Quasi-Modal Equivalence of Canonical Structures

Robert Goldblatt
School of Mathematical and Computing Sciences,
Victoria University, P. O. Box 600, Wellington, New Zealand
Rob.Goldblatt@vuw.ac.nz

10 November 1999

Abstract

A first-order sentence is *quasi-modal* if its class of models is closed under the modal validity preserving constructions of disjoint unions, inner substructures and bounded epimorphic images.

It is shown that all members of the proper class of canonical structures of a modal logic Λ have the same quasi-modal first-order theory Ψ^Λ. The models of this theory determine a modal logic Λ^e which is the largest sublogic of Λ to be determined by an elementary class. The canonical structures of Λ^e also have Ψ^Λ as their quasi-modal theory. In addition there is a largest sublogic Λ^c of Λ that is determined by its canonical structures, and again the canonical structures of Λ^c have Ψ^Λ are their quasi-modal theory. Thus $\Psi^\Lambda = \Psi^{\Lambda^c} = \Psi^{\Lambda^e}$.

Finally, we show that all finite structures validating Λ are models of Ψ^Λ, and that if Λ is determined by its finite structures, then Ψ^Λ is equal to the quasi-modal theory of these structures.

1 Overview

This is a contribution to the problem of fully understanding the role played by so-called *canonical structures* in the model theory of modal and other intensional logics. Each logic Λ defines a sequence

$$\langle\, \mathfrak{S}^\Lambda_\kappa : \kappa \text{ is an infinite cardinal} \,\rangle$$

of such structures, with $\mathfrak{S}^\Lambda_\kappa$ being of size 2^κ. There are many questions about their theories (both modal and first-order), and other relationships between them, that remain unanswered.

A modal formula that is valid in $\mathfrak{S}^{\Lambda}_{\kappa}$ must be a Λ-theorem. Thus the modal theory of each $\mathfrak{S}^{\Lambda}_{\kappa}$ is a sublogic of Λ. In the converse direction we say that Λ is *canonical* if its theorems are validated by all its canonical structures, in which case the modal theory of every $\mathfrak{S}^{\Lambda}_{\kappa}$ is exactly Λ itself. There are logics that are known not to be canonical, because some of their theorems are falsifiable in their canonical structures. Notable examples include the modal logic of the provability predicate of Peano arithmetic, and the tense logic of a continuous temporal ordering, defined by taking the real numbers under their natural order as the model of time.

When $\kappa < \mu$ there is a natural validity-preserving projection of $\mathfrak{S}^{\Lambda}_{\mu}$ onto $\mathfrak{S}^{\Lambda}_{\kappa}$. Then any formula valid in $\mathfrak{S}^{\Lambda}_{\mu}$ will be valid in $\mathfrak{S}^{\Lambda}_{\kappa}$. It seems plausible to conjecture that the converse is true and indeed, more strongly, that validity in just the smallest structure $\mathfrak{S}^{\Lambda}_{\omega}$ is enough to ensure validity in every $\mathfrak{S}^{\Lambda}_{\kappa}$. This would imply that if $\mathfrak{S}^{\Lambda}_{\omega}$ validates Λ then so do all the other structures, and hence Λ is canonical. The motivation for this is the belief that $\mathfrak{S}^{\Lambda}_{\omega}$ is sufficiently "saturated" that any falsification of a formula in a larger canonical structure could be reproduced within $\mathfrak{S}^{\Lambda}_{\omega}$ itself. However these conjectures are unresolved.

A logic Λ is called *elementary* if there exists a collection $\mathcal{K}$ of relational structures (Kripke frames) that is an elementary class, i.e. is axiomatisable in first-order logic, and which *determines* Λ in the sense that the formulae that are valid in all members of $\mathcal{K}$ are precisely the Λ-theorems. It was shown in [2] that every modal logic which is elementary must be canonical, but the *canonicity conjecture* that, conversely, every canonical logic is elementary, has remained open for 25 years. A strong version of this asserts that if Λ is valid in $\mathfrak{S}^{\Lambda}_{\omega}$ then it is elementary. If true, it would confirm the above conjecture that validity in $\mathfrak{S}^{\Lambda}_{\omega}$ implies canonicity.

Given the nature of the size (2^{κ}) of canonical structures, one might well ask whether the answers to some of these questions depend on the nature of the ambient set theory, including the status of the continuum hypothesis. Relevant to this issue is the result in the recent paper [9] that even in a context in which $\mathfrak{S}^{\Lambda}_{\omega}$ and $\mathfrak{S}^{\Lambda}_{\omega_1}$ have the same size, i.e. $2^{\omega} = 2^{\omega_1}$, the two structures cannot be isomorphic when Λ is any sublogic of S5 (and some other cases as well). That only adds pertinence to questions about the logical equivalence or distinguishability of different canonical structures. Can they have different modal theories, or must they all define the same sublogic of Λ? Can they be differentiated by some elementary property, or do they all have the same first-order theory?

Some progress on the link between canonicity and elementarity was made in the author's articles [3, 7] by studying those sentences of first-order logic that are preserved by the three constructions of disjoint unions, inner substructures and bounded epimorphisms. We will dub such sentences *quasi-modal*[1] because these constructions are the primary modal validity preserving operations on structures, and so the quasi-modal sentences include any first-order sentence that defines the class of all structures validating some modal formula. Let Ψ_κ^Λ be the set of quasi-modal sentences that are true in the canonical structure $\mathfrak{S}_\kappa^\Lambda$. Then according to Theorem 11.4.2 of [3], if Λ is determined by *some* elementary class, then it must be determined by the elementary class $Mod\Psi_\omega^\Lambda$ of all models of Ψ_ω^Λ. This suggests that in trying to prove the canonicity conjecture it would be appropriate to focus on showing that if Λ is valid in $\mathfrak{S}_\omega^\Lambda$, then it is also valid in any model of the quasi-modal theory Ψ_ω^Λ of $\mathfrak{S}_\omega^\Lambda$.

In fact if Λ is elementary, then it is determined by the elementary class of models of any Ψ_κ^Λ, as will be shown below (Theorem 6.1). But even when Λ is not elementary we still have a natural elementary class $Mod\Psi_\kappa^\Lambda$ defining a canonical sublogic of Λ for each κ. The principal result of this article is that in fact there is only one elementary class thus defined: it turns out that $\Psi_\kappa^\Lambda = \Psi_\omega^\Lambda$ for all infinite cardinals κ, so the canonical structures of a logic Λ all have exactly the same quasi-modal first-order theory. The models of this theory define a single sublogic Λ^e of Λ that is in fact the the largest elementary sublogic of Λ.

In carrying out this analysis it will also be shown that there is a largest canonical sublogic Λ^c of Λ. If Λ is canonical then $\Lambda^c = \Lambda$, and if the canonicity conjecture is true then $\Lambda^c = \Lambda^e$. In general we know only that $\Lambda^e \subseteq \Lambda^c \subseteq \Lambda$, but will show that all of the canonical structures of all three logics have the same quasi-modal theory: $\Psi_\kappa^{\Lambda^e} = \Psi_\kappa^{\Lambda^c} = \Psi_\kappa^\Lambda = \Psi_\omega^\Lambda$ for all κ.

Finally, we prove that all finite structures validating Λ are models of Ψ_ω^Λ, and that if Λ is determined by its finite structures, then Ψ_ω^Λ is equal to the quasi-modal theory of these structures.

Although the results of this paper are stated for the propositional language of a single unary modality, it should be noted that they adapt readily to hold for *polymodal* logics having n-ary modalities (for various $n \geq 1$) interpreted semantically by $n+1$-ary relations.

[1]The more opaque adjective "pseudo-equational" was used in [3, 7]. "Quasi-modal" seems more evocative of the intended meaning.

2 Validity of Modal Formulae

Let $Var = \{p_\lambda : \lambda \text{ is an ordinal}\}$ be a class of distinct propositional variables. For each infinite cardinal κ, put $Var_\kappa = \{p_\lambda : \lambda < \kappa\}$. The class Fma of *modal formulae* consists of all (finite) formulae generated from members of Var by truth-functional connectives and the modality $\Box$. If V is a subset of Var, we write $Fma(V)$ for the set of formulae generated by V, i.e. those formulae whose variables belong to V. The set of formulae generated by Var_κ will be denoted Fma_κ. These are the *κ-formulae*: the ones who variables all have index less than κ.

Let $\mathcal{L}$ be the first-order language of a single binary predicate. An $\mathcal{L}$-*structure* $\mathfrak{S} = \langle S, R\rangle$ comprises a binary relation R on a set S. $\mathfrak{S}$ is also called a (Kripke) *frame*. A *V-model* $\mathcal{M} = \langle \mathfrak{S}, v\rangle$ on $\mathfrak{S}$ is given by a *valuation* v assigning a subset $v(p)$ of S to each variable p in V. When $V = Var_\kappa$, we call this a *κ-model*. The set $\mathcal{M}(\varphi)$ of points at which the modal formula φ is *true in* $\mathcal{M}$ is then defined inductively for all formulae in $Fma(V)$ by putting $\mathcal{M}(p) = v(p)$, interpreting each truth-functional connective by the appropriate Boolean set operation, and defining $s \in \mathcal{M}(\Box\varphi)$ iff $\{t : sRt\} \subseteq \mathcal{M}(\varphi)$. If $\Diamond = \neg\Box\neg$ is the dual modality to $\Box$, then $s \in \mathcal{M}(\Diamond\varphi)$ iff $\exists t \in \mathcal{M}(\varphi)\,(sRt)$.

Formula φ is *true in model* $\mathcal{M}$ if it is true at all points of $\mathcal{M}$, i.e if $\mathcal{M}(\varphi) = S$. φ is *valid in structure* $\mathfrak{S}$ if it is true in every model on $\mathfrak{S}$ whose valuation includes the variables of φ in its domain. The class of structures in which φ is valid will be denoted $Str\,\varphi$.

For a first-order $\mathcal{L}$-sentence σ, the word "model" will be used as usual to denote any $\mathcal{L}$-structure in which σ is true. $Mod\,\Sigma$ denotes the class of all models of a set Σ of $\mathcal{L}$-sentences. A class $\mathcal{K}$ of structures is *elementary* if it equal to $Mod\,\Sigma$ for some Σ.

3 Logics and Canonical Structures

A *logic* is a subclass Λ of Fma that includes all tautologies and instances of the schema $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and is closed under the inference rules of Modus Ponens, Necessitation (if $\varphi \in \Lambda$ then $\Box\varphi \in \Lambda$); and uniform substitution of a formula for a variable. Members of logic Λ may be called *Λ-theorems*.

A class $\mathcal{K}$ of $\mathcal{L}$-structures defines a logic, namely the class of formulae that are valid in all members of $\mathcal{K}$. We say that logic Λ is *determined by* class $\mathcal{K}$ if it is the logic defined by $\mathcal{K}$ in this way. A given logic may be determined by more than one class. On the other hand there are *incomplete* logics that are not determined by any class of structures.

We can also consider logics within fragments of the form $Fma(V)$ by allowing only formulae with variables from V in the definition of "logic". Thus if Λ is a logic in Fma, then $\Lambda_\kappa = \Lambda \cap Fma_\kappa$ is a logic in Fma_κ. In fact Λ is uniquely characterised by Λ_ω, because a formula has only finitely many variables while Var_ω is infinite. Thus we can associate with any formula φ a substitution instance of it in Fma_ω that is a Λ-theorem iff φ is. This can be used to show that Λ is the only logic in Fma whose restriction to Fma_ω is Λ_ω, and likewise Λ_κ is the only logic in Fma_κ whose restriction to Fma_ω is Λ_ω. Indeed Λ is just the closure of Λ_ω in Fma under substitution, and similarly Λ_κ is the substitution-closure of Λ_ω in Fma_κ. Moreover, a structure validates Λ iff it validates Λ_ω.

A set s of formulae is *Λ-consistent* if $\neg\varphi$ is never a Λ-theorem when φ is the conjunction of finitely many members of s. For a cardinal κ, the *canonical Λ_κ-structure* is $\mathfrak{S}^\Lambda_\kappa = \langle S^\Lambda_\kappa, R^\Lambda_\kappa \rangle$, with S^Λ_κ being the set of all maximally Λ-consistent subsets of Fma_κ, and $sR^\Lambda_\kappa t$ iff $\{\varphi : \Box\varphi \in s\} \subseteq t$. The *canonical Λ_κ-model* is the κ-model $\mathcal{M}^\Lambda_\kappa = \langle \mathfrak{S}^\Lambda_\kappa, v^\Lambda_\kappa \rangle$, where $v^\Lambda_\kappa(p_\lambda) = \{s \in S^\Lambda_\kappa : p_\lambda \in s\}$ for all $\lambda < \kappa$. An inductive argument then shows that $\mathcal{M}^\Lambda_\kappa(\varphi) = \{s \in S^\Lambda_\kappa : \varphi \in s\}$ for all κ-formulae φ. From this it is shown that $\mathcal{M}^\Lambda_\kappa$ *determines* Λ_κ, i.e. a κ-formula is true in $\mathcal{M}^\Lambda_\kappa$ iff it is a Λ-theorem, and hence that any κ-formula valid in $\mathfrak{S}^\Lambda_\kappa$ must be a Λ_κ-theorem. But since validity in a structure is preserved by substitution, it follows that any member of Fma which is valid in $\mathfrak{S}^\Lambda_\kappa$ must be a Λ-theorem.

4 Operations on Classes of Structures

An $\mathcal{L}$-structure $\langle S', R' \rangle$ is a *inner substructure* of $\langle S, R \rangle$ if $S' \subseteq S$, R' is the restriction of R to S', and S' is *R-closed* in the sense that if sRt and $s \in S'$, then $t \in S'$. Inner substructures are also known as *generated subframes*.

A *bounded morphism* $f : \langle S, R \rangle \to \langle S', R' \rangle$ is a function $f : S \to S'$ such that sRt implies $f(s)R'f(t)$, and $f(s)R'u$ implies $\exists t \in S(sRt$ and $f(t) = u)$. If f is surjective, then it is called a *bounded epimorphism*, and $\langle S', R' \rangle$ is a *bounded epimorphic image* of $\langle S, R \rangle$. Bounded morphisms are often called *p-morphisms* in the modal literature.

If $\{\mathfrak{S}_j : j \in J\}$ is a collection of structures, with $\mathfrak{S}_j = \langle S_j, R_j \rangle$, then structure $\mathfrak{S} = \langle S, R \rangle$ is the *bounded union of the* $\mathfrak{S}_j$'s if each $\mathfrak{S}_j$ is an inner substructure of $\mathfrak{S}$, and $S = \bigcup\{S_j : j \in J\}$. $\mathfrak{S}'$ is a *disjoint union* of the $\mathfrak{S}_j$'s if it is the union of a collection $\{\mathfrak{S}'_j : j \in J\}$ of pairwise disjoint isomorphic copies of the $\mathfrak{S}_j$'s, i.e. $\mathfrak{S}'_j \cong \mathfrak{S}_j$, and $S'_j \cap S'_i = \emptyset$ when $j \neq i \in J$. Then each $\mathfrak{S}'_j$ is an inner substructure of $\mathfrak{S}'$, so $\mathfrak{S}'$ is the bounded union of the $\mathfrak{S}'_j$'s.

In order to handle these constructions more conveniently, we introduce some notation for operations on a class $\mathcal{K}$ of structures:

$\mathbb{S}\,\mathcal{K}$	=	the class of isomorphic images of inner substructures of members of $\mathcal{K}$.
$\mathbb{H}\mathcal{K}$	=	the class of bounded epimorphic images of members of $\mathcal{K}$.
$\mathbb{U}\mathsf{d}\,\mathcal{K}$	=	the class of disjoint unions of collections of structures isomorphic to members of $\mathcal{K}$.
$\mathbb{U}\mathsf{b}\mathcal{K}$	=	the class of bounded unions of collections of structures isomorphic to members of $\mathcal{K}$.
$\mathbb{P}\mathsf{u}\,\mathcal{K}$	=	the class of isomorphic images of ultraproducts of collections of structures in $\mathcal{K}$.
$\mathbb{P}\mathsf{w}\,\mathcal{K}$	=	the class of isomorphic images of ultrapowers of structures in $\mathcal{K}$.
$\mathbb{R}\mathsf{u}\,\mathcal{K}$	=	the class of structures $\mathfrak{S}$ having some ultrapower $\mathfrak{S}^J/U$ isomorphic to a member of $\mathcal{K}$. This is the class of *ultraroots* of members of $\mathcal{K}$.

There are many relationships between constructions that can be expressed in this operator notation [7]. For instance the fact that a disjoint union of structures is also a bounded union of (isomorphic copies of) them can be expressed by the observation that $\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{U}\mathsf{b}\,\mathcal{K}$ for any class $\mathcal{K}$.

As a partial converse to this, observe that if $\mathfrak{S}$ is the bounded union of the $\mathfrak{S}_j$'s, and $\mathfrak{S}'$ is their disjoint union, then the isomorphisms $\mathfrak{S}'_j \cong \mathfrak{S}_j$ combine to give a function $\mathfrak{S}' \to \mathfrak{S}$ which is a bounded epimorphism. Thus *a bounded union of structures is a bounded epimorphic image of their disjoint union,* giving

$$\mathbb{U}\mathsf{b}\,\mathcal{K} \subseteq \mathbb{H}\mathbb{U}\mathsf{d}\,\mathcal{K}. \tag{1}$$

A particularly important fact for us is that in general ultraproducts commute with bounded unions:

$$\mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathcal{K} \subseteq \mathbb{U}\mathsf{b}\,\mathbb{P}\mathsf{u}\,\mathcal{K}.$$

A proof of this can be found in [3, Lemma 11.1.2] or [7, Theorem 2.4]. Combined with the above observations it yields

$$\mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\,\mathcal{K}. \tag{2}$$

The three constructions of inner substructures, bounded epimorphic images and disjoint unions all preserve modal validity. In other words, the class $Str\,\varphi$ of structures validating φ is closed under $\mathbb{S}$, $\mathbb{H}$, and $\mathbb{U}\mathsf{d}$.[2] Hence by (1) it is closed under $\mathbb{U}\mathsf{b}$ as well. Validity is also preserved by ultraroots (see [4, Theorem 1.16.2] or [7, Theorem 2.1(10)]), so altogether

$$\mathbb{S}\,Str\,\varphi = \mathbb{H}Str\;\varphi = \mathbb{U}\mathsf{d}\,Str\;\varphi = \mathbb{R}\mathsf{u}\,Str\;\varphi = Str\;\varphi.$$

Truth of first-order sentences is preserved by the operations $\mathbb{P}\mathsf{u}$, $\mathbb{P}\mathsf{w}$, and $\mathbb{R}\mathsf{u}$. In fact a class $\mathcal{K}$ is elementary (i.e. of the form $Mod\,\Sigma$) iff it is closed under ultraproducts and ultraroots: $\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathbb{R}\mathsf{u}\,\mathcal{K} = \mathcal{K}$.

5 Operations on Canonical Structures

At the end of Section 3 it was noted that the canonical model $\mathcal{M}^{\Lambda}_{\kappa}$ determines the modal logic Λ_κ. There is a close relationship between this model and any other κ-model $\mathcal{M}$ that determines Λ_κ. It was shown in [2] that if an elementary extension $\mathcal{M}'$ of $\mathcal{M}$ is sufficiently saturated, then its underlying structure has a bounded epimorphism onto $\mathfrak{S}^{\Lambda}_{\kappa}$. Now such an extension $\mathcal{M}'$ can be constructed as an ultrapower, and so we have the following result.

Lemma 5.1 *If a κ-model $\mathcal{M} = \langle \mathfrak{S}, v\rangle$ determines Λ_κ, then $\mathfrak{S}^{\Lambda}_{\kappa}$ is a bounded epimorphic image of an ultrapower $\mathfrak{S}^J/U$ of $\mathfrak{S}$.*

Proof. We briefly explain the construction. If $\mathfrak{S}^J/U = \langle S^J/U, R_U\rangle$ is any ultrapower of $\mathfrak{S}$, define a κ-model $\mathcal{M}_U = \langle \mathfrak{S}^J/U, v_U\rangle$ by declaring

$$h/U \in v_U(p_\lambda) \quad \text{iff} \quad \{j \in J : h(j) \in v(p_\lambda)\} \in U$$

for all $h/U \in S^J/U$ and $\lambda < \kappa$. Then it can be shown that for any κ-formula φ,

$$h/U \in \mathcal{M}_U(\varphi) \quad \text{iff} \quad \{j \in J : h(j) \in \mathcal{M}(\varphi)\} \in U.$$

From this it follows that any κ-formula true in $\mathcal{M}$ must be true in $\mathcal{M}_U$, and hence in particular that all Λ-theorems are true in $\mathcal{M}_U$. Then for each element s of $\mathfrak{S}^J/U$, the set of κ-formulae

$$f(s) = \{\varphi : s \in \mathcal{M}_U(\varphi)\}$$

[2] This is shown in a number of places, including sections 1.4–1.6 of [4] and section 3.3 of [1]. An algebraic version is given in Theorem 3.7.2(2) of [5].

is maximally Λ-consistent in Fma_κ, and so is an element of $\mathfrak{S}^\Lambda_\kappa$. This construction defines a function $f : \mathfrak{S}^J/U \to \mathfrak{S}^\Lambda_\kappa$, which proves to be a bounded epimorphism if $\mathfrak{S}^J/U$ is ω-saturated. For details of this see Section 3.6 of [5] or Section 11.2 of [3]. $\square$

From this we obtain an important result about the way in which canonical structures can be built from members of a class that determines their logic:

Theorem 5.2 *Let $\mathcal{K}$ be a class of structures that determines the logic Λ. Then $\mathfrak{S}^\Lambda_\kappa \in \mathbb{HP}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$ for any cardinal κ. Moreover, if $\mathcal{K}$ is closed under ultraproducts, then $\mathfrak{S}^\Lambda_\kappa \in \mathbb{HU}\mathsf{d}\,\mathcal{K}$ and Λ is valid in $\mathfrak{S}^\Lambda_\kappa$.*

Proof. Let $\{\varphi_j : j \in J\}$ be the set of all κ-formulae that are not Λ-theorems. Since $\mathcal{K}$ determines Λ, each φ_j is invalid in some member of $\mathcal{K}$, so there is a model $\mathcal{M}_j = \langle \mathfrak{S}_j, v_j \rangle$ with $\mathfrak{S}_j \in \mathcal{K}$ and φ_j not true in $\mathcal{M}_j$. Let $\mathcal{M} = \langle \mathfrak{S}, v \rangle$ be the disjoint union of the models $\mathcal{M}_j$ for all $j \in J$. $\mathcal{M}$ falsifies each non-Λ-theorem, while $\mathfrak{S}$ validates Λ because each $\mathfrak{S}_j$ does. Thus $\mathcal{M}$ determines Λ. By Lemma 5.1 there is an ultrapower $\mathfrak{S}^J/U$ of $\mathfrak{S}$ having $\mathfrak{S}^\Lambda_\kappa$ as a bounded epimorphic image. But $\mathfrak{S}$ is the disjoint union of $\{\mathfrak{S}_j : j \in J\} \subseteq \mathcal{K}$, so $\mathfrak{S} \in \mathbb{U}\mathsf{d}\,\mathcal{K}$, hence $\mathfrak{S}^J/U \in \mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$ and so $\mathfrak{S}^\Lambda_\kappa \in \mathbb{HP}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$ as required.

Now suppose $\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathcal{K}$. Then using result (2) from Section 4 and the fact that $\mathbb{HH} = \mathbb{H}$, we get

$$\mathbb{HP}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{HP}\mathsf{u}\,\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{HHU}\mathsf{d}\,\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathbb{HU}\mathsf{d}\,\mathcal{K},$$

so that indeed $\mathfrak{S}^\Lambda_\kappa \in \mathbb{HU}\mathsf{d}\,\mathcal{K}$. But all members of $\mathcal{K}$ validate Λ, and validity is preserved by $\mathbb{H}$ and $\mathbb{U}\mathsf{d}$, so Λ is validated by all members of $\mathbb{HU}\mathsf{d}\,\mathcal{K}$, including now $\mathfrak{S}^\Lambda_\kappa$. $\square$

Next we consider structural relationships between different canonical structures. First, a simple fact that will be used several times.

Lemma 5.3 *Let Λ be a logic in Fma or in some Fma_κ, and s a set of formulae. If $\Lambda \subseteq s$ and $\{\varphi : \Box\varphi \in s\} \subseteq t$, then $\Lambda \subseteq t$.*

Proof. If $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$ by the Necessitation rule, so $\Box\varphi \in s$ and hence $\varphi \in t$. $\square$

Theorem 5.4 *If Λ^* is a sublogic of Λ, i.e. $\Lambda^* \subseteq \Lambda$, then $\mathfrak{S}^\Lambda_\kappa$ is an inner substructure of $\mathfrak{S}^{\Lambda^*}_\kappa$.*

Proof. A set that is maximally Λ-consistent in Fma_κ must also be maximally Λ^*-consistent in Fma_κ[3]. Thus S^Λ_κ is a subset of $S^{\Lambda^*}_\kappa$ and R^Λ_κ is the restriction of $R^{\Lambda^*}_\kappa$ to S^Λ_κ.

If s belongs to S^Λ_κ and $sR^{\Lambda^*}_\kappa t$, then $\Lambda_\kappa \subseteq s$ and so Lemma 5.3 gives $\Lambda_\kappa \subseteq t$. This shows that t is maximally Λ-consistent in Fma_κ, i.e. $t \in S^\Lambda_\kappa$. Therefore S^Λ_κ is $R^{\Lambda^*}_\kappa$-closed. $\square$

Theorem 5.5 *If $\kappa < \mu$, then $\mathfrak{S}^\Lambda_\kappa$ is a bounded epimorphic image of $\mathfrak{S}^\Lambda_\mu$.*

Proof. We have $Fma_\kappa \subseteq Fma_\mu$. If s is a set of μ-formulae, put $f(s) = s \cap Fma_\kappa$. If s is maximally Λ-consistent in Fma_μ, then $f(s)$ will be maximally Λ-consistent in Fma_κ. But if u is maximally Λ-consistent in Fma_κ, then it is Λ-consistent in Fma_μ and so extends to a maximally Λ-consistent set s in Fma_μ with $s \cap Fma_\kappa = u$. Hence $f : S^\Lambda_\mu \to S^\Lambda_\kappa$ is surjective.

If $sR^\Lambda_\mu t$, then $\Box\varphi \in s \cap Fma_\kappa$ implies $\varphi \in t \cap Fma_\kappa$, so $f(s)R^\Lambda_\kappa f(t)$. Finally, suppose $s \in S^\Lambda_\mu$ and $f(s)R^\Lambda_\kappa u$ in S^Λ_κ. Let

$$t_0 = \{\varphi \in Fma_\mu : \Box\varphi \in s\} \cup u.$$

Now if t_0 is not Λ-consistent, then since the two sets that make up t_0 are each closed under conjunction there would be formulae φ, ψ with $\Box\varphi \in s$, $\psi \in u$, and $(\varphi \to \neg\psi) \in \Lambda$. Then $(\Box\varphi \to \Box\neg\psi) \in \Lambda_\mu \subseteq s$, so $\Box\neg\psi \in s$ as $\Box\varphi \in s$. But $\Box\neg\psi$ is a κ-formula, so it belongs to $f(s)$, and hence $\neg\psi \in u$ as $f(s)R^\Lambda_\kappa u$. Since $\psi \in u$, this contradicts the Λ-consistency of u. Therefore t_0 must be Λ-consistent, and so extends to a set $t \in S^\Lambda_\mu$ which includes u, whence $f(t) = u$, and has $sR^\Lambda_\mu t$. $\square$

6 Quasi-Modal $\mathcal{L}$-Sentences

An $\mathcal{L}$-sentence will be called *quasi-modal* if it has the syntactic form $\forall x\rho$, with ρ being an $\mathcal{L}$-formula that is constructed from amongst atomic formulae and the constants $\bot$ (False) and $\top$ (True) using at most the connectives $\wedge$ (conjunction), $\vee$ (disjunction), and the *bounded* universal and existential quantifier forms $\forall z(yRz \to \tau)$ and $\exists z(yRz \wedge \tau)$ with $y \neq z$.

Any quasi-modal sentence σ is preserved by $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{U}\mathsf{d}$ (and hence by $\mathbb{U}\mathsf{b}$ as well). Conversely, any $\mathcal{L}$-sentences that is preserved by those three

[3]Here it is useful to know that s is maximally Λ-consistent in Fma_κ iff every Λ_κ-theorem is in s, and for each κ-formula φ, exactly one of $\varphi, \neg\varphi$ is in s.

operations is logically equivalent to a quasi-modal sentence. More generally, if the class of models of a *set* Σ of $\mathcal{L}$-sentences is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{U}\mathsf{d}$, then $Mod\,\Sigma = Mod\,\Sigma^*$ for some set Σ^* of quasi-modal sentences. A proof of this may be found in [5, Section 4].

For a class $\mathcal{K}$ of structures, let $\Psi^{\mathcal{K}}$ be the *quasi-modal theory of* $\mathcal{K}$, i.e. the set of all quasi-modal $\mathcal{L}$-sentences that are true in all members of $\mathcal{K}$. Then the class $Mod\Psi^{\mathcal{K}}$ of all models of $\Psi^{\mathcal{K}}$ is an elementary class including $\mathcal{K}$. When $\mathcal{K}$ itself is elementary, the members of $Mod\Psi^{\mathcal{K}}$ can be constructed from $\mathcal{K}$ by using operations that preserve validity of modal formulae. In fact $\mathcal{K}$ only needs to be closed under ultraproducts for this to be so. The result, which is proven in [7, Section 7], is that

$$\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathcal{K} \quad \text{implies} \quad Mod\Psi^{\mathcal{K}} = \mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{H}\mathbb{S}\,\mathcal{K}. \tag{3}$$

Now let Ψ^{Λ}_{κ} be the set of all quasi-modal $\mathcal{L}$-sentences that are true in the canonical structure $\mathfrak{S}^{\Lambda}_{\kappa}$. Our goal is to show that all canonical structures for Λ have the same quasi-modal theory, i.e. $\Psi^{\Lambda}_{\kappa} = \Psi^{\Lambda}_{\mu}$ for all cardinals κ, μ. The following result will be needed for this.

Theorem 6.1 *Let Λ be a logic that is determined by a class $\mathcal{K}$ of structures that is closed under ultraproducts. Then for all κ, Λ is determined by the class $Mod\Psi^{\Lambda}_{\kappa}$ of all models of Ψ^{Λ}_{κ}.*

Proof. Suppose $\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathcal{K}$ and $\mathcal{K}$ determines Λ. All members of $\mathcal{K}$ validate Λ, and validity is preserved by $\mathbb{R}\mathsf{u}$, $\mathbb{U}\mathsf{b}$, $\mathbb{H}$ and $\mathbb{S}$, so by (3) any Λ-theorem is valid in all members of $Mod\Psi^{\mathcal{K}}$. By Theorem 5.2, $\mathfrak{S}^{\Lambda}_{\kappa} \in \mathbb{H}\mathbb{U}\mathsf{d}\,\mathcal{K}$, so any quasi-modal sentence true throughout $\mathcal{K}$ will be true in $\mathfrak{S}^{\Lambda}_{\kappa}$, i.e. $\Psi^{\mathcal{K}} \subseteq \Psi^{\Lambda}_{\kappa}$. Thus $Mod\Psi^{\Lambda}_{\kappa} \subseteq Mod\Psi^{\mathcal{K}}$, and so any Λ-theorem is valid in all members of $Mod\Psi^{\Lambda}_{\kappa}$. But by definition $\mathfrak{S}^{\Lambda}_{\kappa} \in Mod\Psi^{\Lambda}_{\kappa}$, so any formula valid in all members of $Mod\Psi^{\Lambda}_{\kappa}$ will be valid in $\mathfrak{S}^{\Lambda}_{\kappa}$ and hence a Λ-theorem. $\square$

7 The Largest Elementary Sublogic

A logic Λ will be called *elementary* if it is determined by the elementary class $Mod\,\Sigma$ of all models of some set Σ of $\mathcal{L}$-sentences. Theorem 6.1 implies that an elementary logic Λ is determined by the elementary class $Mod\Psi^{\Lambda}_{\kappa}$ for any κ. We will show that these classes are all equal, and that in general they give the best elementary approximation to a logic.

Theorem 7.1 *For any logic Λ, the logic Λ^e determined by $Mod\Psi_\omega^\Lambda$ is the largest sublogic of Λ that is elementary.*

Proof. Λ^e is elementary by definition. Since $\mathfrak{S}_\omega^\Lambda$ is a Ψ_ω^Λ-model, any Λ^e-theorem is valid in $\mathfrak{S}_\omega^\Lambda$, hence is a Λ-theorem, so $\Lambda^e \subseteq \Lambda$.

Now let Λ^* be any sublogic of Λ that is elementary. We want to show $\Lambda^* \subseteq \Lambda^e$. Since $\mathfrak{S}_\omega^\Lambda$ is an inner substructure of $\mathfrak{S}_\omega^{\Lambda^*}$ (Theorem 5.4), every quasi-modal sentence true in $\mathfrak{S}_\omega^{\Lambda^*}$ is true in $\mathfrak{S}_\omega^\Lambda$, which means that $\Psi_\omega^{\Lambda^*}$ is included in Ψ_ω^Λ. Therefore

$$Mod\Psi_\omega^\Lambda \subseteq Mod\Psi_\omega^{\Lambda^*}.$$

But by Theorem 6.1, since Λ^* is elementary it is determined by $Mod\Psi_\omega^{\Lambda^*}$, so Λ^* is valid in all members of $Mod\Psi_\omega^\Lambda$, hence is included in Λ^e. $\square$

Theorem 7.2 *For any logic Λ, the quasi-modal $\mathcal{L}$-theory of any of its canonical structures $\mathfrak{S}_\kappa^\Lambda$ is equal to that of $\mathfrak{S}_\omega^\Lambda$, i.e. $\Psi_\kappa^\Lambda = \Psi_\omega^\Lambda$, and hence $Mod\Psi_\kappa^\Lambda = Mod\Psi_\omega^\Lambda$.*

Proof. Firstly, $\mathfrak{S}_\omega^\Lambda$ is a bounded epimorphic image of $\mathfrak{S}_\kappa^\Lambda$ (Theorem 5.5), so quasi-modal sentences true in $\mathfrak{S}_\kappa^\Lambda$ are true in $\mathfrak{S}_\omega^\Lambda$, i.e. $\Psi_\kappa^\Lambda \subseteq \Psi_\omega^\Lambda$.

For the converse, since $Mod\Psi_\omega^\Lambda$ determines Λ^e, Theorem 5.2 gives

$$\mathfrak{S}_\kappa^{\Lambda^e} \in \mathbb{H}\mathbb{U}\mathsf{d}\, Mod\Psi_\omega^\Lambda = Mod\Psi_\omega^\Lambda \tag{4}$$

(Ψ_ω^Λ being preserved by $\mathbb{H}$ and $\mathbb{U}\mathsf{d}$). But $\mathfrak{S}_\kappa^\Lambda$ is an inner substructure of $\mathfrak{S}_\kappa^{\Lambda^e}$ (Theorem 5.4), so from (4) we get

$$\mathfrak{S}_\kappa^\Lambda \in \mathbb{S}\, Mod\Psi_\omega^\Lambda = Mod\Psi_\omega^\Lambda.$$

Thus $\mathfrak{S}_\kappa^\Lambda$ is a model of Ψ_ω^Λ, implying $\Psi_\omega^\Lambda \subseteq \Psi_\kappa^\Lambda$. $\square$

Corollary 7.3 *The canonical structures of the logic Λ^e all have the same quasi-modal $\mathcal{L}$-theory as those of Λ, i.e. $\Psi_\kappa^{\Lambda^e} = \Psi_\omega^\Lambda$ for all κ.*

Proof. By result (4) above, $\mathfrak{S}_\kappa^{\Lambda^e}$ is a model of Ψ_ω^Λ, implying $\Psi_\omega^\Lambda \subseteq \Psi_\kappa^{\Lambda^e}$. Conversely, since $\mathfrak{S}_\kappa^\Lambda$ is an inner substructure of $\mathfrak{S}_\kappa^{\Lambda^e}$, we get $\Psi_\kappa^{\Lambda^e} \subseteq \Psi_\kappa^\Lambda$. But by the Theorem, $\Psi_\kappa^\Lambda = \Psi_\omega^\Lambda$. $\square$

$Mod\,\Psi_\omega^\Lambda$ is by no means the only elementary class to determine Λ^e. In fact any class $\mathcal{K}$ satisfying

$$\mathfrak{S}_\omega^{\Lambda^e} \in \mathcal{K} \subseteq Mod\,\Psi_\omega^\Lambda$$

determines Λ^e, and amongst these is any class of the form $Mod\,\Sigma$ with Σ being a set of $\mathcal{L}$-sentences that includes Ψ_ω^Λ and has $\mathfrak{S}_\omega^{\Lambda^e}$ as a model. Some notable example of such Σ are:

1. The set of all $\mathcal{L}$-sentences true in $\mathfrak{S}_\omega^{\Lambda^e}$. In this case $Mod\,\Sigma$ is the class of structures elementarily equivalent to $\mathfrak{S}_\omega^{\Lambda^e}$.

2. The set of all $\mathbb{S}$-sentences true in $\mathfrak{S}_\omega^{\Lambda^e}$. An $\mathbb{S}$-sentence is any $\mathcal{L}$-sentence that is constructed from amongst $\bot$, $\top$, atomic formulae and negations of atomic formulae by using at most $\wedge$, $\vee$, bounded existential quantifiers, and arbitrary universal quantifiers. These are, up to logical equivalence, precisely the sentences that are preserved by inner substructures [5, 4.2.5(4)].

3. The set of all $\mathbb{H}$-sentences true in $\mathfrak{S}_\omega^{\Lambda^e}$, these being constructed from amongst $\bot$, $\top$, and atomic formulae by at most $\wedge$, $\vee$, and arbitrary universal and existential quantifiers. They are, up to logical equivalence, precisely the sentences that are preserved by bounded epimorphic images. [5, 4.2.5(5)].

8 The Largest Canonical Sublogic

A logic Λ is *canonical* if it is valid in all its canonical structures. Theorem 5.2 states that being determined by a $\mathbb{P}\mathsf{u}$-closed class is enough to make Λ valid in all $\mathfrak{S}_\kappa^\Lambda$'s. In particular, *every elementary logic is canonical.*

Theorem 8.1 *For any logic Λ, there is a largest sublogic Λ^c of Λ that is canonical. Moreover, Λ^e is a sublogic of Λ^c.*

Proof. Let $\{\Lambda^j : j \in J\}$ be the collection of all canonical sublogics of Λ. For each κ, define the structure $\mathfrak{S}_\kappa = \langle S_\kappa, R_\kappa\rangle$ to be the intersection of the canonical structures $\mathfrak{S}_\kappa^{\Lambda^j}$ for all $j \in J$. Thus

$$S_\kappa = \bigcap\{S_\kappa^{\Lambda^j} : j \in J\}$$

and $sR_\kappa t$ iff $s, t \in S_\kappa$ and $\{\varphi : \Box\varphi \in s\} \subseteq t$. Define Λ^c to be the logic determined by the class $\{\mathfrak{S}_\kappa : \kappa \text{ an infinite cardinal}\}$.

Now $\mathfrak{S}_\kappa^\Lambda$ is an inner substructure of each $\mathfrak{S}_\kappa^{\Lambda^j}$ (as $\Lambda^j \subseteq \Lambda$), and so is an inner substructure of $\mathfrak{S}_\kappa$. Thus any formula valid in all $\mathfrak{S}_\kappa$'s will be valid in all $\mathfrak{S}_\kappa^\Lambda$'s, giving $\Lambda^c \subseteq \Lambda$.

Next we observe that for any given $j \in J$, each $\mathfrak{S}_\kappa$ is an inner substructure of $\mathfrak{S}_\kappa^{\Lambda^j}$, ensuring similarly that Λ^j is a sublogic of Λ^c. To see this, let s be an element of S_κ and suppose that $sR_\kappa^{\Lambda^j}t$ in $\mathfrak{S}_\kappa^{\Lambda^j}$, i.e. $\{\varphi : \Box\varphi \in s\} \subseteq t$. Then for each $i \in J$ we have $s \in S_\kappa^{\Lambda^i}$, so $\Lambda_\kappa^i \subseteq s$, hence $\Lambda_\kappa^i \subseteq t$ by Lemma 5.3, implying $t \in S_\kappa^{\Lambda^i}$. Since this holds for all $i \in J$ it follows that $t \in S_\kappa$, showing that S_κ is $R_\kappa^{\Lambda^j}$-closed as desired.

It remains to prove that Λ^c is canonical. We show that each of its canonical structures $\mathfrak{S}_\kappa^{\Lambda^c}$ is an inner substructure of $\mathfrak{S}_\kappa$, so formulae valid in $\mathfrak{S}_\kappa$ are valid in $\mathfrak{S}_\kappa^{\Lambda^c}$, showing that $\mathfrak{S}_\kappa^{\Lambda^c}$ does indeed validate Λ^c. Firstly, if s belongs to $S_\kappa^{\Lambda^c}$ then $\Lambda_\kappa^c \subseteq s$, so for any $j \in J$, $\Lambda_\kappa^j \subseteq s$ as Λ^j is a sublogic of Λ^c, making $s \in S_\kappa^{\Lambda^j}$. Thus $s \in S_\kappa$, establishing that $\mathfrak{S}_\kappa^{\Lambda^c}$ is a substructure of $\mathfrak{S}_\kappa$. Finally, $S_\kappa^{\Lambda^c}$ is R_κ-closed for the same reason as in the previous paragraph: if $s \in S_\kappa^{\Lambda^c}$ and $sR_\kappa t$, then $\Lambda_\kappa^c \subseteq t$ by Lemma 5.3, implying $t \in S_\kappa^{\Lambda^c}$.

This completes the proof that Λ^c is the largest canonical sublogic of Λ. But the sublogic Λ^e is elementary, and therefore canonical, so it is equal to one of the Λ^j's and hence is included in Λ^c. $\Box$

Theorem 8.2 *The canonical structures of the logic Λ^c all have the same quasi-modal $\mathcal{L}$-theory as those of Λ, i.e. $\Psi_\kappa^{\Lambda^c} = \Psi_\omega^\Lambda$ for all κ.*

Proof. Since $\Lambda^e \subseteq \Lambda^c \subseteq \Lambda$, $\mathfrak{S}_\kappa^\Lambda$ is an inner substructure of $\mathfrak{S}_\kappa^{\Lambda^c}$, and $\mathfrak{S}_\kappa^{\Lambda^c}$ is an inner substructure of $\mathfrak{S}_\kappa^{\Lambda^e}$. Preservation of quasi-modal sentences by $\mathbb{S}$ then implies $\Psi_\kappa^{\Lambda^e} \subseteq \Psi_\kappa^{\Lambda^c} \subseteq \Psi_\kappa^\Lambda$. But by 7.2 and 7.3 we have $\Psi_\kappa^{\Lambda^e} = \Psi_\kappa^\Lambda = \Psi_\omega^\Lambda$, so altogether, for all κ,

$$\Psi_\kappa^{\Lambda^e} = \Psi_\kappa^{\Lambda^c} = \Psi_\kappa^\Lambda = \Psi_\omega^\Lambda.$$

$\Box$

9 Finite Λ-Structures and $Mod\Psi_\omega^\Lambda$

We conclude with some results that further demonstrate the centrality of Ψ_ω^Λ to the model theory of Λ. Let Fin_Λ be the class of finite Λ-structures, i.e. finite structures that validate Λ. Write Ψ_{fin}^Λ for the set of quasi-modal sentences that are true in all members of Fin_Λ.

It will be shown that if a quasi-modal sentence is true in the canonical structure $\mathfrak{S}^{\Lambda}_{\omega}$, then it is true in all finite Λ-structures; and that the converse holds if Λ is determined by Fin_{Λ}.

Theorem 9.1 *Every finite Λ-structure is a model of Ψ^{Λ}_{ω}, and so $\Psi^{\Lambda}_{\omega} \subseteq \Psi^{\Lambda}_{fin}$.*

Proof. Take $\mathfrak{S} = \langle S, R \rangle \in Fin_{\Lambda}$, in order to show $\mathfrak{S} \in Mod\Psi^{\Lambda}_{\omega}$. For each element s of $\mathfrak{S}$, let $\mathfrak{S}_s$ be the inner substructure of $\mathfrak{S}$ generated by s. This is the substructure based on the set $\{s' \in S : sR^*s'\}$, where R^* is the reflexive transitive closure of R. Now $\mathfrak{S}$ is the bounded union of all these $\mathfrak{S}_s$'s, and $Mod\Psi^{\Lambda}_{\omega}$ is closed under bounded unions, so it suffices to show that each $\mathfrak{S}_s$ is in $Mod\Psi^{\Lambda}_{\omega}$.

Let the elements of $\mathfrak{S}_s$ be $s = s_0, \dots, s_n$ for some finite n. Let $\mathcal{M}$ be any model on $\mathfrak{S}_s$ having $\mathcal{M}(p_j) = \{s_j\}$ for all $j \leq n$, and put

$$t = \{\varphi \in Fma_{\omega} : s \in \mathcal{M}(\varphi)\},$$

the set of all ω-formulae that are true in $\mathcal{M}$ at s. Then since $\mathfrak{S}_s$ validates Λ, t is a maximally Λ-consistent subset of Fma_{ω}, i.e. a point in the canonical structure $\mathfrak{S}^{\Lambda}_{\omega}$. Let $\mathfrak{T}$ be the inner substructure of $\mathfrak{S}^{\Lambda}_{\omega}$ generated by t. We will show that there is a bounded epimorphism f from $\mathfrak{T}$ onto $\mathfrak{S}_s$. Since $\mathfrak{S}^{\Lambda}_{\omega}$ is a model of Ψ^{Λ}_{ω}, preservation of Ψ^{Λ}_{ω} under $\mathbb{S}$ and $\mathbb{H}$ guarantees that $\mathfrak{T}$ and then $\mathfrak{S}_s$ is a model of Ψ^{Λ}_{ω} as desired.

The following modal formulae are true at s in $\mathcal{M}$, and hence belong to t. (Here $\Box^m$ denotes a sequence of $\Box$'s of length m.)

$$\begin{array}{ll} \Box^m(p_0 \vee \cdots \vee p_n) & \text{for all } m < \omega, \\ \Box^m\neg(p_j \wedge p_k) & \text{for all } m < \omega \text{ and all } 0 \leq j \neq k \leq n. \end{array}$$

Using these formulae it can be shown that each member of $\mathfrak{T}$ contains exactly one of the variables $p_0, \dots, p_n$. Also, for each $j \leq n$, there is some m_j that the formula $\Diamond^{m_j} p_j$ is true at s, hence a member of t, ensuring that p_j belongs to some member of $\mathfrak{T}$. Thus putting

$$f(u) = s_j \quad \text{iff} \quad p_j \in u$$

gives a well-defined function from $\mathfrak{T}$ onto $\mathfrak{S}_s$. With the help of further formulae true at s it is then seen that f is a bounded morphism, as desired. Formulae that suffice for this are:

$$\begin{array}{ll} \Box^m(p_j \to \Box\neg p_k) & \text{for all } m < \omega, \text{ if not } s_jRs_k, \\ \Box^m(p_j \to \Diamond p_k) & \text{for all } m < \omega, \text{ if } s_jRs_k. \end{array}$$

$\Box$

Corollary 9.2 *If Λ is an elementary logic, then it is valid in any ultraproduct of finite Λ-structures.*

Proof. The Theorem shows that $Fin_\Lambda \subseteq Mod\Psi_\omega^\Lambda$. Thus $\mathbb{P}\mathsf{u}Fin_\Lambda \subseteq Mod\Psi_\omega^\Lambda$. But if Λ is elementary, then by Theorem 6.1, every Λ-theorem is valid in all members of $Mod\Psi_\omega^\Lambda$, hence in all members of $\mathbb{P}\mathsf{u}Fin_\Lambda$. □

Since every elementary logic is canonical, one way to show that a logic is not elementary would be to show that it is not canonical. But this strategy is obviously not available if we want to try to show that some *canonical* logic is not elementary. Corollary 9.2 gives a possible strategy that is independent of canonicity: in order to show that no elementary class whatsoever could determine Λ it is enough to exhibit a particular set of finite structures that validate Λ and an ultraproduct of them that does not.

Theorem 9.3 *Suppose that Λ is determined by Fin_Λ. Then a quasi-modal sentence that is true in all finite Λ-structures must be true in $\mathfrak{S}_\omega^\Lambda$, and so $\Psi_{fin}^\Lambda = \Psi_\omega^\Lambda$.*

Proof.
If Fin_Λ determines Λ, then by Theorem 5.2

$$\begin{aligned} \mathfrak{S}_\omega^\Lambda &\in \mathbb{H}\mathbb{P}\mathsf{w}\mathbb{U}\mathsf{d}\,Fin_\Lambda \\ &\subseteq \mathbb{H}\mathbb{P}\mathsf{w}\mathbb{U}\mathsf{d}\,Mod\Psi_{fin}^\Lambda \\ &= Mod\Psi_{fin}^\Lambda. \end{aligned}$$

Thus $\mathfrak{S}_\omega^\Lambda$ is a model of Ψ_{fin}^Λ, implying $\Psi_{fin}^\Lambda \subseteq \Psi_\omega^\Lambda$. Equality follows by 9.1. □

Corollary 9.4 *If an elementary logic Λ is determined by Fin_Λ, then it is determined by $Mod\Psi_{fin}^\Lambda$.*

Proof. $Mod\Psi_{fin}^\Lambda = Mod\Psi_\omega^\Lambda$, and $Mod\Psi_\omega^\Lambda$ determines Λ. □

References

[1] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic.* Oxford University Press, 1997.

[2] K. Fine. Some Connections between Elementary and Modal Logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North-Holland, 1975.

[3] Robert Goldblatt. Elementary Logics are Canonical and Pseudo-Equational. In [6], pages 243–257.

[4] Robert Goldblatt. Metamathematics of Modal Logic. In [6], pages 9–79.

[5] Robert Goldblatt. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, 44:173–242, 1989.

[6] Robert Goldblatt. *Mathematics of Modality.* CSLI Lecture Notes No. 43. Center for the Study of Language and Information, Stanford, California, 1993. Distributed by Cambridge University Press.

[7] Robert Goldblatt. Elementary Generation and Canonicity for Varieties of Boolean Algebras with Operators. *Algebra Universalis*, 34:551–607, 1995.

[8] Timothy J. Surendonk. *Canonicity for Intensional Logics.* PhD thesis, Australian National University, Canberra, September 1998.

[9] Timothy J. Surendonk. On Isomorphisms Between Canonical Frames. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic, Volume 1*, CSLI Lecture Notes No. 87, pages 249–268. CSLI Publications, Stanford, California, 1998. See also Chapter 9 of [8].