Constant Modal Logics and Canonicity

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Abstract

If a modal logic is valid in its canonical frame, and its class of validating frames is invariant under bounded epimorphisms (or equivalently, closed under images of bisimulations), then the logic is axiomatizable by variable-free formulas. Hence its class of frames is first-order definable.

1 Introduction

Every normal propositional modal logic L has a *canonical frame* $\mathcal{F}_L$ whose points are the maximally L-consistent sets of formulas. This construction was introduced by Lemmon and Scott [7], and dubbed 'canonical' by Krister Segerberg, who used it to great effect in proving completeness theorems for a wide range of important intensional logics (see [8, 9, 10, 11] et alia). This contribution to the metatheory of $\mathcal{F}_L$ is written as a tribute to Krister on his 70th birthday.

$\mathcal{F}_L$ has the property that any formula that it validates must be an L-theorem. If the converse holds, i.e. if $\mathcal{F}_L$ validates L, then L is called *canonical.* A closely related notion is that of an *elementarily determined* logic, one that is determined by (i.e. sound and complete for) some collection of Kripke frames that is an *elementary class*, i.e. consists of all models of some set of sentences in the first-order language of a binary relation.

It has been known since the early 1970's that every elementarily determined logic is canonical [1], but only recently was it shown that the two notions are distinct. We now know [5, 4] that there are uncountably many canonical logics that are not elementarily determined. As Ian Hodkinson has observed, this fact is appealing in showing that the world is more complex, with more diverse phenomena, than it would otherwise have been.

But it is also attractive to the mathematician in raising the challenge of discovering new conditions under which a canonical logic does turn out to be elementarily determined. One natural strengthening of canonicity is the property

$$\text{(E):}\quad \mathcal{F}_L \equiv \mathcal{F} \text{ implies } \mathcal{F} \models L,$$

where $\equiv$ is the relation of *elementary equivalence* that holds between two frames when they satisfy the same first-order sentences. Condition (E) implies that L is elementarily determined by the class $\{\mathcal{F} : \mathcal{F}_L \equiv \mathcal{F}\}$ of all models of the first-order theory of $\mathcal{F}_L$. In fact it has been shown by the author [2, Theorem 11.3.1] that the property of L being elementarily determined is *equivalent* to (E). So the examples of [5, 4] show that there are canonical logics for which (E) fails, and the mathematician's question becomes: "*when* does a canonical logic satisfy (E)?"

In this paper we give one answer: when L is *invariant under bounded epimorphisms*. This means that if there is a bounded morphism $\mathcal{F} \twoheadrightarrow \mathcal{G}$ from frame $\mathcal{F}$ onto frame $\mathcal{G}$, then $\mathcal{F}$ validates L if, and only if, $\mathcal{G}$ validates L. Bounded epimorphisms are the frame versions of the *pseudo-epimorphisms* between Kripke models that were introduced by Segerberg [8] (and abbreviated to 'p-morphisms' in [10, 11]). Validity of modal formulas is always *preserved* by bounded epimorphisms – which is why these morphisms have become central to modal model theory. So the invariance just described is equivalent to the requirement that L is *reflected* by bounded epimorphisms, i.e. $\mathcal{G} \models L$ implies $\mathcal{F} \models L$ whenever there exists a bounded epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{G}$. It is also equivalent to L being preserved under *surjective bisimulations*, as we explain later.

Here is a sketch of a proof that every canonical logic that is invariant under bounded epimorphisms must satisfy (E). If $\mathcal{F}_L \equiv \mathcal{F}$, then by the Keisler-Shelah Ultrapower Theorem there exists an ultrafilter U such that the ultrapowers $\mathcal{F}_L/U$ and $\mathcal{F}/U$ are isomorphic. Moreover these ultrapowers can be taken to have a certain saturation, and in that case there exists a natural bounded epimorphism $\mathcal{F}_L/U \twoheadrightarrow \mathcal{F}_L$. Thus if $\mathcal{F}_L \models L$, then $\mathcal{F}_L/U \models L$ as L is invariant under bounded epimorphisms, hence $\mathcal{F}/U \models L$ as $\mathcal{F}_L/U \cong \mathcal{F}/U$, and so $\mathcal{F} \models L$ as modal validity is always preserved by ultraroots.

However, it could be said that this argument uses a bulldozer to fashion a goat track. Keisler's original proof of the Ultrapower Theorem depended on the generalized continuum hypothesis, and Shelah's proof without GCH was a tour de force requiring lengthy and difficult work with the combinatorics of filters. For our problem, simpler constructions are possible. The main result of this paper is that if L is canonical and invariant under bounded epimorphisms, then L has a set of axioms that are *constant*, i.e. have no propositional variables. But the frames validating a constant formula are defined by a simple first-order condition. This condition is satisfied by $\mathcal{F}_L$ when the constant formula is an L-theorem, and is preserved by elementary equivalence. (E) follows readily from these facts.

2 Framework

Let *Var* be a countably infinite set of propositional variables. The set *Fma* of propositional modal *formulas* consists of all (finite) formulas generated from members of Var and a propositional constant $\perp$ by the usual Boolean connectives and the modality $\Box$. A *logic* is a subset L of *Fma* such that $\perp \notin L$ and L includes all tautologies and instances of the schema $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and is closed under the inference rules of modus ponens, necessitation (if $\varphi \in L$ then $\Box\varphi \in L$) and uniform substitution of a formula for a variable. If $\varphi \in L$ we may write $L \vdash \varphi$ and call φ an *L-theorem*. Thus we have $L \nvdash \perp$. The smallest (intersection) of all logics is customarily denoted K. We write KΔ for the smallest logic containing a given set Δ of formulas.

A *frame* $\mathcal{F} = (W, R)$ consists of a binary relation R on a non-empty set W. A *model* $\mathcal{M}$ on $\mathcal{F}$ is a function assigning a subset $\mathcal{M}(p)$ of W to each variable $p \in Var$. This function is extended inductively to all formulas by putting $\mathcal{M}(\perp) = \emptyset$, interpreting each Boolean connective by the appropriate Boolean set operation, and defining

$$\mathcal{M}(\Box\varphi) = \{s \in W : \{t : sRt\} \subseteq \mathcal{M}(\varphi)\}.$$

We may write $\mathcal{M}, s \models \varphi$ to mean $s \in \mathcal{M}(\varphi)$.

Formula φ is *true in model* $\mathcal{M}$, denoted $\mathcal{M} \models \varphi$, when $\mathcal{M}(\varphi) = W$; and is *valid in frame* $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, when φ is true in every model based on $\mathcal{F}$. The formulas that are valid in all frames are precisely the theorems of K. $\mathcal{F} \models \Delta$ means that $\mathcal{F} \models \varphi$ for all $\varphi \in \Delta$. If a logic L is equal to KΔ for some $\Delta \subseteq Fma$, then $\mathcal{F} \models L$ iff $\mathcal{F} \models \Delta$, because if $\mathcal{F} \models \Delta$ then $\{\varphi : \mathcal{F} \models \varphi\}$ is a logic that includes Δ and hence includes KΔ.

We write $Fr(L)$ for the class $\{\mathcal{F} : \mathcal{F} \models L\}$ of all frames validating L. L is *complete* if there exists some class $\mathcal{C}$ of frames that *determines* L in the sense that, in general, $L \vdash \varphi$ if and only if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{C}$. Equivalently, L is complete iff it is detemined by $Fr(L)$.

3 Constant Formulas

Let Γ denote the set of all formulas that are *constant*, i.e. contain no variables. The members of Γ are generated from $\bot$ by the Boolean connectives and $\Box$. A constant formula φ has a fixed interpretation in any frame, because $\mathcal{M}(\varphi) = \mathcal{M}'(\varphi)$ for any models $\mathcal{M}$ and $\mathcal{M}'$ on the same frame. Thus it makes sense to write $\mathcal{F}, s \models \varphi$ for constant φ, meaning that $\mathcal{M}, s \models \varphi$ for some, hence every, model $\mathcal{M}$ on $\mathcal{F}$.

A *constant logic* is a logic L with a set of constant axioms, i.e. $L = \mathrm{K}\Delta$ for some set $\Delta \subseteq \Gamma$. Equivalently, L is constant iff $L = \mathrm{K}(L \cap \Gamma)$.

Each $\varphi \in \Gamma$ can be translated into a formula $\varphi^*(x)$ in the first-order language of frames with x as its only free variable. This is defined inductively by taking $\bot^*(x)$ to be $x \neq x$, translating the Boolean connectives as themselves, and taking $(\Box\varphi)^*(x)$ to be

$$\forall y(xRy \to \varphi^*(y/x)),$$

where y is distinct from x and x is free for y in φ^*. Then in general

$$\mathcal{F}, s \models \varphi \quad \text{iff} \quad \mathcal{F} \models \varphi^*[s],$$

where the notation '$\mathcal{F} \models \varphi^*[s]$' means that φ^* is satisfied in $\mathcal{F}$ in the usual Tarskian sense for first-order logic when the variable x is assigned the value s. Thus

$$\mathcal{F} \models \varphi \quad \text{iff} \quad \mathcal{F} \models \forall x \varphi^*,$$

showing that the frames validating a constant formula form an elementary class.

Of course there are many formulas that contain variables but have a fixed interpretation in any frame in the manner defined above. These include the valid formulas like $\Box(\varphi \wedge \psi) \to \Box\varphi$, the unsatisfiable ones like $\neg\Box(\varphi \to \varphi)$, and less obvious examples like $\Box(\Box\neg\varphi \to \neg\Box\varphi)$. But every such formula is equivalent to a constant one in a strong sense. If we write $\varphi^\top$ for the constant formula obtained by replacing every variable in φ by $\top := \bot \to \bot$, then it can be shown [2, Theorem 1.14.5] that if φ has a fixed interpretation in each frame, then the formula $\varphi \leftrightarrow (\varphi^\top)$ is valid in all frames, so in general $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}, s \models \varphi^\top$ iff $\mathcal{F}, s \models \varphi^\top$ for all points s in all models $\mathcal{M}$ on all frames $\mathcal{F}$.

4 Bisimulations

A *bisimulation* from frame $\mathcal{F} = (W, R)$ to frame $\mathcal{F}' = (W', R')$ is a binary relation $\rho \subseteq W \times W'$ satisfying the following 'back-and-forth' property: for all $s \in W$ and $s' \in W'$, if $s\rho s'$ then

$$s'R't' \quad \text{implies} \quad \exists t(sRt \text{ and } t\rho t'),$$

and

$$sRt \quad \text{implies} \quad \exists t'(s'R't' \text{ and } t\rho t').$$

A straightforward induction on the formation of formulas shows that if φ is *constant*, then in general

$$s\rho s' \quad \text{implies} \quad [\mathcal{F}, s \models \varphi \text{ iff } \mathcal{F}', s' \models \varphi].$$

This in turn implies that if ρ is *surjective*, i.e. its image is W', then $\mathcal{F} \models \varphi$ implies $\mathcal{F}' \models \varphi$ for all $\varphi \in \mathit{\Gamma}$. Thus validity of a constant formula is *preserved by bisimulation images*. It also implies that if ρ is *total*, i.e. its domain is W, then $\mathcal{F}' \models \varphi$ implies $\mathcal{F} \models \varphi$ for all $\varphi \in \mathit{\Gamma}$. Thus validity of a constant formula is *reflected by total bisimulations*.

A *bounded morphism* $f : \mathcal{F} \to \mathcal{F}'$ can be defined as a function $f : W \to W'$ whose graph $\{(s, f(s)) : s \in W\}$ is a (total) bisimulation. This is equivalent to the usual definition that

$$f(s)R't' \quad \text{iff} \quad \exists t(sRt \text{ and } f(t) = t').$$

If f is surjective, then it is called a bounded *epi*morphism. Thus we have that validity of a constant formula is *invariant under* (i.e. preserved and reflected by) bounded epimorphisms.

This result does not hold for non-constant formulas. As first shown by Segerberg [8], validity of an arbitrary formula is preserved by bounded epimorphic images. But it may not be reflected by them. To take a familiar example, let $\mathcal{F} = (\{0, 1, 2, \dots\}, <)$ and let $\mathcal{F}'$ be a one-element reflexive frame, hence a bounded epimorphic image of $\mathcal{F}$. The formula $\Box p \to p$ is valid in $\mathcal{F}'$ but not in $\mathcal{F}$.

To clarify the situation for non-constant formulas, we need the notion of $\mathcal{F}$ being an *inner subframe* of $\mathcal{F}'$. This means that $W \subseteq W'$, R is the restriction of R' to W, and W is closed under R': if $s \in W$ and $sR't'$, then $t' \in W$ (which is equivalent to the inclusion function $W \hookrightarrow W'$ being a bounded morphism.) Validity of all modal formulas is preserved by inner subframes, so in this situation $\mathcal{F}' \models \varphi$ implies $\mathcal{F} \models \varphi$ for all $\varphi \in \mathit{Fma}$.

Theorem 4.1 *For any formula $\varphi \in Fma$, the following are equivalent:*

(1) *φ is preserved by bisimulation images.*

(2) *φ is invariant under total surjective bisimulations.*

(3) *φ is invariant under bounded epimorphic images.*

Proof. (1) implies (2): If ρ is a bisimulation from $\mathcal{F}$ to $\mathcal{F}'$, then its inverse $\rho^{-1} = \{(s', s) : s\rho s'\}$ is a bisimulation from $\mathcal{F}'$ to $\mathcal{F}$ whose image is the domain of ρ. Thus if (1) holds and ρ is total, then under ρ^{-1}, $\mathcal{F}' \models \varphi$ implies $\mathcal{F} \models \varphi$.

(2) implies (3): the graph of a bounded epimorphism is a total surjective bisimulation.

(3) implies (1). Let ρ be a bisimulation from $\mathcal{F}$ to $\mathcal{F}'$. We make ρ itself into a frame $\mathcal{R} = (\rho, R_\rho)$ by putting, for all (s, s') and (t, t') in ρ,

$$(s, s')R_\rho(t, t') \quad \text{iff} \quad sRt \text{ and } s'R't'.$$

Then the back-and-forth properties of ρ ensure that the projection maps $\pi : \rho \to W$ and $\pi' : \rho \to W'$ are bounded morphisms $\mathcal{R} \to \mathcal{F}$ and $\mathcal{R} \to \mathcal{F}'$. The image W_π of π is a subset of W that is R-closed: if $s \in W_\pi$ and sRt, then $t \in W_\pi$. Thus W_π is the base of an *inner subframe* $\mathcal{F}_\pi$ of $\mathcal{F}$ that is a bounded epimorphic image of $\mathcal{R}$ under π. Hence if $\mathcal{F} \models \varphi$ then $\mathcal{F}_\pi \models \varphi$ and so if (3) holds we get $\mathcal{R} \models \varphi$. But then if ρ is surjective, so is π', and hence $\mathcal{F}' \models \varphi$. Altogether, this shows that if (3) holds, then $\mathcal{F} \models \varphi$ implies $\mathcal{F}' \models \varphi$ whenever there is a surjective bisimulation from $\mathcal{F}$ to $\mathcal{F}'$. ■

5 Canonical Frames

For any logic L, a set s of formulas is *L-consistent* if $L \nvdash \neg\varphi$ whenever φ is a conjunction of finitely many members of s. A *maximally L-consistent* set s is one that is L-consistent but has no proper L-consistent extension: this is equivalent to requiring that $L \subseteq s$ and for all $\varphi \in Fma$, exactly one of φ and $\neg\varphi$ belongs to s. The *canonical L-frame* is $\mathcal{F}_L = (W_L, R_L)$, where

$$W_L = \{s \subseteq Fma : s \text{ is maximally } L\text{-consistent}\}$$

and

$$sR_Lt \quad \text{iff} \quad \{\varphi \in Fma : \Box\varphi \in s\} \subseteq t.$$

Every L-consistent set extends to a maximally L-consistent one, and this implies that the only formulas that belong to every member of W_L are the L-theorems. The assumption that $L \nvdash \bot$ implies that L is L-consistent, and hence that W_L is non-empty.

The *canonical model* $\mathcal{M}_L$ is defined by $\mathcal{M}_L(p) = \{s \in W_L : p \in s\}$ for all $p \in \mathit{Var}$. Its fundamental property is that $\mathcal{M}_L(\varphi) = \{s \in W_L : \varphi \in s\}$ for all $\varphi \in \mathit{Fma}$. From this it follows that

$$\mathcal{M}_L \models \varphi \quad \text{iff} \quad L \vdash \varphi,$$

hence $\mathcal{F}_L \models \varphi$ implies $L \vdash \varphi$. L is a *canonical logic* if, conversely, $\mathcal{F}_L \models L$. A formula φ will be called *canonical* if $\mathrm{K}\varphi$, the smallest logic containing φ, is canonical. Since $\mathcal{F}_L$ is a subframe of $\mathcal{F}_{\mathrm{K}\varphi}$ whenever $L \vdash \varphi$ [3, Theorem 5.4], this is equivalent to requiring that $\mathcal{F}_L \models \varphi$ whenever $L \vdash \varphi$.

Lemma 5.1 *Every constant formula is canonical. Hence every constant logic is canonical.*

Proof. Let $\varphi \in \mathit{\Gamma}$. Then $L \vdash \varphi$ implies $\mathcal{M}_L \models \varphi$, and so $\mathcal{F}_L \models \varphi$ as φ is constant. Thus φ is canonical.

Hence if $L = \mathrm{K}\Delta$ with $\Delta \subseteq \mathit{\Gamma}$, then the members of Δ are all canonical, so $\mathcal{F}_L \models \Delta$ and hence $\mathcal{F}_L \models L$, i.e. L is canonical. ■

Extending this last proof a little further: if $L \vdash \varphi$ and φ is constant, then $\mathcal{F}_L \models \forall x \varphi^*$, i.e. $\mathcal{F}_L$ satisfies the first-order sentence defining the elementary class of frames validating φ.

6 The Constant Canonical Frame

For any logic L, let $G_L = \{s \cap \mathit{\Gamma} : s \in W_L\}$, and define a frame $\mathcal{G}_L = (G_L, R_{\mathit{\Gamma}})$ by putting

$$u R_{\mathit{\Gamma}} v \quad \text{iff} \quad \{\varphi \in \mathit{\Gamma} : \Box\varphi \in u\} \subseteq v$$

for all $u, v \in G_L$. Thus $\mathit{\Gamma}_L$ is a version of the canonical L-frame restricted to the language $\mathit{\Gamma}$ of constant formulas. Note that if $u \in G_L$ and $\varphi \in \mathit{\Gamma} - u$, then $\neg\varphi \in u$. The fact that W_L is non-empty ensures that G_L is too.

Lemma 6.1 *If $L^{\#}$ is any logic with $L \cap \mathit{\Gamma} \subseteq L^{\#} \subseteq L$, then $\mathcal{G}_L$ is a bounded epimorphic image of $\mathcal{F}_{L^{\#}}$.*

Proof. For $s \in W_{L^\#}$, put $f(s) = s \cap \Gamma$.

First we show $f(s) \in G_L$. Let $u = s \cap \Gamma$. If u were not L-consistent, there would be some finite $v \subseteq u$ with $L \vdash \neg \bigwedge v$, hence $\neg \bigwedge v \in L \cap \Gamma \subseteq L^\# \subseteq s$, which would contradict the $L^\#$-consistency of s as $v \subseteq s$. Hence u is L-consistent and so extends to a maximally L-consistent set $t \in W_L$. Thus $u \subseteq t \cap \Gamma$. But if $\varphi \in t \cap \Gamma$, then $\varphi \in s$ or else $\neg\varphi \in s \cap \Gamma = u \subseteq t$ contrary to the L-consistency of t, so $\varphi \in u$. Altogether then, $f(s) = u = t \cap \Gamma \in G_L$.

Next we show f maps *onto* G_L. For if $u \in G_L$, then $u = s \cap \Gamma$ for some $s \in W_L$, and then $s \in W_{L^\#}$ as $L^\# \subseteq L \subseteq s$, hence $f(s)$ is defined and equal to u.

It remains to show f is a bounded morphism. If $sR_{L^\#}t$, then $\{\varphi \in Fma : \Box\varphi \in s\} \subseteq t$, hence $\{\varphi : \Box\varphi \in s \cap \Gamma\} \subseteq t \cap \Gamma$, showing $f(s)R_\Gamma f(t)$. Finally, suppose $s \in W_{L^\#}$ and $f(s)R_\Gamma u \in G_L$. We have to show that there exists some $t \in W_{L^\#}$ with $sR_{L^\#}t$ and $f(t) = u$. Put

$$t_0 = \{\varphi \in Fma : \Box\varphi \in s\} \cup u.$$

Now if t_0 were not $L^\#$-consistent then, since the two sets that make up t_0 are each closed under conjunction, there would be formulas φ, ψ with $\Box\varphi \in s$, $\psi \in u$ and $L^\# \vdash \varphi \to \neg\psi$. Then $(\Box\varphi \to \Box\neg\psi) \in L^\# \subseteq s$, so $\Box\neg\psi \in s$ as $\Box\varphi \in s$. But then $\Box\neg\psi \in s \cap \Gamma = f(s)$, implying $\neg\psi \in u$ as $f(s)R_\Gamma u$, contrary to the L-consistency of u. Thus t_0 is $L^\#$-consistent, so there is some $t \in W_{L^\#}$ with $t_0 \subseteq t$. Then $sR_{L^\#}t$ and $u \subseteq t$, hence $u \subseteq t \cap \Gamma$. But if $\varphi \in t \cap \Gamma$, then $\varphi \in u$ or else $\neg\varphi \in u \subseteq t$ contrary to the $L^\#$-consistency of t. So $u = t \cap \Gamma = f(t)$ as desired. ■

It is natural to ask if there can be an $L^\#$ as in this Lemma that is a proper sublogic of L. If not, then L is $\mathrm{K}(L \cap \Gamma)$, the smallest logic containing the constant set $L \cap \Gamma$. Hence L is canonical and invariant under bounded epimorphisms. In fact these last two conditions are equivalent to constancy of L as we now show in our main result:

Theorem 6.2 *For any logic L, the following are equivalent:*

(1) *L is constant.*

(2) *L is canonical and preserved by bisimulation images.*

(3) *L is canonical and invariant under total surjective bisimulations.*

(4) *L is canonical and invariant under bounded epimorphic images.*

Proof. (1) implies (2): If L is constant, then L is canonical by Lemma 5.1. Also validity of L is preserved by bisimulation images since validity of Δ is thus preserved (where $L = \mathrm{K}\Delta$ with $\Delta \subseteq \Gamma$), as explained in Section 4.

The equivalence of (2), (3), and (4) follows from Theorem 4.1. We complete the Theorem by showing that (4) implies (1). Let L be canonical and invariant under bounded epimorphic images. We prove (1) by showing that $L = \mathrm{K}(L \cap \Gamma)$.

By canonicity $\mathcal{F}_L \models L$. By Lemma 6.1 with $L^{\#} = L$, there is a bounded epimorphism $\mathcal{F}_L \twoheadrightarrow \mathcal{G}_L$, so then $\mathcal{G}_L \models L$.

But by the same Lemma 6.1 with $L^{\#} = \mathrm{K}(L \cap \Gamma)$, there is a bounded epimorphism $\mathcal{F}_{\mathrm{K}(L\cap\Gamma)} \twoheadrightarrow \mathcal{G}_L$, so by the assumed invariance of L we then get $\mathcal{F}_{\mathrm{K}(L\cap\Gamma)} \models L$. Thus if $\varphi \in L$ then $\mathcal{F}_{\mathrm{K}(L\cap\Gamma)} \models \varphi$ and so $\mathrm{K}(L \cap \Gamma) \vdash \varphi$. This shows that $L = \mathrm{K}(L \cap \Gamma)$ as desired, proving that L is a constant logic. ■

This Theorem verifies the claim made in the Introduction: if a logic L is canonical and invariant under bounded epimorphic images, then it is constant and its class $\{\mathcal{F} : \mathcal{F} \models L\}$ of validating frames is the elementary class

$$\{\mathcal{F} : \mathcal{F} \models \forall x \varphi^* \text{ for all } \varphi \in L \cap \Gamma\}.$$

This class is closed under elementary equivalence and includes $\mathcal{F}_L$, showing that condition (E) holds for L.

Finally we consider the necessity of the canonicity condition in Theorem 6.2. First, note that the proof of the Theorem is not specific to monomodal logics: it works for logics containing any number of modalities. This makes it easy to see that there can be logics that are invariant under bounded epimorphic images but not canonical, hence not constant. For example, the *incomplete* bimodal (tense) logic of [12] is not valid in any frames, hence is vacuously invariant under bounded epimorphic images and not canonical. In fact we have the following consequence of Theorem 6.2.

Corollary 6.3 *Let a logic L be invariant under bounded epimorphic images. Then the following are equivalent.*

(1) *L is canonical.*

(2) *L is constant.*

(3) *L is complete and determined by an elementary class of frames.*

Proof. (1) implies (2): by Theorem 6.2.

(2) implies (3): a constant logic is determined by the class of all models of the first-order theory $\{\forall x \varphi^* : \varphi \in L \cap \Gamma\}$.

(3) implies (1): this is Theorem 3 of [1], and holds for any logic. ■

This leaves us with a question: is there a logic that is complete and invariant under bounded epimorphic images, but not constant? Such a logic can be neither canonical nor determined by any elementary class.

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