Mereocompactness and duality for mereotopological spaces

Robert Goldblatt and Matt Grice

Abstract Mereotopology studies relations between regions of space, including the contact relation. It leads to an abstract notion of Boolean contact algebra which has been shown to be representable as an algebra of regular closed subsets of a compact topological space. Here we define mereotopological spaces and their mereomorphisms, and construct a dual equivalence between the category of Boolean contact algebras and a category of mereotopological spaces that have a property we call mereocompactness, strictly stronger than ordinary compactness.

This is a further illustration of the kind of duality that has been widely used in the semantic analysis of propositional logics, and which has been a significant theme in the research of J. Michael Dunn.

Keywords: Boolean algebra, clan, compact, contact relation, duality, mereocompact, mereotopology, regular closed, ultrafilter

Introduction

Duality in the semantic analysis of propositional logics has been a significant theme in the research of J. Michael Dunn. It is involved in his gaggle theory, whose development motivated the construction of a new topological duality for general lattices (Hartonas and Dunn, 1997). It underlies the framework of a number of topics he has worked on, including: the representation of quasi-Boolean algebras (Dunn, 1982) and positive modal algebras (Dunn, 1995); the modelling of negation using information states (Dunn, 1993); the representation of relation algebras over Routley–

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Meyer structural models for relevant logics (Dunn, 2001); the relational semantics for linear logic and other substructural logics (Allwein and Dunn, 1993; Dunn et al., 2005). There are chapters on duality in his two most recent books (Dunn and Hardegree, 2001; Bimbó and Dunn, 2008).

This notion of duality is a certain relationship between two kinds of model: *algebraic* and *structural*. In an algebraic model, propositional formulas denote elements of an algebra whose fundamental operations interpret logical connectives. In a structural model, formulas denote subsets of some background set that carries relational and/or topological structure. The elements of the set are viewed as possible worlds/situations/information states/temporal instants etc., and certain of its subsets are taken to be propositions. The structure gives rise to connective-interpreting operations on propositions, so a structural model S has an associated algebra S^+ of propositions. S^+ is the *dual* of S.

In the opposite direction, representation theorems are applied to show that an algebra A has a *dual structure* A_+ such that A is isomorphic to the algebra $(A_+)^+$ of propositions of A_+. Topological properties may be used to characterise the propositions of A_+ and to identify which structures are (isomorphic to) the duals of algebras, or equivalently which structures S are isomorphic to their double dual structure $(S^+)_+$. For example, the duals of Boolean algebras are the Stone spaces, and the propositions of such spaces are the clopen (closed-and-open) subsets. The duals of distributive lattices can be described as the spectral spaces, with their propositions being the compact open subsets; or as the Priestley spaces, with clopen down-sets as propositions.

Ultimately, duality is a category-theoretic notion, taking the form of a pair of contravariant functors that constitute a *dual equivalence* between a category of algebras and a category of structures.[1]

The purpose of the present paper is to add another brick to the pyramid of ideas on duality, in the context of *mereotopology*. This is an approach to the abstract geometry of space, based on *regions* rather than points, in which there is a primitive relation of *contact* between regions. It originates in philosophical work in the early 20th century by de Laguna (1922) on postulates for a "can connect" relation between "solids," and by Whitehead (1929) on an "extensive connection" relation between regions. Its name derives from the word "mereology," devised by Leśniewski in the 1920's to refer to his theory of the part-whole relation. In more recent times the study of such theories has received impetus from theoretical computer science, since they provide a framework for qualitative spatial reasoning, as embodied in the Region Connection Calculus (Randell and Cohn, 1989; Randell et al., 1992) that was built on an axiomatisation of Whitehead's theory by Clarke (1981).[2]

[1] For categories $\mathbf{A}$ of algebras and $\mathbf{S}$ of structures, if the assignments $A \mapsto A_+$ and $S \mapsto S^+$ extend to contravariant functors from $\mathbf{A}$ to $\mathbf{S}$ and vice versa, then this constitutes a dual equivalence when the composition of these functors in either order gives functors that are naturally isomorphic to the identity functors on $\mathbf{A}$ and $\mathbf{S}$, respectively. This perspective on models of propositional logic was introduced (for modal algebras and Kripke frames) in the first author's thesis (Goldblatt, 1974). The concept of natural isomorphism is explained in the present article at the end of Section 3.

[2] Further work on the Region Connection Calculus is surveyed in (Cohn et al., 1997).

In a topological interpretation, regions can be taken to be *regular closed* subsets of a topological space, with such sets being in contact if they have a non-empty intersection. Alternatively, regions may be *regular open* sets, in contact when their closures intersect. These two kinds of regular sets form Boolean algebras that have distinct operations but are isomorphic.

Düntsch and Winter (2005) studied an axiomatic notion of Boolean contact algebra and showed that such an algebra could be represented by embedding it into the regular-closed-subset algebra of a T_1 topological space. Dimov and Vakarelov (2006a) extended this by dropping an extensionality requirement on the contact relation and representing the resulting algebras in spaces that are T_0, compact and semi-regular (the latter meaning that the regular closed sets form a closed basis for the topology).[3] They gave an example in (Dimov and Vakarelov, 2006b) to show that those three topological conditions do not suffice to characterise the dual spaces of contact algebras.

Our aim here is to lift these results to a full categorical duality, making Boolean contact algebras into a category and identifying a suitable dual category of "mereotopological" spaces with "mereomorphisms" between them. To characterise the duals of contact algebras we define a new notion of *mereocompactness* (see Section 4) which is strictly stronger than ordinary topological compactness.

We will work through the details of this programme according to the following steps, which serve as a summary of the paper.

- Define the category **BCA** of Boolean contact algebras whose morphisms are the Boolean homomorphisms that *reflect contact* (or equivalently, preserve non-tangential inclusion).
- Define a mereotopological space S as an ordinary topological space with a distinguished Boolean sub-algebra of its regular-closed-set algebra that is a closed basis for the topology, and which forms the dual contact algebra S^+ of S.
- Define the category **MS** of mereotopological spaces, whose *mereomorphisms* $\theta\colon S_1 \to S_2$ are functions whose pullback action provides a Boolean algebra homomorphism $S_2^+ \to S_1^+$.
- Construct a contravariant functor $\Phi\colon \mathbf{MS} \to \mathbf{BCA}$ having $\Phi(S) = S^+$.
- Adapt the representation theory of (Düntsch and Winter, 2005; Dimov and Vakarelov, 2006a) to associate with each contact algebra A a T_0 mereotopological space A_+ such that A is isomorphic in **BCA** to $(A_+)^+$.
- Define *mereocompactness* for a mereotopological space. Show that the dual space A_+ of any contact algebra is mereocompact, and that an arbitrary space S is isomorphic in **MS** to its double dual $(S^+)_+$ iff it is mereocompact and T_0.
- Construct a contravariant functor $\Theta\colon \mathbf{BCA} \to \mathbf{MS*}$ having $\Theta(A) = A_+$, where **MS*** is the category of mereocompact T_0 spaces.
- Show that the categories **BCA** and **MS*** are dually equivalent, by showing that $\boldsymbol{\Phi*} \circ \Theta$ is naturally isomorphic to the identity functor on **BCA**, while $\Theta \circ \boldsymbol{\Phi*}$ is

[3] See the Introductions of papers (Düntsch and Winter, 2005; Dimov and Vakarelov, 2006a) for an overview of the background literature on region-based theories of space.

naturally isomorphic to the identity functor on **MS***, where $\Phi^*\colon \mathbf{MS^*} \to \mathbf{BCA}$ is the restriction of Φ to **MS***.

In the final section we explore alternative versions and consequences of the notion of mereocompactness.

1 Contact Algebras

We use the notation $(B,+,\cdot,-,0,1)$ for an abstract Boolean algebra on a set B, with operations $+$ of join, $\cdot$ of meet and $-$ of complement; and least element 0 and greatest element 1 under the partial ordering $\leq$ that has $x \leq y$ iff $x+y=y$ iff $x\cdot y=x$. We may denote this algebra by its underlying set B.

A *contact relation* on a Boolean algebra is a binary relation C on B satisfying the following axioms.

C1. xCy implies $x,y \neq 0$.
C2. xCy implies yCx.
C3. $xC(y+z)$ iff xCy or xCz.
C4. $x \neq 0$ implies xCx.

Such a C is always *monotonic* in each variable: if xCy, $x \leq x'$ and $y \leq y'$, then $x'Cy'$. Each Boolean algebra has a smallest contact relation $\{\,(a,b)\colon a\cdot b \neq 0\,\}$ and a largest one $\{\,(a,b)\colon a \neq 0 \neq b\,\}$.

A *Boolean contact algebra*, or BCA, is a pair $A=(B_A,C_A)$ with C_A a contact relation on Boolean algebra B_A. We may also denote such an algebra in the form $A_i=(B_i,C_i)$ where i is some suitable label, or as $A'=(B',C')$ etc.

We define a *homomorphism* $f\colon A \to A'$ of contact algebras, or *BCA-morphism*, to be a homomorphism $f\colon B \to B'$ of Boolean algebras such that, for all $x,y \in B$,

$$f(x)C'f(y) \text{ implies } xCy.$$

Thus a BCA-morphism *reflects contact*. Equivalently it preserves separation in the sense that if elements are not in contact in A, then their f-images are not in contact in A'.

A relation $\ll$ of *non-tangential inclusion* is defined on any contact algebra by putting $x \ll y$ iff not $xC(-y)$. A BCA-morphism can then be characterised as a Boolean algebra homomorphism that preserves non-tangential inclusion in the sense that

$$x \ll y \text{ implies } f(x) \ll' f(y).$$

The axioms (C1)–(C4) for a contact relation can be equivalently formulated entirely as properties of the relation $\ll$ (Dimov and Vakarelov, 2006a, p. 214).

It is readily seen that the functional composition of two BCA-morphisms is a BCA-morphism, and that the identity function on a contact algebra is a BCA-morphism. Thus the Boolean contact algebras and their morphisms are the objects and arrows of a concrete category, which we denote **BCA**.

Category theory gives us a definition of an *isomorphism* of contact algebras: a BCA-morphism $f\colon A \to A'$ is an isomorphism when there exists a BCA-morphism $g\colon A' \to A$, the *inverse* of f, such that each of the compositions $g \circ f$ and $f \circ g$ is the identity morphism on its domain.

Theorem 1.1. *A BCA-morphism $f\colon A \to A'$ is an isomorphism if, and only if, it is bijective and preserves contact in the sense that xCy implies $f(x)C'f(y)$.*

Proof. Let $f\colon A \to A'$ be an isomorphism, with inverse BCA-morphism $g\colon A' \to A$ as above. Since $g \circ f$ and $f \circ g$ are identity functions it follows that f is bijective. If xCy, then since $x = g(f(x))$ and $y = g(f(y))$, it follows that $f(x)C'f(y)$ as g reflects contact.

Conversely, suppose $f\colon A \to A'$ is a bijective BCA-morphism preserving contact. As a bijection, f has an inverse $g\colon B_{A'} \to B_A$. It is a fact of universal algebra that *the inverse of a bijective homomorphism of algebras is itself a homomorphism.*[4] So in this case g is a Boolean algebra homomorphism. If $g(x)C_A g(y)$, then $f(g(x))C_{A'}f(g(y))$ as f preserves contact, hence $xC_{A'}y$. This shows that g reflects contact and so is a BCA-morphism $A' \to A$, providing the inverse in **BCA** that ensures f is an isomorphism. □

2 Mereotopological spaces

Let (X,τ) be a topological space, comprising a topology τ on set X. We typically denote the space just as X. Let cl_X and int_X be the closure and interior operators induced on subsets of X by its topology. A subset a of X is *regular closed* if it is equal to the closure of its interior: $a = \mathrm{cl}_X(\mathrm{int}_X(a))$. The set $\mathrm{RC}(X)$ of all regular closed subsets of X forms a Boolean algebra in which $a+b = a\cup b$, $a\cdot b = \mathrm{cl}_X(\mathrm{int}_X(a\cap b))$, $-a = \mathrm{cl}_X(X\backslash a)$, $0 = \emptyset$ and $1 = X$.

There is a contact relation C_X on $\mathrm{RC}(X)$ defined by putting aC_Xb iff $a\cap b \neq \emptyset$.[5] Thus $(\mathrm{RC}(X), C_X)$ is a Boolean contact algebra in which 'in contact' means to have a non-empty intersection. The non-tangential inclusion relation on this algebra has $a \ll b$ iff $a \subseteq \mathrm{int}(b)$.

By a *mereotopological space* we mean a pair $S = (X_S, M_S)$ where X_S is a topological space and M_S is a subalgebra of the Boolean algebra $\mathrm{RC}(X_S)$ of regular closed subsets of X_S, such that M_S is a closed basis for X_S. This last condition on M_S means that every closed subset of X_S is an intersection of a collection of members of M_S. That M_S is a subalgebra of $\mathrm{RC}(X_S)$ means that it is closed under the Boolean algebra operations of $\mathrm{RC}(X_S)$, and hence is itself a Boolean algebra under these operations.

Remark 2.1. Mereotopology can equivalently be approached from the point of view of sets a that are *regular open* in the sense that $a = \mathrm{int}(\mathrm{cl}(a))$. A set is regular open

[4] Henkin et al., 1971, 0.2.9.

[5] Note that $a\cdot b \subseteq a\cap b$ for regular closed a and b, so $a\cap b \neq \emptyset$ is a weaker assertion here than $a\cdot b \neq 0$.

iff its complement $X\backslash a$ is regular closed. The set $\mathrm{RO}(X)$ of regular open subsets of X is a Boolean algebra in which $a+b = \mathrm{int}(\mathrm{cl}(a \cup b))$, $a \cdot b = a \cap b$ and $-a = \mathrm{int}(X\backslash a)$. Its natural contact relation D_X has aD_Xb iff $\mathrm{cl}(a) \cap \mathrm{cl}(b) \neq \emptyset$ iff there is a point that is *close to* both a and b. The map $a \mapsto \mathrm{cl}(a)$ is a **BCA**-isomorphism from $(\mathrm{RO}(X), D_X)$ onto $(\mathrm{RC}(X), C_X)$ (Dimov and Vakarelov, 2006a, Example 2.1). Pratt-Hartmann (2007, Definition 2.5) defines a *mereotopology* over a topological space X as a Boolean sub-algebra of $\mathrm{RO}(X)$ that is a basis for the topology.

A *semiregular* topological space is one that has a basis of regular open sets, or equivalently has a closed basis of regular closed sets. For instance, the real line $\mathbb{R}$ with its standard topology is semi-regular since its intervals (x,y) are regular open and form a basis. If a space X is semiregular, then $(X, \mathrm{RC}(X))$ is a mereotopological space as defined here.

Contact algebras, especially those of the form $(\mathrm{RC}(X), C_X)$, can be used to model logics based on propositional languages with a binary connective C and possibly other connectives, including modalities. Work in this direction can be found in (Lutz and Wolter, 2006; Kontchakov et al., 2008; Nenov and Vakarelov, 2008; Vakarelov, 2007) as well as in some chapters of the *Handbook of Spatial Logic* (Aiello et al., 2007).

Now we define mereomorphisms. If $S_1 = (X_1, M_1)$ and $S_2 = (X_2, M_2)$ are mereotopological spaces, a *mereomorphism* $\theta : S_1 \to S_2$ is a function $\theta : X_1 \to X_2$ whose pullback action on members of M_2 is a Boolean algebra homomorphism from M_2 to M_1. This means that for each subset $a \subseteq X_2$ with $a \in M_2$, the pre-image $\theta^{-1}(a) = \{x \in X_1 : \theta(x) \in a\}$ belongs to M_1, and the map $M_2 \to M_1$ acting by $a \mapsto \theta^{-1}(a)$ is a Boolean algebra homomorphism.

Lemma 2.1. *Every mereomorphism is continuous.*

Proof. Let θ be a mereomorphism as above, and b a closed subset of X_2. Then $b = \bigcap_{i \in I} a_i$ for some $a_i \in M_2$, since M_2 is a closed basis for the space X_2. So $\theta^{-1}b = \bigcap_{i \in I} \theta^{-1}a_i$, with each $\theta^{-1}a_i$ belonging to M_1 and hence being (regular) closed in X_1. Therefore $\theta^{-1}b$ is closed.

This shows that under $\theta : X_1 \to X_2$, pre-images of closed sets are closed, implying that θ is continuous. $\square$

Remark 2.2. The map $a \mapsto \theta^{-1}(a)$ always preserves Boolean joins (=unions), so for it to be a Boolean homomorphism it is sufficient that it preserve Boolean complements: $\theta^{-1}(-a) = -\theta^{-1}(a)$. But for $\theta : X_1 \to X_2$ to be a mereomorphism, it is not sufficient in general that it be continuous and pull back members of M_2 to members of M_1, as we show next.

Example 2.3. Let S be the mereotopological space $(\mathbb{R}, \mathrm{RC}(\mathbb{R}))$, where $\mathbb{R}$ is the real line with its standard topology. Let θ be the constant function having $\theta(x) = 0$ for all $x \in \mathbb{R}$. Then $\theta : \mathbb{R} \to \mathbb{R}$ is continuous, and if $a \in \mathrm{RC}(\mathbb{R})$ then $\theta^{-1}(a)$ is either $\mathbb{R}$ or $\emptyset$ accordingly as $0 \in a$ or not, so $\theta^{-1}(a) \in \mathrm{RC}(\mathbb{R})$.

However θ is not a mereomorphism, because the map $a \mapsto \theta^{-1}(a)$ does not preserve the Boolean complement operation on $\mathrm{RC}(\mathbb{R})$. For example, let a be the regular closed interval $[0,\infty)$. Then $-a = (-\infty,0]$, so $\theta^{-1}(-a) = \mathbb{R}$. But also $\theta^{-1}(a) = \mathbb{R}$, so $-\theta^{-1}(a) = \emptyset \neq \theta^{-1}(-a)$.

Observe also that the map $a \mapsto \theta^{-1}(a)$ does not preserve Boolean meets. With $a = [0,\infty)$ as above we have $\theta^{-1}(a \cdot -a) = \theta^{-1}(\emptyset) = \emptyset$, whereas $\theta^{-1}(a) \cdot \theta^{-1}(-a) = \mathbb{R} \cdot \mathbb{R} = \mathbb{R}$.

The identity function on a mereotopological space is a mereomorphism, and the functional composition of two mereomorphisms is a mereomorphism. Thus we have a category **MS** of mereotopological spaces and mereomorphisms.

In the next result we use the notation $\theta[a]$ for the direct image $\{\,\theta(x) \colon x \in a\,\}$ of a subset a of the domain of θ. If θ is a bijection with inverse σ, then $\theta[a] = \sigma^{-1}(a)$.

Theorem 2.2. *A mereomorphism $\theta \colon S_1 \to S_2$ is an isomorphism in the category **MS** if, and only if, it is a bijection that has $\theta[a] \in M_2$ for all $a \in M_1$.*

Proof. Let θ be an isomorphism. This means that there is an inverse mereomorphism $\sigma \colon S_2 \to S_1$ such that each of the compositions $\sigma \circ \theta$ and $\theta \circ \sigma$ is the identity morphism on its domain. The existence of σ ensures that θ is a bijection. For each $a \in M_1$ we have $\theta[a] = \sigma^{-1}(a) \in M_2$ as σ is a mereomorphism.

Conversely, assume θ is a bijective mereomorphism having $\theta[a] \in M_2$ for all $a \in M_1$. As a bijection, θ has an inverse function $\sigma \colon X_2 \to X_1$. For each $a \in M_1$ we have $\sigma^{-1}(a) = \theta[a] \in M_2$, showing that the map $a \mapsto \sigma^{-1}(a)$ pulls members of M_1 back to members of M_2. Since θ is a mereomorphism, the map $b \mapsto \theta^{-1}(b)$ is a Boolean algebra homomorphism from M_2 to M_1. But this map is bijective, with inverse $a \mapsto \sigma^{-1}(a)$, since $\sigma^{-1}(\theta^{-1}(b)) = b$ and $\theta^{-1}(\sigma^{-1}(a)) = a$. As the inverse of a bijective homomorphism of algebras is itself a homomorphism, it follows that $a \mapsto \sigma^{-1}(a)$ is a Boolean algebra homomorphism.

This shows that σ is a mereomorphism $S_2 \to S_1$ and provides the inverse in **MS** that ensures θ is an isomorphism. $\square$

An isomorphism in **MS** might well be called a *mereo-isomorphism*. By Lemma 2.1 such a map is a *homeomorphism*, i.e., is a continuous bijection with a continuous inverse. However, a mereomorphism that is a homeomorphism need not be a mereo-isomorphism:

Example 2.4. Let $\mathbb{Q}$ be the set of rational numbers, and for each $p,q \in \mathbb{Q}$ with $p < q$, let a_{pq} be the regular closed subset $(-\infty,p] \cup [q,\infty)$ of $\mathbb{R}$. Put $M_0 = \{\, a_{pq} \colon p,q \in \mathbb{Q}\,\}$, and let M be the Boolean subalgebra of $\mathrm{RC}(\mathbb{R})$ generated by M_0. Since M_0 is countable, so too is M, and therefore M is a proper subset of $\mathrm{RC}(\mathbb{R})$.

Now every open subset of $\mathbb{R}$ is a union of open intervals (p,q) with rational endpoints, so every closed subset is an intersection of members of M_0. Thus M is a closed basis for the standard topology on $\mathbb{R}$. Hence $(\mathbb{R}, \mathrm{RC}(\mathbb{R}))$ and $(\mathbb{R}, M)$ are distinct mereotopological spaces based on the same topological space $\mathbb{R}$.

Let θ be the identity function on $\mathbb{R}$. Then θ is a homeomorphism, and is a mereomorphism because the map $a \mapsto \theta^{-1}(a)$ is the inclusion homomorphism of M into $\mathrm{RC}(\mathbb{R})$. However θ is not a mereo-isomorphism, by Theorem 2.2, as there are (uncountably many) elements $a \in \mathrm{RC}(\mathbb{R}))$ such that $\theta[a] = a \notin M$.

We now define the dual of a mereotopological space S by putting $S^+ = (M_S, C_S)$ where C_S is the intersect relation on the Boolean algebra M_S, i.e., aC_Sb iff $a \cap b \neq \emptyset$, for all $a, b \in M_S$. Then S^+ is a Boolean contact algebra.

For each mereomorphism $\theta\colon S_1 \to S_2$, define $\theta^+\colon M_2 \to M_1$ by putting $\theta^+(a) = \theta^{-1}(a)$. The definition of mereomorphism ensures that $\theta^+(a) \in M_1$ for all $a \in M_2$, and that θ^+ is a Boolean algebra homomorphism. Moreover, if $\theta^+(a)C_{S_1}\theta^+(b)$, then $\theta^{-1}(a) \cap \theta^{-1}(b) \neq \emptyset$, hence $a \cap b \neq \emptyset$ and so $aC_{S_2}b$. Thus θ^+ reflects contact as well, making it a BCA-morphism $S_2^+ \to S_1^+$.

Now given a pair of composable mereomorphisms

$$S_1 \xrightarrow{\theta_1} S_2 \xrightarrow{\theta_2} S_3,$$

we obtain the composable BCA-morphisms

$$S_1^+ \xleftarrow{\theta_1^+} S_2^+ \xleftarrow{\theta_2^+} S_3^+,$$

for which it can be shown that $\theta_1^+ \circ \theta_2^+ = (\theta_2 \circ \theta_1)^+$ (because $\theta_1^{-1} \circ \theta_2^{-1} = (\theta_2 \circ \theta_1)^{-1}$). Also, if θ is the identity mereomorphism on S, i.e., the identity function on X_S, then θ^+ is the identity function on M_S, hence is the identity BCA-morphism on S^+.

Thus the assignments $\Phi(S) = S^+$ and $\Phi(\theta) = \theta^+$ form a *contravariant functor* $\Phi\colon \mathbf{MS} \to \mathbf{BCA}$ from the category of mereotopological spaces to the category of Boolean contact algebras.

Our next task is to construct a functor in the opposite direction.

3 Representation by clans

If $A = (B_A, C_A)$ is a Boolean contact algebra, then a *clan* of A is a non-empty subset Γ of B_A such that:

K1. $0 \notin \Gamma$.
K2. $x \in \Gamma$ and $x \leq y$ implies $y \in \Gamma$.
K3. $x + y \in \Gamma$ implies $x \in \Gamma$ or $y \in \Gamma$.
K4. $x, y \in \Gamma$ implies xCy.

A non-empty Γ satisfying K1–K3 is called a *grill*.[6] So a clan is a grill for which any two members are in contact. It is readily seen that any ultrafilter of B_A is a clan (Dimov and Vakarelov, 2006a). For, it is standard that an ultrafilter satisfies K1–K3, so is a grill. For K4, if x and y belong to an ultrafilter, then $x \cdot y \neq 0$, so $x \cdot y C_A x \cdot y$ by C4. Since $x \cdot y \leq x, y$ and C_A is monotonic in each variable, it follows that xC_Ay.

Let X_A be the set of all clans of A. For each $x \in B_A$ let $f_A(x) = \{\Gamma \in X_A : x \in \Gamma\}$. The function f_A is injective, for if $x \neq y$, then say $x \not\leq y$, so there is an ultrafilter U

[6] Grills originate with Choquet (1947) and clans with Thron (1973).

of B_A that contains x but not y. Then U is a clan belonging to $f_A(x)$ but not $f_A(y)$, showing that $f_A(x) \neq f_A(y)$.

Let $M_A = \{ f_A(x) \colon x \in B_A \}$. Now as Lemma 5.1(i) of (Dimov and Vakarelov, 2006a) states, f_A has the properties $f_A(0) = \emptyset$, $f_A(1) = X_A$, $f_A(x+y) = f_A(x) + f_A(y)$. So M_A contains $\emptyset$ and X_A and is closed under finite unions. This is enough to ensure that M_A is a closed basis for a topology on X_A whose closed subsets of X_A are the intersections of collections of members of M_A. We now view X_A as a space under this topology.

It is proved in (Dimov and Vakarelov, 2006a) that X_A is compact and T_0. We will show in Section 4 below that compactness follows from our stronger mereocompactness property (see Theorem 4.2). The T_0 separation property is that for any pair of distinct points there is an open neighbourhood of one that excludes the other. To show this for X_A, let Γ and Δ be distinct clans of A. Then there is an element of one that does not belong to the other, say $x \in \Gamma$ and $x \notin \Delta$. Then $X_A \setminus f_A(x)$ is an open neighbourhood of Δ that excludes Γ.

By Lemma 5.3(ii) of (Dimov and Vakarelov, 2006a), each set $f_A(x)$ is regular closed in the space X_A, so f_A maps B_A into the Boolean algebra $\mathrm{RC}(X_A)$. Moreover, Lemma 5.3(i) of (Dimov and Vakarelov, 2006a) shows that $f_A(-x) = \mathrm{cl}_{X_A}(X_A \setminus f_A(x)) = -f_A(x)$ in $\mathrm{RC}(X_A)$. So together with its above listed properties, we see that f_A is a Boolean algebra homomorphism into $\mathrm{RC}(X_A)$, making its image M_A a subalgebra of $\mathrm{RC}(X_A)$.

Thus the structure $A_+ = (X_A, M_A)$ is a mereotopological space. This is the dual space of the algebra A.

Theorem 3.1. *A is isomorphic to the contact algebra $(A_+)^+$ in the category* ***BCA***.

Proof. By definition, $(A_+)^+ = (M_A, C_{A_+})$, where C_{A_+} is the intersect relation on the Boolean set algebra M_A. We have already observed that f_A is an injective Boolean algebra homomorphism, and it maps B_A onto M_A. By (Dimov and Vakarelov, 2006a, Proposition 3.3(i)) we have that xC_Ay iff there is a clan Γ of A with $x, y \in \Gamma$, which is equivalent to $f_A(x) \cap f_A(y) \neq \emptyset$, i.e., to $f_A(x)\, C_{A_+} f_A(y)$. Hence f_A preserves and reflects contact.

Altogether this shows that $f_A \colon B_A \to M_A$ is a bijective BCA-morphism preserving contact, so is an isomorphism from A to $(A_+)^+$ by Theorem 1.1. □

Now for any BCA-morphism $f \colon A \to A'$, define a function f_+ on $X_{A'}$ by putting, for each clan Γ of A', $f_+(\Gamma) = f^{-1}(\Gamma) = \{ x \in B_A \colon f(x) \in \Gamma \}$.

Theorem 3.2. *f_+ is a mereomorphism from A'_+ to A_+.*

Proof. First we need that f_+ is a function from $X_{A'}$ to X_A, i.e., that $f^{-1}(\Gamma)$ is a clan of A when Γ is a clan of A'. First, $f^{-1}\Gamma$ is non-empty because $f(1_A) = 1_{A'} \in \Gamma$ and so $1_A \in f^{-1}\Gamma$. Next, the grill properties K1–K3 lift from Γ to $f^{-1}\Gamma$ because f preserves least elements, the partial orders $\leq$, and joins. For K4, if $x, y \in f^{-1}\Gamma$, then $f(x)C'f(y)$ as Γ has K4, hence xCy as f reflects contact. Thus $f^{-1}(\Gamma)$ is indeed a clan.

Also we require that pulling back along f_+ gives a Boolean homomorphism from M_A to $M_{A'}$. For an arbitrary element $f_A(x)$ of M_A, we have, for any $\Gamma \in X_{A'}$, that $\Gamma \in (f_+)^{-1}(f_A(x))$ iff $x \in f^{-1}(\Gamma)$ iff $\Gamma \in f_{A'}(f(x))$. This shows that for any $x \in B_A$,

$$(f_+)^{-1}(f_A(x)) = f_{A'}(f(x)), \tag{1}$$

confirming that $(f_+)^{-1}$ maps M_A into $M_{A'}$. Then equation (1) and the fact that f, f_A and $f_{A'}$ are all Boolean homomorphisms allow us to infer that $(f_+)^{-1}$ preserves Boolean complements, because

$$(f_+)^{-1}(-f_A(x)) = (f_+)^{-1}(f_A(-x)) = f_{A'}(f(-x)) = -f_{A'}(f(x)) = -(f_+)^{-1}(f_A(x)).$$

As already noted in Remark 2.2, that suffices to ensure that $(f_+)^{-1}$ is a Boolean homomorphism. $\square$

Now given a pair of composable BCA-morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3,$$

we obtain the composable mereomorphisms

$$A_{1+} \xleftarrow{f_{1+}} A_{2+} \xleftarrow{f_{2+}} A_{3+},$$

for which it can be shown that $f_{1+} \circ f_{2+} = (f_2 \circ f_1)_+$. Also, if f is the identity BCA-morphism on A, then f_+ is the identity mereomorphism on A_+.

Thus the assignments $\Theta(A) = A_+$ and $\Theta(f) = f_+$ form a contravariant functor $\Theta\colon \mathbf{BCA} \to \mathbf{MS}$. For any BCA-morphism $f\colon A \to A'$, equation (1) implies that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[d, "f"'] & (A_+)^+ = \Phi(\Theta(A)) \arrow[d, "(f_+)^+ = \Phi(\Theta(f))"] \\
A' \arrow[r, "f_{A'}"] & (A'_+)^+ = \Phi(\Theta(A'))
\end{tikzcd}$$

This means, by definition, that the assignment $A \mapsto f_A$ for all BCA's A constitutes a *natural transformation* from the identity functor on **BCA** to the functor $\Phi \circ \Theta\colon \mathbf{BCA} \to \mathbf{BCA}$ that assigns to each contact algebra A its "double dual" $(A_+)^+$. The morphisms f_A are the *components* of this natural transformation. In general, a natural transformation is called a *natural isomorphism* if its components are isomorphisms (i.e., invertible morphisms) in their ambient category. Thus in our present situation, as the components f_A are all mereo-isomorphisms (Theorem 3.1), it follows that $\Phi \circ \Theta$ is *naturally isomorphic* to the identity functor.[7]

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[7] See (Mac Lane, 1998, I.4) for the theory of natural transformations and isomorphisms.

4 Mereocompactness

The functor $\Theta \circ \Phi \colon \mathbf{MS} \to \mathbf{MS}$ is not naturally isomorphic to the identity functor on $\mathbf{MS}$, because a mereotopological space S need not be isomorphic to its double dual $(S^+)_+$. For instance, $(S^+)_+$ can be of higher cardinality that S. As an example, let $S = (X, \mathrm{RC}(X))$ where X is a discrete space of any infinite cardinality κ. Then $\mathrm{RC}(X)$ is the powerset algebra of X, of cardinality 2^κ, having 2^{2^κ} ultrafilters. Since ultrafilters are clans, it follows that $(S^+)_+$ is of cardinality 2^{2^κ}.

Note that the question of whether S is isomorphic to its double dual is equivalent to the question of whether it is isomorphic to the dual of something. For if $S \cong A_+$, then $S^+ \cong (A_+)^+ \cong A$ (Theorem 3.1), and so $(S^+)_+ \cong A_+ \cong S$.

We now explore conditions under which a space $S = (X_S, M_S)$ *is* isomorphic to $(S^+)_+$. For each $x \in X_S$, define $\rho_S(x) = \{\, a \in M_S \colon x \in a \,\}$.

Theorem 4.1. (1) *$\rho_S(x)$ is a clan of the algebra $S^+ = (M_S, C_S)$, hence a point of the space X_{S^+}.*

(2) *$\rho_S \colon X_S \to X_{S^+}$ is a mereomorphism $S \to (S^+)_+$.*

(3) *ρ_S is injective if, and only if, X_S is T_0.*

(4) *For any mereomorphism $\theta \colon S \to S'$, the following diagram commutes:*

(2)
$$\begin{tikzcd} S \arrow[r, "\rho_S"] \arrow[d, "\theta"'] & (S^+)_+ \arrow[d, "(\theta^+)_+"] \\ S' \arrow[r, "\rho_{S'}"] & (S'^+)_+ \end{tikzcd}$$

Proof. (1) That $\rho_S(x)$ satisfies K1–K3 is routine. For K4 recall that C_S is the intersect relation, and note that if $a_1, a_2 \in \rho_S(x)$, then $x \in a_1 \cap a_2$, so $a_1 C_S a_2$.

(2) First we need to have ρ_S^{-1} pulling back members of $M_{S^+} = \{\, f_{S^+}(a) \colon a \in M_S \,\}$ to members of M_S. But for any $a \in M_S$ we have

(3)
$$\rho_S^{-1}(f_{S^+}(a)) = a,$$

hence $\rho_S^{-1}(f_{S^+}(a)) \in M_S$ as required. Equation (3) holds since $x \in \rho_S^{-1}(f_{S^+}(a))$ iff $\rho_S(x) \in f_{S^+}(a)$ iff $a \in \rho_S(x)$ iff $x \in a$.

With the help of (3) for $-a$ and for a we reason that

$$\rho_S^{-1}(-f_{S^+}(a)) = \rho_S^{-1}(f_{S^+}(-a)) = -a = -\rho_S^{-1}(f_{S^+}(a))$$

so ρ_S^{-1} preserves Boolean complements. That is enough to make it a Boolean homomorphism (Remark 2.2), completing the proof that ρ_S is a mereomorphism.

(3) Suppose X_S is T_0. Then if $x, y \in X_S$ with $x \neq y$, there is an open set containing one but not the other, hence its complement is a closed set containing one but not the other. Since M_S is a closed basis for X_S there must then be a member a

of M_S containing one but not the other, so a belongs either to $\rho_S(x)\backslash\rho_S(y)$ or to $\rho_S(y)\backslash\rho_S(x)$. In either case $\rho_S(x) \neq \rho_S(y)$, showing ρ_S is injective.
For the converse, let ρ_S be injective. If $x \neq y$, there is some $a \in M_S$ such that a belongs either to $\rho_S(x)\backslash\rho_S(y)$ or to $\rho_S(y)\backslash\rho_S(x)$. Then the complement of a is an open set containing one of x and y but not the other. This shows that distinct points of X_S do not have the same open neighbourhoods, which is the T_0 property.

(4) For each $x \in X_S$ we have $\rho_{S'}(\theta(x)) = \{\, a \in M_{S'} \colon \theta(x) \in a \,\}$, while

$$(\theta^+)_+(\rho_S(x)) = (\theta^+)^{-1}(\rho_S(x)) = \{\, a \in M_{S'} \colon \theta^+(a) \in \rho_S(x) \,\}.$$

But $\theta^+(a) = \theta^{-1}(a) \in \rho_S(x)$ iff $x \in \theta^{-1}(a)$ iff $\theta(x) \in a$. So $\rho_{S'}(\theta(x)) = (\theta^+)_+(\rho_S(x))$ as required for the diagram to commute. $\square$

Now define a *mereocompact space* to be a mereotopological space S satisfying the following property:

For every $\Gamma, \Delta \subseteq M_S$ with Γ a *clan* of S^+, if $\bigcap \Gamma \subseteq \bigcup \Delta$ then $\Gamma \cap \Delta \neq \emptyset$.

Theorem 4.2. (1) *S is mereocompact iff $\rho_S \colon X_S \to X_{S^+}$ is surjective.*
(2) *Every mereocompact space is compact.*
(3) *If A is any Boolean contact algebra, then A_+ is mereocompact.*

Proof. (1) Let S be mereocompact. Take any $\Gamma \in X_{S^+}$, i.e., Γ is a clan of the algebra S^+. Put $\Delta = M_S \backslash \Gamma$. Then $\Gamma \cap \Delta = \emptyset$ so by mereocompactness $\bigcap \Gamma \nsubseteq \bigcup \Delta$. Hence there is some $x \in \bigcap \Gamma \backslash \bigcup \Delta$. Then $\rho_S(x) = \Gamma$. This shows ρ_S is surjective.
Conversely, suppose ρ_S is surjective. Let $\Gamma, \Delta \subseteq M_S$ with Γ a clan, and $\bigcap \Gamma \subseteq \bigcup \Delta$. Then $\Gamma = \rho_S(x)$ for some $x \in S$, hence $x \in \bigcap \Gamma$. Thus there is some $\delta \in \Delta$ with $x \in \delta$. Hence $\delta \in \rho_S(x)$, so $\delta \in \Gamma \cap \Delta \neq \emptyset$. This shows S is mereocompact.

(2) Let S be mereocompact. For compactness of X_S it suffices to show that any collection of closed sets with the *finite intersection property* has non-empty intersection. (Recall that collection M has the finite intersection property if each finite subcollection of M has non-empty intersection). But since M_S is a closed basis for X_S, it is enough to prove this for subcollections of M_S. So take any $M \subseteq M_S$ with the finite intersection property. Then M extends to an ultrafilter U of the powerset algebra $\mathcal{P}(X_S)$ of all subsets of X_S. Let $\Gamma = U \cap M_S$. Then Γ is a clan of S^+: the fact that U is a grill of $\mathcal{P}(X_S)$ ensures that Γ is a grill of M_S, and if $a, b \in \Gamma$, then $a, b \in U$ and so $a \cap b \neq \emptyset$, i.e., aC_Sb.
Now put $\Delta = \emptyset$ in the definition of mereocompactness of S. Since Γ is a clan and $\Gamma \cap \emptyset = \emptyset$, it follows that $\bigcap \Gamma \neq \bigcup \emptyset = \emptyset$. Since $M \subseteq \Gamma$, this implies $\bigcap M \neq \emptyset$ as required.

(3) Recall that $A_+ = (X_A, M_A)$ with X_A the set of clans of A and $M_A = \{\, f_A(x) \colon x \in B_A \,\}$. Take subsets Γ, Δ of M_A with Γ a clan of $(A_+)^+ = (M_A, C_{A_+})$. Put $J = f_A^{-1}\Gamma = \{\, x \in B_A \colon f_A(x) \in \Gamma \,\}$. Then J is a clan of A by the proof of Theorem 3.2, so $J \in X_A$.
Now for any $f_A(x) \in \Gamma$ we have $x \in J$ and so $J \in f_A(x)$. Thus $J \in \bigcap \Gamma$. So if $\bigcap \Gamma \subseteq \bigcup \Delta$ then there is some $f_A(y) \in \Delta$ such that $J \in f_A(y)$. But then $y \in J$,

implying $f_A(y) \in \Gamma$. Hence $f_A(y) \in \Gamma \cap \Delta$, showing that $\Gamma \cap \Delta$ is non-empty. This proves mereocompactness of A_+. $\square$

Mereocompactness is a strictly stronger property than compactness:

Example 4.1. A topological space is said to be *strongly compact* (Rasiowa and Sikorski, 1963, p. 101) if it is not covered by open proper subsets, i.e., if every open cover of the space must include the space itself as a member. Equivalently, this means that the intersection of any set of non-empty closed subsets is non-empty, which is a much stronger condition than compactness. Any topological space X has a *one-point strong compactification* (Rasiowa and Sikorski, 1963, p. 102) obtained by adding a new point π to X and declaring that the open subsets of $X \cup \{\pi\}$ are $X \cup \{\pi\}$ itself and all the open subsets of X. Then the closed subsets of $X \cup \{\pi\}$ are $\emptyset$ and all sets of the form $b \cup \{\pi\}$ with b a closed subset of X. Thus π belongs to every non-empty closed set in $X \cup \{\pi\}$, ensuring strong compactness. The regular closed subsets of $X \cup \{\pi\}$ are $\emptyset$ and all sets $b \cup \{\pi\}$ with b a *non-empty* regular closed subset of X.

Now let X be the three-element set $3 = \{0,1,2\}$ with the discrete topology, and put $S = (X_S, M_S)$ with $X_S = X \cup \{\pi\}$ and

$$M_S = \mathrm{RC}(X \cup \{\pi\})) = \{\emptyset\} \cup \{b \cup \{\pi\} : \emptyset \neq b \subseteq 3\}.$$

$S^+ = (M_S, C_S)$ is an eight-element Boolean contact algebra (see Figure 1) in which

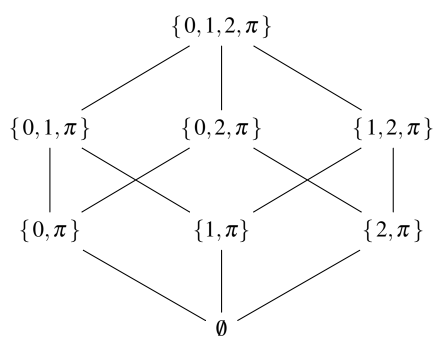

Fig. 1 The algebra $\mathrm{RC}(X \cup \{\pi\})$ with $X = \{0,1,2\}$.

any two non-empty members are C_S-related, i.e., intersect, since they contain π. For each $x \in 3$, the *point-clan* $\rho_S(x) = \{a \in M_S : x \in a\}$ is the principal ultrafilter of S^+ generated by the atom $\{x,\pi\}$, and contains the four elements $\{x,\pi\}$, $\{x,y,\pi\}$, $\{x,z,\pi\}$, $\{x,y,z,\pi\}$, where y,z are the two elements of 3 other than x. $\rho_S(\pi)$ is the seven-element set $\{b \cup \{\pi\} : \emptyset \neq b \subseteq 3\}$.

Let $\Gamma = \rho_S(0) \cup \rho_S(1)$. Then Γ is a clan of S^+ and is in fact the six-element set $M_S \backslash \{\{2,\pi\}, \emptyset\}$. Thus $\Gamma \neq \rho_S(w)$ for all $w \in 3 \cup \{\pi\}$, so the function $\rho_S : X_S \to X_{S^+}$

is not surjective. Hence while X_S is strongly compact, S is not mereocompact by Theorem 4.2 (1).

The idea of this counter-example was prompted by Example 4.2 of (Dimov and Vakarelov, 2006b), which exhibited a compact semiregular T_0 space X whose RC-algebra has a clan that is not equal to any point-clan by taking $X = \mathbb{R} \cup \{\pi\}$ (with a different description of its topology and without the discussion of strong compactness). We make use of the finiteness of our example in the final section below.

Although mereocompactness implies compactness of the underlying topology, it is independent of *strong* compactness. To see this, let S_1 be any mereotopological space containing non-empty regular closed sets a, b that are disjoint. E.g., $S_1 = (\mathbb{R}, \mathrm{RC}(\mathbb{R}))$ with a, b any two disjoint closed intervals. Let $A = S_1^+$. Then a and b are not in contact in A, so A_+ is a mereocompact space in which $f_A(a)$ and $f_A(b)$ are non-empty closed subsets of X_A that are disjoint. Thus A_+ is not strongly compact.

We now establish the conditions under which a space is isomorphic to its double dual:

Theorem 4.3. *The mereomorphism $\rho_S\colon S \to (S^+)_+$ is a mereo-isomorphism if, and only if, S is mereocompact and T_0.*

Proof. By Theorems 4.2 (1) and 4.1 (3), ρ_S is a bijection from X_S onto X_{S^+}, the set of all clans of S^+, iff S is mereocompact and T_0.

Now let S be mereocompact and T_0. To prove that the bijection ρ_S is a mereo-isomorphism, it suffices by Theorem 2.2 to prove that for each $a \in M_S$, the direct image $\rho_S[a]$ belongs to M_{S^+}.

But by Theorem 3.1 with $A = S^+$, the BCA-isomorphism f_{S^+} between S^+ and its double dual maps M_S onto M_{S^+}, with $f_{S^+}(a)$ being the set of all clans of S^+ that contain a. Since ρ_S is surjective, any clan of S^+ is equal to $\rho_S(x)$ for some $x \in X_S$. Thus

$$f_{S^+}(a) = \{\rho_S(x)\colon x \in X_S \quad \text{and} \quad a \in \rho_S(x)\} = \{\rho_S(x)\colon x \in a\} = \rho_S[a].$$

So $\rho_S[a] = f_{S^+}(a) \in M_{S^+}$ as required. □

Now let **MS*** be the full subcategory of **MS** whose objects are the mereocompact T_0 spaces. For each Boolean contact algebra A, the dual space $\Theta(A) = A_+$ is mereocompact and T_0, so we can view Θ as a functor from **BCA** into **MS***. In the opposite direction, let $\Phi^*\colon \mathbf{MS^*} \to \mathbf{BCA}$ be the restriction of functor Φ to **MS***. From our earlier analysis, $\Phi^* \circ \Theta\colon \mathbf{BCA} \to \mathbf{BCA}$ is naturally isomorphic to the identity functor on **BCA**.

The commuting diagram (2) in Theorem 4.1 shows that the mereo-isomorphisms ρ_S for all **MS***-objects S form the components of a natural isomorphism between the identity functor on **MS*** and the functor $\Theta \circ \Phi^*\colon \mathbf{MS^*} \to \mathbf{MS^*}$ that assigns to each mereocompact T_0 space S its double dual $(S^+)_+$. These properties of Θ and Φ^* establish that the category of Boolean contact algebras is *dually equivalent* to the category of mereocompact T_0 spaces. That is the principal result of this paper.

There are further results in the topological representation of BCA's, for instance concerning the notion of an *extensional contact algebra* (ECA) as a BCA satisfying

$$\forall z\,(xCz \ \text{ iff } \ yCz) \text{ implies } x = y.$$

By restricting the points of the representing space to be *maximal* clans, it was shown in (Düntsch and Winter, 2005; Dimov and Vakarelov, 2006a) that any ECA is embeddable into the RC-algebra of a space that is compact, T_1 and weakly regular, the later meaning that the space is semi-regular and any non-empty open set a has a non-empty open subset b with $\mathrm{cl}(b) \subseteq a$. It is left to the interested reader to extend this result to a full duality for ECA's, and to do likewise for other classes of BCA's discussed in the literature.

5 Variations on mereocompactness

We conclude by giving an alternative formulation of mereocompactness, and exploring some consequences that have been used in other duality theories to characterise dual spaces of algebras.

Let the notation $\Gamma \subseteq_{\mathrm{f}} \Gamma'$ mean that Γ is a *finite* subset of Γ'. Consider the property

(μ_0) For every $\Gamma, \Delta \subseteq M_S$ with Γ a *clan* of S^+, if $\bigcap \Gamma \subseteq \bigcup \Delta$ then there exists a $\gamma \in \Gamma$ and a $\Delta_0 \subseteq_{\mathrm{f}} \Delta$ such that $\gamma \subseteq \bigcup \Delta_0$.

(μ_0) is equivalent to mereocompactness. For if $\gamma \in \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$ with $\gamma \subseteq \bigcup \Delta_0$, then $\bigcup \Delta_0 \in \Gamma$ by K2 for Γ, and so by K3 there is some $\delta \in \Delta_0$ with $\delta \in \Gamma$, hence $\delta \in \Gamma \cap \Delta \neq \emptyset$. Conversely, if there is a $\gamma \in \Gamma \cap \Delta$, then $\gamma \subseteq \bigcup \{\gamma\}$ and $\{\gamma\} \subseteq_{\mathrm{f}} \Delta$.

Next consider

(μ_1) For every $\Gamma, \Delta \subseteq M_S$ with $\bigcap \Gamma \subseteq \bigcup \Delta$, there exist sets $\Gamma_0 \subseteq_{\mathrm{f}} \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$ such that $\bigcap \Gamma_0 \subseteq \bigcup \Delta_0$.

This property holds when M_S is the dual algebra of clopen subsets of the Stone space of a Boolean algebra, and is a consequence of, indeed equivalent to, the compactness of that Stone space. μ_1 also holds when M_S is the algebra of compact open subsets of the dual space of a distributive lattice, and has been used as one of the characterising properties of such spaces (Balbes and Dwinger, 1974, Chapter IV).

In our mereotopological setting, μ_1 follows from mereocompactness of S. To see why, suppose that $\bigcap \Gamma_0 \nsubseteq \bigcup \Delta_0$ for all sets $\Gamma_0 \subseteq_{\mathrm{f}} \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$. Then $\Gamma \cup \{X_S \backslash \delta \colon \delta \in \Delta\}$ has the finite intersection property and so extends to an ultrafilter U of the powerset algebra $\mathcal{P}(X_S)$ that includes Γ and is disjoint from Δ. Then $\Gamma' = U \cap M_S$ is a clan of S^+ that includes Γ. But if $\bigcap \Gamma \subseteq \bigcup \Delta$, then $\bigcap \Gamma' \subseteq \bigcap \Gamma \subseteq \bigcup \Delta$, so from mereocompactness of S we infer $\Gamma' \cap \Delta \neq \emptyset$, contradicting $U \cap \Delta = \emptyset$. Hence $\bigcap \Gamma \nsubseteq \bigcup \Delta$, confirming that μ_1 holds for S.

μ_1 is in fact weaker than mereocompactness. It holds trivially whenever M_S is finite, so it holds in the finite space S of Example 4.1, which is not mereocompact.

Now we modify μ_1 to the statement

(μ_2) For every $\Gamma, \Delta \subseteq M_S$ with $\bigcap \Gamma \subseteq \bigcup \Delta$, there exist sets $\Gamma_0 \subseteq_{\mathrm{f}} \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$ such that $\bigwedge \Gamma_0 \subseteq \bigcup \Delta_0$,

where $\bigwedge \Gamma_0 = \mathrm{cl}(\mathrm{int}(\bigcap \Gamma_0))$ is the Boolean meet of Γ_0 in S^+. Since $\bigwedge \Gamma_0 \subseteq \bigcap \Gamma_0$, it is evident that ($\mu_2$) follows from ($\mu_1$).

Property μ_2 is itself equivalent to requiring that

(μ_3) for all ultrafilters Γ of S^+, for all $\Delta \subseteq M_S$, $\bigcap \Gamma \subseteq \bigcup \Delta$ implies $\Gamma \cap \Delta \neq \emptyset$.

Proof. Assume μ_2 and take any $\Gamma, \Delta \subseteq M_S$ such that Γ is an ultrafilter and $\bigcap \Gamma \subseteq \bigcup \Delta$. Then there exist sets $\Gamma_0 \subseteq_{\mathrm{f}} \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$ such that $\bigwedge \Gamma_0 \subseteq \bigcup \Delta_0$. As a filter, Γ is closed under finite meets and closed upwards under $\subseteq$, so then $\bigcup \Delta_0 \in \Gamma$. Since Γ satisfies K3 it follows that $\delta \in \Gamma$ for some $\delta \in \Delta_0$. Hence $\delta \in \Gamma \cap \Delta \neq \emptyset$, proving μ_3.

Conversely assume μ_3, take any $\Gamma, \Delta \subseteq M_S$ and suppose that $\bigwedge \Gamma_0 \nsubseteq \bigcup \Delta_0$ for all $\Gamma_0 \subseteq_{\mathrm{f}} \Gamma$ and $\Delta_0 \subseteq_{\mathrm{f}} \Delta$. Then $\Gamma \cup \{ -\delta \colon \delta \in \Delta \}$ has the *finite meet property*, i.e., each of its finite subsets has non-zero meet in S^+—which means non-empty meet. Hence $\Gamma \cup \{ -\delta \colon \delta \in \Delta \}$ extends to an ultrafilter Γ' of S^+ that is disjoint from Δ. By μ_3, since $\Gamma' \cap \Delta = \emptyset$ we immediately get $\bigcap \Gamma' \nsubseteq \bigcup \Delta$. But $\bigcap \Gamma' \subseteq \bigcap \Gamma$, so then $\bigcap \Gamma \nsubseteq \bigcup \Delta$. This proves μ_2. $\square$

μ_3 is in turn equivalent to the condition

(μ_4) Each ultrafilter of S^+ is equal to $\rho_S(x)$ for some $x \in X$.

The equivalence of μ_3 and μ_4 follows by the same reasoning that shows that mereocompactness is equivalent to the surjectivity of ρ_S (Theorem 4.2).

μ_4 has the immediate consequence that every ultrafilter of S^+ has non-empty intersection. This consequence is weaker than μ_4, because it is also implied by topological compactness whereas μ_4 is not. In fact μ_4 is not even implied by strong compactness.

Example 5.1. We show that μ_4 fails in $S = (X, \mathrm{RC}(X))$ where X is the strong compactification $\mathbb{R} \cup \{ \pi \}$ of $\mathbb{R}$ (see Example 4.1). Hence $\mu_0 - \mu_3$ also fail in this space.

For each real number r, let $a_r = [r, \infty) \cup \{ \pi \} \in \mathrm{RC}(X)$. Put $\Gamma = \{ a_r \colon r \in \mathbb{R} \}$. Then $a_r \subseteq a_s$ iff $s \leq r$, so if $\emptyset \neq \Gamma_0 \subseteq_{\mathrm{f}} \Gamma$, then $\bigwedge \Gamma_0 = a_r \neq \emptyset$ where a_r is the $\subseteq$-least member of Γ_0. Hence Γ has the finite meet property and so extends to an ultrafilter U of S^+.

Now $\bigcap \Gamma = \{ \pi \}$, since $s \notin a_r$ for any real $s < r$. Hence $\bigcap U = \{ \pi \}$. If we had $U = \rho_S(x)$ for some x, then $x \in \bigcap U$ and so $x = \pi$ and $U = \rho_S(\pi)$. But this is impossible as $\rho_S(\pi)$ is $\mathrm{RC}(X) \setminus \{ \emptyset \}$ and is not an ultrafilter. Indeed it is not even a filter since it contains both a_r and its complement $-a_r = (-\infty, r] \cup \{ \pi \}$ but does not contain their meet $a_r \wedge -a_r = \emptyset$. So U violates μ_4.

We can also see directly that U violates μ_3: since π belongs to every non-empty member of $\mathrm{RC}(X)$ we have $\bigcap U \subseteq \bigcup (\mathrm{RC}(X) \setminus U)$ while $U \cap (\mathrm{RC}y(X) \setminus U) = \emptyset$.

It seems possible that μ_2 could be weaker than μ_1. To show this it would suffice to exhibit a mereotopological space S that satisfies any of μ_2–μ_4 and has a subset of M_S with the finite intersection property but empty intersection.

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