuity, and the Riemann curvature a delta-function at the brane. The dynamics of the brane can then be investigated in this Z_2-doubled manifold, and once the dynamical equations and their solutions have been investigated the second surplus copy of spacetime can quietly be forgotten.

In particular, all the calculations we have performed for the spherically symmetric wormholes of this paper apply equally well to spherically symmetric holes in spacetime (not black holes, actual voids in the manifold), with the edge of the hole being a brane — we deduce the existence of a large class of stable void solutions, and an equally large class of unstable voids that either collapse to form black holes, or explode to engulf the entire universe.

Equally well, the baby universes of the preceding section can, under this new physical interpretation of the relevant mathematics, be used to investigate finite volume universes with boundary. The bulk of the physical universe now lies in the range $r \in (0, a)$, and the edge of the universe is located at a. Again, we deduce the existence of a large class of stable baby universes with boundary, and an equally large class of unstable baby universes that either collapse to singularity, or explode to provide arbitrarily large universes. Note that these particular exploding universes are *not* FLRW universes, and are not suitable cosmologies for our own universe. Nevertheless, this notion of using a brane as an actual physical boundary of spacetime is an issue of general applicability, and we hope to return to this topic in future publications.

5. Discussion

The main point of this paper is that in the brane picture there is nothing wrong with the notion of a *negative* brane tension, and that once branes of this type are allowed to contribute to the stress-energy, the class of solutions is greatly enhanced, now including many quite peculiar beasts not normally considered to be part of standard general relativity. As specific examples, the energy condition violations caused by negative tension branes allow one to construct classical traversable wormholes, at least some of which (as we have seen) are actually dynamically stable. Now for spherically symmetric wormholes of the type considered in this paper, attempting to cross from one universe to the other requires the traveller to cross the brane, a process which is likely to prove disruptive of the traveller's internal structure, well being, and overall health. This problem, or rather the no-brane analog of this problem, was already considered by Morris and Thorne in their pioneering work on traversable wormholes [8]. A possible resolution comes from the fact that spherical symmetry is a considerable idealization: One of the present authors has demonstrated that if one uses negative tension cosmic strings instead of negative tension domain walls, then it is possible to construct traversable wormhole spacetimes that do not possess spherical symmetry, and contain perfectly reasonable paths from one asymptotic region to the other that do not involve personal encounters with any form of "exotic matter" [40]. (See also the extensive discussion in [9].) In a brane context this means we should consider the possibility of a negative tension $(1+1)$-dimensional brane in $(3+1)$-dimensional spacetime.

Now the peculiarities attendant on widespread violations of the energy conditions are not limited to violations of topological censorship; as we have seen there is also the possibility of violating (evading) the singularity theorem. If this is not enough, then it should be borne in mind that without some form of energy condition we do not have a positive mass *theorem*. (Looking out into our own universe, we do have a positive mass *observation*, but it would be nice to be able to deduce this from general principles.) A discussion of some of the peculiarities attendant on negative asymptotic mass can be found in the early work of Bondi [41], and a possible observational signal (particular types of caustics in the light curves due to gravitational lensing) has been pointed out by Cramer et al. [42]. Finally, energy condition violations are also the *sine qua non* for the Alcubierre "warp drive" [43].

In summary, all of these somewhat peculiar geometries, which were investigated within the general relativity community more with a view to understanding the limitations of general relativity (and more specifically, of semiclassical general relativity) than in the expectation that they actually exist in reality, are now seen to automatically be part and parcel of the brane models currently being considered as semi-phenomenological models of empirical reality.

Acknowledgements

The research of CB was supported by the Spanish Ministry of Education and Culture (MEC). MV was supported by the US Department of Energy. We wish to thank Harvey Reall and Sumit Das for their comments and interest.

References

[1] L. Randall, R. Sundrum, A large mass hierarchy from a small extra dimension, Phys. Rev. Lett. 83 (1999) 3370, hep-ph/9905221.
[2] L. Randall, R. Sundrum, An alternative to compactification, Phys. Rev. Lett. 83 (1999) 4690, hep-th/9906064.
[3] E.E. Flanagan, R.M. Wald, Phys. Rev. D 54 (1996) 6233.
[4] C. Barcelo, M. Visser, Traversable wormholes from massless conformally coupled scalar fields, Phys. Lett. B 466 (1999) 127, gr-qc/9908029.
[5] M. Visser, C. Barcelo, Energy conditions and their cosmological implications, gr-qc/0001099.
[6] C. Barcelo, M. Visser, Scalar fields, energy conditions, and traversable wormholes, gr-qc/0003025.
[7] L.A. Anchordoqui, S.E. Perez Bergliaffa, The world is not enough, gr-qc/0001019.
[8] M.S. Morris, K.S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. 56 (1988) 395.
[9] M. Visser, Lorentzian Wormholes: From Einstein to Hawking, American Institute of Physics, Woodbury, 1995.
[10] H. Kawai, D.C. Lewellen, S.H. Tye, Construction of four-dimensional fermionic string models, Phys. Rev. Lett. 57 (1986) 1832.
[11] H. Kawai, D.C. Lewellen, S.H. Tye, Construction of fermionic string models in four dimensions, Nucl. Phys. B 288 (1987) 1.

[12] H. Kawai, D.C. Lewellen, S.H. Tye, Four-dimensional type II strings and their extensions: type-III strings, Phys. Lett. B 191 (1987) 63.
[13] H. Kawai, D.C. Lewellen, S.H. Tye, Construction of four-dimensional fermionic string models with a generalized supercurrent, Int. J. Mod. Phys. A 3 (1988) 279.
[14] S. Chaudhuri, H. Kawai, S.H. Tye, Chiral bosonic supercurrents and four-dimensional fermionic string models, Nucl. Phys. B 322 (1989) 373.
[15] J. Polchinski, An Introduction to the Bosonic String, String Theory, Vol. 1, Cambridge University Press, England, 1998.
[16] J. Polchinski, Superstring Theory and Beyond, String Theory, Vol. 2, Cambridge University Press, England, 1998.
[17] C. Csaki, M. Graesser, L. Randall, J. Terning, Cosmology of brane models with radion stabilization, hep-ph/9911406.
[18] C. Csaki, J. Erlich, T.J. Hollowood, J. Terning, Holographic RG and cosmology in theories with quasi-localized gravity, hep-th/0003076.
[19] C. Csaki, J. Erlich, T.J. Hollowood, Y. Shirman, Universal aspects of gravity localized on thick branes, hep-th/0001033.
[20] J. Cline, C. Grojean, G. Servant, Inflating intersecting branes and remarks on the hierarchy problem, Phys. Lett. B 472 (2000) 302, hep-ph/9909496.
[21] C. Grojean, J. Cline, G. Servant, Supergravity inspired warped compactifications and effective cosmological constants, hep-th/9910081.
[22] J.M. Cline, C. Grojean, G. Servant, Cosmological expansion in the presence of extra dimensions, Phys. Rev. Lett. 83 (1999) 4245, hep-ph/9906523.
[23] J.M. Cline, Cosmological expansion in the Randall–Sundrum warped compactification, hep-ph/0001285, Cosmo99 Proceedings.
[24] I.I. Kogan, S. Mouslopoulos, A. Papazoglou, G.G. Ross, J. Santiago, A three three-brane universe: new phenomenology for the new millennium?, hep-ph/9912552.
[25] T. Li, Classification of 5-dimensional space–time with parallel 3-branes, hep-th/9912182.
[26] P. Kanti, I.I. Kogan, K.A. Olive, M. Pospelov, Single-brane cosmological solutions with a stable compact extra dimension, hep-ph/9912266.
[27] M. Visser, Traversable wormholes from surgically modified Schwarzschild space–times, Nucl. Phys. B 328 (1989) 203.
[28] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, N. Kaloper, Infinitely large new dimensions, Phys. Rev. Lett. 84 (2000) 586, hep-th/9907209.
[29] Z. Chacko, A.E. Nelson, A solution to the hierarchy problem with an infinitely large extra dimension and moduli stabilization, hep-th/9912186.
[30] C. Grojean, T self-dual transverse space and gravity trapping, hep-th/0002130.
[31] D. Hochberg, M. Visser, The null energy condition in dynamic wormholes, Phys. Rev. Lett. 81 (1998) 746, gr-qc/9802048.
[32] W. Israel, Singular hypersurfaces and thin shells in general relativity, Nuovo Cimento B 44 (10) (1966) 1–14; Nuovo Cimento B 48 (10) (1967) 463–463 (Errata).
[33] K. Lanczos, Untersuching über flächenhafte verteiliung der materie in der Einsteinschen gravitationstheorie, 1922, unpublished.
[34] K. Lanczos, Flächenhafte verteiliung der materie in der Einsteinschen gravitationstheorie, Ann. Phys. (Leipzig) 74 (1924) 518–540.
[35] N. Sen, Über dei grenzbedingungen des schwerefeldes an unstetig keitsflächen, Ann. Phys. (Leipzig) 73 (1924) 365–396.
[36] A. Chamblin, M.J. Perry, H.S. Reall, Non-BPS D8-branes and dynamic domain walls in massive IIA supergravities, JHEP 9909 (1999) 014, hep-th/9908047.
[37] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, England, 1973.

[38] E. Poisson, Black-hole interiors and strong cosmic censorship, gr-qc/9709022; in: L.M. Burko, A. Ori (Eds.), The Internal Structure of Black Holes and Spacetime Singularities, Institute of Physics Press, Bristol, 1997.
[39] E.E. Flanagan, Quantum mechanical instabilities of Cauchy horizons in two dimensions: a modified form of the blueshift instability mechanism, gr-qc/9711066; in: L.M. Burko, A. Ori (Eds.), The Internal Structure of Black Holes and Spacetime Singularities, Institute of Physics Press, Bristol, 1997.
[40] M. Visser, Traversable wormholes: some simple examples, Phys. Rev. D 39 (1989) 3182.
[41] H. Bondi, Negative mass in general relativity, Rev. Mod. Phys. 29 (1957) 423.
[42] J.G. Cramer, R.L. Forward, M.S. Morris, M. Visser, G. Benford, G.A. Landis, Natural wormholes as gravitational lenses, Phys. Rev. D 51 (1995) 3117, astro-ph/9409051.
[43] M. Alcubierre, The warp drive: hyper-fast travel within general relativity, Class. Quantum Grav. 11 (1994) L73.