AN APPROXIMATE INVENTORY MODEL BASED ON DIMENSIONAL ANALYSIS

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Dimensional analysis is used to develop a power law solution that gives approximately optimal (s, S) policies for a periodic review stochastic inventory model. The solution is similar in form to Ehrhardt's Power Law Model and uses his structure for s, as direct methods of fitting proved unsuccesful.

It gives marginally better results than Ehrhardt's [1,2] on the same data sets that he used.

Keywords: approximations, inventory theory, dimensional analysis, power law model

1. Introduction

The inventory model to be studied assumes a periodic-review, single product, structure with independent stochastic demand with mean μ, and standard deviation σ, in each period of length t, and an infinite planning horizon. There is a constant leadtime of an integer number of periods,L, a fixed setup cost, K, a linear purchase cost, c, and linear holding, h, and penalty costs, p, based on ending period stock levels. There is total backlogging of unfilled demand. A discount factor is applied to future cash flows. The objective is to choose an order policy that minimises the long-run equivalent average cost per period.

It is known from the results of Scarf [3] and Iglehart [4] that for this model the (s, S) policy is optimal. That is, at the beginning of a period, if the entering stock is (strictly) less than s, order $(S - s)$, otherwise do not order.

Veinott and Wagner's [5] algorithm to obtain the optimal value of $(S - s)$ for a given set of parameters involves a search over both s and S and is impracticable for general use when one has many thousands of items to be controlled. A faster algorithm by Federgruen and Zipkin [6] might be practical for a few A items but probably not for all the items in an inventory.

Thus it is attractive to develop approximate algorithms that will get close enough to the optimal (s, S) policy for practical work with little computation. Ehrhardt [1] and Ehrhardt and Mosier [2] had proposed another approximate method in the form of a formula for $(S - s)$ which is a power function of the model parameters. Porteous [7] has studied a number of such approximation algorithms and concludes that two he describes, the MCR and the exponential-tail algorithms, are good enough for use in practice. In this paper we examine a model of a similar form to that of Ehrhardt, but based on a dimensional analysis approach to inventory problems. This approach has the effect of limiting the search for good models to those that are dimensionally homogeneous.

In the next two sections we examine the Ehrhardt models and an alternative approach [8] based on dimensional analysis [9,10] that results in a modified version of the power-type of model. Then we describe experience in obtaining parameters for the model and end by comparing the approximations.

2. approximate inventory policy algorithms

Ehrhardt's [1] Power Approximation was developed by considering a limiting form of an optimal policy and devising a power form for $D = S - s$ and S (or s) that could be fitted by regression to a set of known optimal policies. To get a data set, Ehrhardt varied the parameters, μ, K, p, and (σ^2/μ), keeping h constant. Different values of (σ^2/μ) implied different shapes for the discrete demand distribution. These formed 288 sets of data for each of which the optimal values of (S, s) were found using the Veinott-Wagner algorithm. The formula for s included an approximation to the inverse of the tail of the loss function.

The model was modified in the later paper with Mosier [2] to correct two flaws. First, to prevent $D \to 0$ when the leadtime demand variance $\sigma_L \to 0$, the form $(1 + \sigma_L^2/\mu^2)$ was used instead of (σ_L^2/μ^2). Second, to correct a dimensional flaw that had the effect of changing the model values if the units of demand were changed, the exponent in the demand term was constrained during the regression so that there was one less exponent to find. The resulting Revised Power Approximation has the following form:

$$D = S - s = 1.30\mu^{0.494}(\frac{K}{h})^{0.506}(1 + \frac{\sigma_L^2}{\mu^2})^{0.116} \quad (1)$$

$$z = (Dh/p\sigma_L)^{0.5} \quad (2)$$

$$s = 0.973\mu_L + \sigma_L[0.183/z + 1.603 - 2.192z] \quad (3)$$

(for values of D/μ less than 1.5, a different formula is used to calculate s)

The expression for D is very close to that for the traditional EOQ model except for the slight differences in the powers of the parameters and the addition of the variance term. The similarity is not surprising since it was derived from a model with the EOQ form. The formula for s is derived from an approximation to the tail of the demand distribution. z is dimensionless.

3. A dimensional analysis approach

Any physical relationship must be dimensionally homogeneous; one can sum a number of terms of costs but cannot add kgs and costs in the same formula [9,10]. From this "Principle of Homogeneity", it can be seen by simple division that any such relationahip can be expressed as a function of dimensionless products of involved quantities. This may seem a commonplace statement but it constrains the form of any relationship including such formulae as the inventory approximations.

There is some value in expressing the relationship in terms of these dimensionless products (groups, or numbers) since they (1) are independent of the units in which variables in the problem are measured and (2) often indicate the relative magnitude of physical effects in the real situation. In some areas the dimensionless numbers are of such great importance that they have been named after illustrious workers in the field.

A relationship can be graphed in dimensionless form much more compactly than as a function of the whole range of the original parameters. The resulting graphs are valid over wide ranges in parameter values and the parameters of the graphs are independent of the units used.

Usually there are fewer dimensionless products than original parameters and variables. Further, they have the effect of constraining the way that the original parameters and variables enter into the solution. It is therefore sensible, in our search for the proper form of our models, to use these dimensionless products as our fundamental variables. If we intend to carry out numerical work, for example to fit approximate models by means of regression, or experimental work involving factorial designs, it is better to work in terms of these dimensionless products since there are fewer of them and they better describe the phenomena to be studied.

Since we know that the models can be expressed in terms of these dimensionless numbers it gives us a start on the form of models to be studied where, as in the case here, we need to choose a good approximation over a wide range of situations. We start by listing all the different dimensionless products that can be formed by the variables and parameters provided in the problem statement. We can obtain the full set of independent dimensionless products from these. There are usually $N - m$ of them, where there are N variables and parameters and m fundamental dimensions. In this case there are 3 fundamental dimensions: time $[T]$, quantity $[U]$, in which we measure the amount of material to be ordered, and money [\$], in which we measure costs and profits.

For inventory problems with the variables given in this problem we find the following list of dimensionless products are suitable (Vignaux [9] derives a similar set for the continuous review case).

$$(\frac{K}{ht\mu}), (\frac{p}{h}), (\frac{D}{\mu}), (\frac{\sigma}{\mu}), (L) \tag{4}$$

The first product, $w = K/ht\mu$, is related to the 'Wilson' number which reflects the balance between holding cost and ordering cost. Here we have assumed that h has dimensions $[\$U^{-1}T^{-1}]$. w can be expressed in terms of the demand rate, $M = \mu/t$ $[UT^{-1}]$ in the form $KM/h\mu^2$. It appears in the traditional EOQ solution as

$$D/\mu = \sqrt{2}\sqrt{w} \tag{5}$$

Cancellation of the μ on each side results in the EOQ formula for order quantity.

(p/h) is a scale factor representing the balance between penalty and stock holding cost, (σ/μ) is the coefficient of variation of demand in a single period and L is the leadtime (in units of review period). It is also convenient to assume a form for the convolution of demand over the leadtime, L, and define the leadtime variance as $\sigma_L^2 = (1 + L)\sigma^2[U^2]$ and also to recognise, as Ehrhardt did in his model, that, for a periodic review model, the leadtime L needs to be represented as $(1 + L)$.

As long as our model uses these products in some combination, we can be sure that it will remain dimensionally homogeneous. We cannot be sure of the exact form of the model but can make some preliminary investigations to see what the general nature of the model must be. Since we are looking for an approximation of $D = S - s$, our first approach to the model might be:

$$(D/\mu) = fn\{w, (p/h), (\sigma/\mu), L\} \tag{6}$$

One would conjecture a similar form for s/μ or S/μ.

4 Models for D

As long as the model uses these dimensionless numbers, or combinations of them, it can take any form. We can start our search for an appropriate model by seeing how it must behave as parameters move to their limits.

As $\sigma/\mu \to 0$, D must remain finite and the model becomes deterministic. If we are to retain the product form of the model, σ/μ therefore probably appears in the form $1 + \sigma/\mu$ or $1 + \sigma^2/\mu^2$.

The deterministic solution gives $D/\mu = \sqrt{2}\sqrt{w}\sqrt{1 + h/p}$. We would expect that the term involving p/h goes to 1 as $h/p \to 0$. So an appropriate form, as a first approximation, might be $(1+h/p)^\alpha$, a generalisation of the deterministic form.

We will further assume that L appears in the dimensionless form $1 + L$.

In order to ensure that our model also fitted these extreme cases of small σ/μ we added additional data to the 288-point Ehrhardt data set with this ratio set to zero giving 384 points in total. FOr these additional points, the optimal policies and costs were computed from the deterministic model. Using regression, we tested a variety of combinations of such structures on the data set and were led to the following version:

$$(D/\mu) = 1.22w^{0.53}(1 + h/p)^{0.38}(1 + \sigma/\mu)^{0.10}(1 + L)^{0.06} \tag{7}$$

This had, for the data set of $n = 384$, a value of $R^2 = 0.977$ compared to 0.971 for Ehrhardt [2]. It differs from his solution in that the effect of h/p is considerably larger presumably because of the values with $h/p = 0$.

5. Models for s

Examination of the optimal (s, S) solutions for the given data set reveals that some of the s values are negative and a purely power form of model cannot be used. A power form for $S = s + D$ is feasible but was found to give very poor fits (R^2 between 40 and 50%).

Moving from the power law form to a simple polynomial regression, on the basis that, at least for small values of the parameters, such a fit should approximate most functions, gave similar poor results (R^2 between 45 and 50%).

When the data were split into groups corresponding to the different distributions (essentially, the parameter σ^2/μ for the negative binomial distribution used by Ehrhardt) the fits for the individual groups were much better (R^2 between 67 and 72%) but still not useful enough.

We finally decided that the more complicated Ehrhardt version for s as a function of dimensionless z and σ_L/μ_L would have to be used.

A regression gave the following result:

$$s/\mu_L = 0.995 + (\sigma_L/\mu_L)[0.188/z + 0.939 - 2.11z] \tag{8}$$

This has an excellent fit with an R^2 of 0.999 compared with 0.998 obtianed by Ehrhardt.

6. Discussion

The model described in the previous two sections gives a marginally better fit for D than Ehrhardt's (an R^2 of 0.977 compared with 0.971 on the data set we used). On examining the residuals we find a pattern that indicates that the difficulties occur for small values of σ/μ and as a function of μ. We conjecture that this is due to changes in the shape of the demand distribution and that further work should concentrate on this aspect.

This effect of the demand distribution shape may also explain some large residuals we obtained for the fitted s when high p/h and σ/μ ratios occurred.

The model gives a marginal improvment on Ehrhardt's when the average cost rate, C [$\$/T$] obtained using the model (s, D) is compared with that for the known optimal policy, C^*. The average value of the percentage error $(C - C^*)/C^*$ was 0.40% versus 0.47% for the Ehrhardt model with a corresponding decrease in the error frequencies.

We set out to look for a power law based on dimensional methods that would improve on the Ehrhardt model. It was natural to compare them using his data set, enhanced in our case by a group of special cases. It is clear that this data, while balanced in terms of the original variable values is quite unbalanced in terms of the dimensionless numbers that the model is based on. A more appropriate data set would be obtained by carrying out a factorial design with independent variations of these numbers and testing the models on that basis. For practical use the approximations should work well for regions of the data space that correspond to real data and this may not distribute over the whole volume of dimensionless parameter space with equal density. We may need to choose the subspace that corresponds best to practice and fit our models to data equally spread over that or alternatively use some form of weighted regression.

7. Conclusion

This work has shown that it is possible to use dimensional arguments to develop models that are reasonable approximations to the true situation, and are also dimensionally correct.

The power approximation finally developed in this paper (equations 7,2, and 8), although it owes much to the Ehrhardt model, gives marginally better policies and could be substituted for it in in applications. It is little different in results, but has the advantage that it is soundly based on dimensional arguments. In particular, its parameters are known to be invariant to changes in the units of measurement of demand, time, or cost. Their actual values, as they are obtained from regression over a particular set of data, will depend on that set being representative of the practical circumstances in which the model is to be used.

There is therefore some point in repeating the experiment using a balanced set of values in the dimensionless numbers rather than with the original model variables. We would expect only small changes in the model parameters, but it might fit over a wider range of practical situations.

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