Hybrid Steepest Descent Method for Variational Inequality Problem over Fixed Point Sets of Certain Quasi-Nonexpansive Mappings

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This talk is based on a joint work with

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We are trying to solve: in Real Hilbert Sp $\mathcal{H}$

Variational Inequality Problem over $Fix(T)$

For given $T : \mathcal{H} \to \mathcal{H}$ and $\Theta : \mathcal{H} \to \mathbb{R}$ (Convex func.),

Find

$u^* \in Fix(T) := \{x \in \mathcal{H} \mid T(x) = x\}$ **closed convex**

s.t. $\langle u - u^*, \Theta'(u^*)\rangle \geq 0, \ \forall u \in Fix(T).$

For T: **Convex Projection** $\Rightarrow$ **Gradient Projection Method**
(Goldstein'64/Levitin&Polyak'66)

We propose Hybrid Steepest Descent Method

- $T : \mathcal{H} \to \mathcal{H}$ **Nonexpansive Mapping**

(Yamada et al '96— / Deutsch & Yamada '98 / Yamada '01)

Appl: Convexly Constrained Inverse Problems

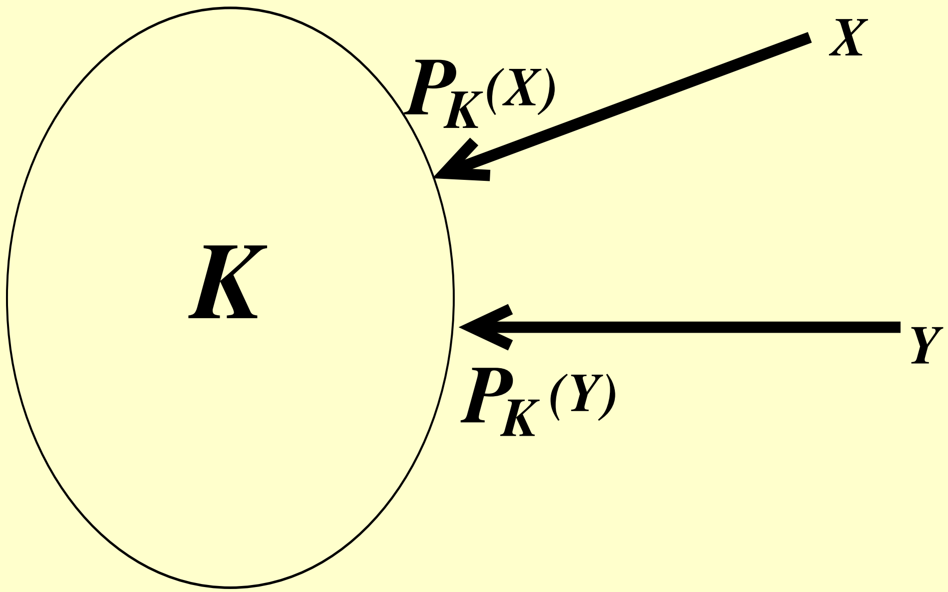
- $T : \mathcal{H} \to \mathcal{H}$ **Quasi-Nonexpansive**(Yamada&Ogura'03)

Appl: Optimization of Fixed Point of Subgradient Projector

Part 1

Background / Preliminaries

Original Idea of Gradient Projection Method



Convex Projection: Basic Properties

- $\|P_K(\boldsymbol{x}) - P_K(\boldsymbol{y})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|, \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$
- $Fix(P_K) := \{\boldsymbol{x} \in \mathcal{H} \mid P_K(\boldsymbol{x}) = \boldsymbol{x}\} = K$
- K must be simple to compute P_K.

Gradient Projection Method (1964—)

$$u_{n+1} := P_K\left(u_n - \lambda_{n+1}\Theta'(u_n)\right),$$
$$n = 0, 1, 2, \ldots$$

— under certain conditions —

converges (strongly / weakly) to a solution to

Smooth Convex Optimization Problem (P1)

Minimize $\Theta : \mathcal{H} \to \mathbb{R}$ G-differentiable convex func.

Subject to $x \in K\ (\subset \mathcal{H})$ closed convex set

where $\mathcal{H}$: Real Hilbert Space

NOTE: $u^* \in K$ is a solution of (P1)

$\Leftrightarrow u^* \in K$ satisfies $\left\langle u - u^*, \Theta'(u^*)\right\rangle \geq 0,\ \forall u \in K$.

Part 2

Hybrid Steepest Descent Method

From Projection to

Nonexpansive Mapping /
Quasi-Nonexpansive Mapping

$T : \mathcal{H} \to \mathcal{H}$ is called **κ-Lipschitzian** if $\exists \kappa > 0$ s.t.

$$\|T(x) - T(y)\| \le \kappa \|x - y\| \text{ for all } x, y \in \mathcal{H}.$$

If $\kappa = 1$

- $T : \mathcal{H} \to \mathcal{H}$ is **Nonexpansive mapping**.
- $Fix\,(T) := \{x \in \mathcal{H} \mid T(x) = x\}$ is **closed convex**.

$$\Downarrow$$

- Generalization $\kappa < 1 \Rightarrow \kappa < 1$ or $\kappa = 1$ broadens **Fixed Point Theory** significantly.
- **Many choices of T s.t. $Fix(T) = K$**, e.g.,

$$Fix\left(\sum_{i=1}^{m} w_i T_i\right) = \bigcap_{i=1}^{m} Fix(T_i) \text{ if } \bigcap_{i=1}^{m} Fix(T_i) \neq \emptyset.$$

Is It Possible to Extend from

Gradient Projection Method

$$v_{n+1} := P_K\left(v_n - \lambda_{n+1}\Theta'(v_n)\right)$$

to

$$\underline{v_{n+1} := T\left(v_n - \lambda_{n+1}\Theta'(v_n)\right)}$$

where $T : \mathcal{H} \to \mathcal{H}$: **Nonexpansive Mapping**

for Minimizing Θ
over $Fix(T)$?

To Answer to the Question, we introduce

Hybrid Steepest Descent Method (Yamada et al, 1996—)

$$u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n))$$

where $T:\mathcal{H} \to \mathcal{H}$: **Nonexpansive Mapping**

This is because

- $v_n := T(u_n)$ is generated by

$$\underline{v_{n+1} := T\left(v_n - \lambda_{n+1}\Theta'(v_n)\right)}$$

and

- If $\text{s-}\lim_{n\to\infty} u_n = u^* \in Fix(T)$

$$\Rightarrow \text{s-}\lim_{n\to\infty} v_n = u^* \in Fix(T)$$

In short,

Hybrid Steepest Descent Method (Yamada2001):

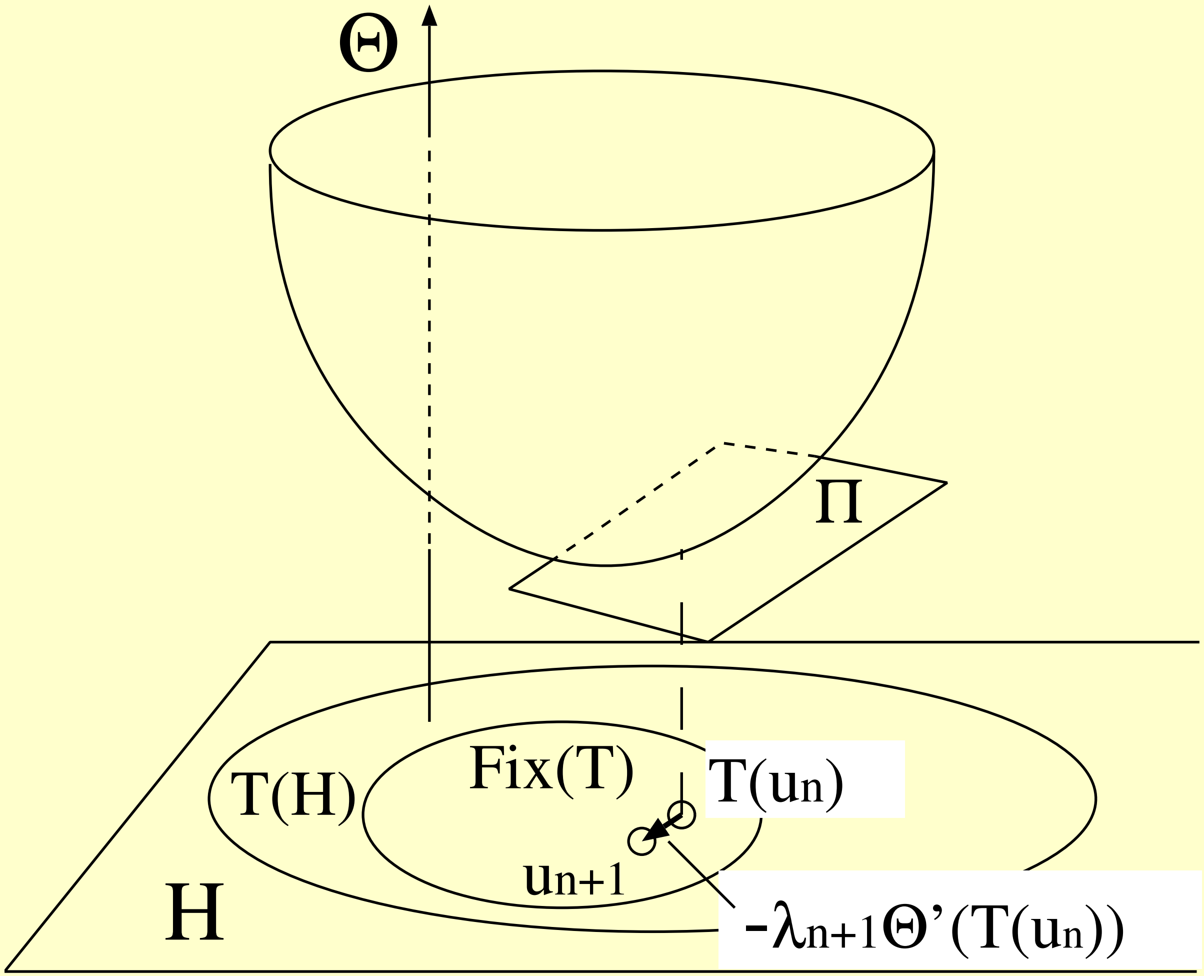
$$\underline{u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n))}$$

can minimize Θ over $Fix(T)$,

where

$T : \mathcal{H} \rightarrow \mathcal{H}$: nonexpansive, and
$(\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}^+$: slowly decreasing.

Sequence Generation by
Hybrid Steepest Descent Method



Hybrid Steepest Descent Method (Yamada 2001)

Suppose that

(a) $T : \mathcal{H} \to \mathcal{H}$: Nonexp. mapping,

(b) $\Theta : \mathcal{H} \to \mathbb{R}$:Convex function,

(c) Θ': Lipschitzian & Strongly monotone over $T(\mathcal{H})$,

(d) $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ satisfies

(i) $\lim_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n\geq 1} \lambda_n = \infty$, (iii) $\sum_{n\geq 1} |\lambda_n - \lambda_{n+1}| < \infty$.

$$\Downarrow$$

$u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n))$ satisfies

s-$\lim_{n\to\infty} u_n = u^* \in \arg \inf_{x\in Fix(T)} \Theta(x)$. (Unique)

If we specially choose $\Theta(x) := \frac{1}{2}\|x-a\|^2$ in the **Hybrid Steepest Descent Method**,

$$\Downarrow$$

Halpern ('67), P.L.Lions ('77), Wittmann ('92)

$$u_{n+1} := \lambda_{n+1} a + (1-\lambda_{n+1}) T(u_n),$$

converges strongly to $P_{Fix(T)}(a)$, where

$T : \mathcal{H} \to \mathcal{H}$: nonexpansive, and

$(\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}^+$: slowly decreasing.

More general cyclic versions were given by

P.L. Lions (1977) and H.H. Bauschke (1996)

Generalization of Θ

Θ': **Lipschitzian** & **Paramonotone** (Ogura,Yamada 2002)

Robust Hybrid Steepest Descent Method

(Yamada, Ogura, Shirakawa 2002)

$$\underline{u_{n+1} := T_{(n)}(u_n) - \lambda_{n+1}\Theta'\left(T_{(n)}(u_n)\right)}$$

$$\text{where } T_{(n)} := (1 - t_{n+1})I + t_{n+1}T$$

is gifted with **notable numerical robustness.**

For detail, see

Contemporary Mathematics 313 (2002)

Convexly Constrained Generalized Inverse Problem

Let $K \subset \mathcal{H}$: *a closed convex set,*

$\Psi : \mathcal{H} \to \mathbb{R}$: *the 1st convex function,*

satisfying

$$K_\Psi := \arg\inf_{x \in K} \Psi(x) \neq \emptyset.$$

Then the problem is

$$\textit{Find a point } x^* \in \arg\inf_{x \in K_\Psi} \Theta(x) =: \Gamma(\neq \emptyset),$$

where $\Theta : \mathcal{H} \to \mathbb{R}$ *is the 2nd convex function.*

Suppose that $\Psi' : \mathcal{H} \to \mathcal{H}$ (G-derivative) is γ-Lipschitzian.

$$\Downarrow$$

Apply **Hybrid Steepest Descent Method**

$$u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n)),$$

$$\left[T := P_K(I - \nu\Psi'), \quad \forall \nu \in (0, 2/\gamma]\right]$$

Solves the Problem, i.e., $\lim_{n\to\infty} d(u_n, \Gamma) = 0$.

NOTE: **Projected Landweber Iteration** (Eicke 1992):

$$v_{n+1} := P_K\left(\lambda_{n+1}A^*\boldsymbol{b} + \beta_n(I - \lambda_{n+1}A^*A)v_n\right)$$

is the simplest realization for $\Theta(x) := \frac{1}{2}\|x\|^2$ and $\Psi(x) := \frac{1}{2}\|A(x) - b\|_o^2$ ($A : \mathcal{H} \to \mathcal{H}_o$: bdd linear).

Part 3

Hybrid Steepest Descent Method

From Nonexpansive Mapping to
Quasi-Nonexpansive Mapping

Quasi-Nonexpansive Mapping

$T : \mathcal{H} \to \mathcal{H}$ is called *Quasi-Nonexpansive* if

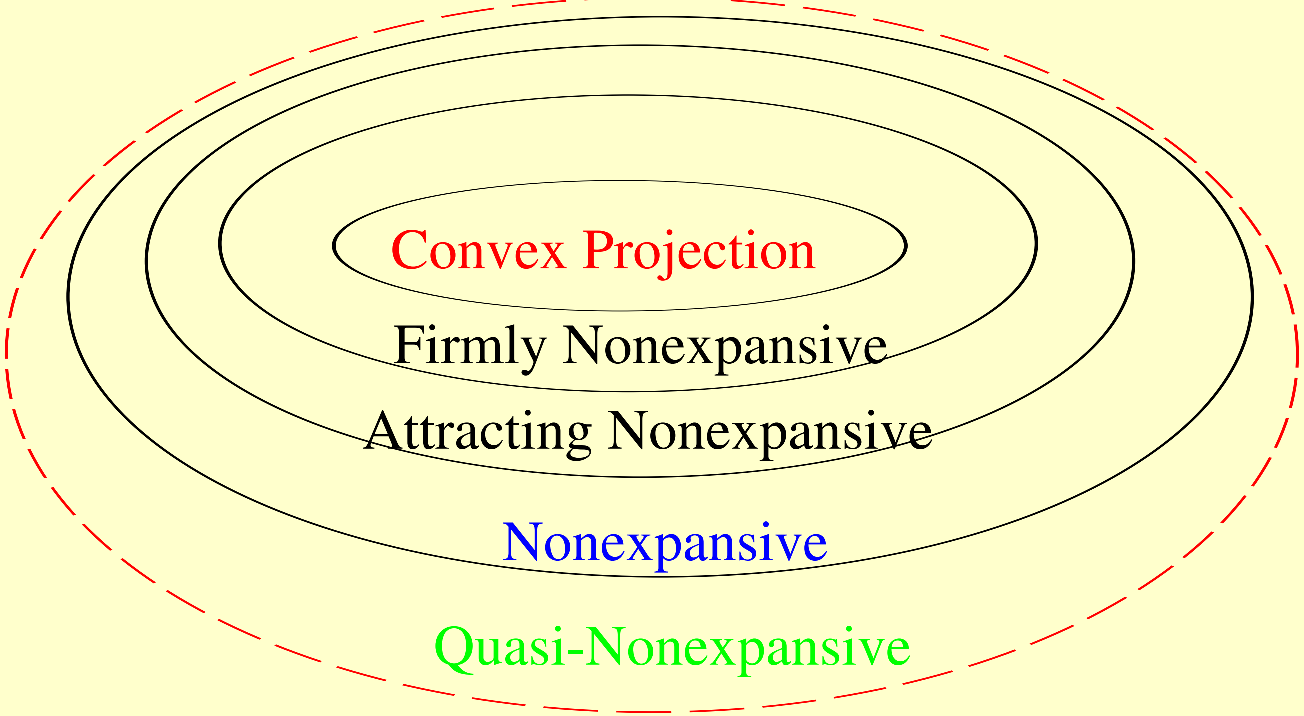
$$\|T(\boldsymbol{x}) - T(\boldsymbol{f})\| \leq \|\boldsymbol{x} - \boldsymbol{f}\|, \ \forall (\boldsymbol{x}, \boldsymbol{f}) \in \mathcal{H} \times Fix(T).$$

In this case,

$$Fix\,(T) := \{x \in \mathcal{H} \mid T(x) = x\}$$

is **closed convex set**.

Quasi-nonexpansive mapping T is not necessarily continuous.



Next Example shows

The level set of continuous convex function can be expressed as

Fixed Point Set of

<u>Simple Quasi-Nonexpansive Mapping.</u>

Example (Subgradient Projection $T_{sp(\Phi)}$)

Subgradient Projection for Cont. convex function Φ

$$T_{sp(\Phi)} : x \mapsto \begin{cases} x - \dfrac{\Phi(x)}{\|g(x)\|^2} g(x) & \text{if } \Phi(x) > 0 \\ x & \text{if } \Phi(x) \leq 0, \end{cases}$$

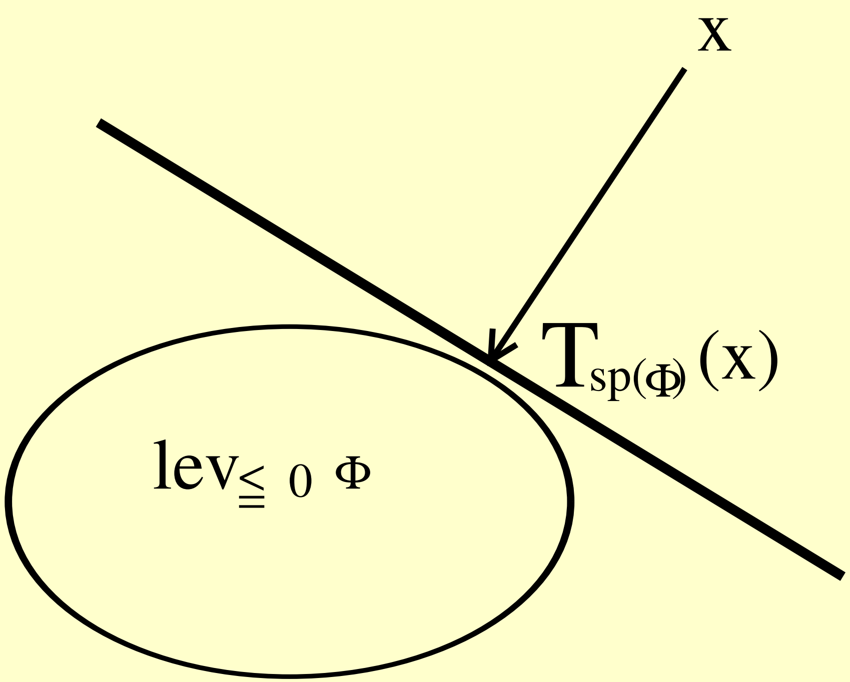
where $g(x) \in \partial\Phi(x)$: subgradient of Φ at $x \in \mathcal{H}$.

$$\Downarrow$$

See for example (Bauschke & Combettes '01)

- $T_{sp(\Phi)}$: ($\frac{1}{2}$-averaged) quasi-nonexpansive,
- $Fix(T_{sp(\Phi)}) = \{x \in \mathcal{H} \mid \Phi(x) \leq 0\} =: \text{lev}_{\leq 0}\Phi$

Subgradient Projection : Approximation of Convex Projection



$$Fix\left(T_{sp(\Phi)}\right) = lev_{\leq 0}(\Phi)$$

Is It Possible to Extend from

$$\underline{u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n))}$$

where $T : \mathcal{H} \to \mathcal{H}$: **Nonexpansive**

to

$$\underline{u_{n+1} := T(u_n) - \lambda_{n+1}\Theta'(T(u_n))}$$

where $T : \mathcal{H} \to \mathcal{H}$: **Quasi-Nonexpansive**

for Minimizing Θ
over $Fix(T)$?

Quasi-shrinking (Yamada & Ogura '03)

Let $T : \mathcal{H} \to \mathcal{H}$: quasi-nonexpansive with $Fix(T) \cap C \neq \emptyset$ for $\exists C (\subset \mathcal{H})$: closed convex set.

$$\Downarrow$$

$T : \mathcal{H} \to \mathcal{H}$ is called *quasi-shrinking* on C if

$$D : r \in [0, \infty) \mapsto \begin{cases} \inf_{u \in \triangleright(Fix(T), r) \cap C} d(u, Fix(T)) - d(T(u), Fix(T)) & \text{if } \triangleright(Fix(T), r) \cap C \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

satisfies $D(r) = 0 \Leftrightarrow r = 0$.

where $\triangleright(Fix(T), r) := \{x \in \mathcal{H} \mid d(x, Fix(T)) \geq r\}$.

Hybrid Steepest Descent Method (Quasi-Nonexpansive)

Suppose that
(a) $T : \mathcal{H} \to \mathcal{H}$: **Quasi-Nonexpansive**,
(b) Θ': κ-Lipschitzian & η-Strongly monotone over $T(\mathcal{H})$,
(c) $\exists f \in Fix(T)$, s.t. T is **quasi-shrinking** on

$$C_f(u_0) := \left\{ x \in \mathcal{H} \mid \|x - f\| \le \max\left(\|u_0 - f\|, \frac{\|\mu \mathcal{F}(f)\|}{1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}} \right) \right\}$$

where $\mu \in (0, \frac{2\eta}{\kappa^2})$,

$$\Downarrow$$

With $(\lambda_n)_{n\ge1} \subset [0,1]$ s.t. (i) $\lim_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n\ge1} \lambda_n = \infty$,

$$\underline{u_{n+1} := T(u_n) - \lambda_{n+1}\mu\Theta'(T(u_n))}$$ satisfies

$$\text{s-}\lim_{n\to\infty} u_n = u^* \in \arg\inf_{x\in Fix(T)} \Theta(x) \text{ (Unique)}$$

Proposition

Suppose $\Phi : \mathcal{H} \to \mathbb{R}$ (cont. convex function) satisfies

- $\text{lev}_{\leq 0}\Phi \neq \emptyset$ and
- $\partial\Phi$ bounded.

Define

$$T_\alpha := (1-\alpha)I + \alpha T_{sp(\Phi)} \quad (\alpha \in (0,2)).$$

Then

(a) If $\dim(\mathcal{H}) < \infty$, $\Longrightarrow$
T_α : quasi-shrinking on any bdd closed convex C satisfying $C \cap \text{lev}_{\leq 0}\Phi \neq \emptyset$.

(b) If $\Phi' \in \partial\Phi$: Uniformly monotone over $\mathcal{H}$, $\Longrightarrow$
T_α : quasi-shrinking on any bdd closed convex C satisfying $C \cap \text{lev}_{\leq 0}\Phi \neq \emptyset$.

Hybrid Steepest Descent Method (for $T_{sp(\Phi)}$)

Suppose that

(a) $\Phi : \mathcal{H} \to \mathbb{R}$: **Cont. Convex**, $\mathrm{lev}_{\leq 0}\Phi \neq \emptyset$ & $\partial\Phi$: bdd,

Let $T_\alpha := (1-\alpha)I + \alpha T_{sp(\Phi)}$ $(\alpha \in (0,2))$.

(b) Θ': κ-Lipschitzian & η-Strongly monotone over $T_\alpha(\mathcal{H})$,

$$\Downarrow$$

When $\dim(\mathcal{H}) < \infty$

With $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ s.t. (i) $\lim_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n\geq 1} \lambda_n = \infty$,

$\underline{u_{n+1} := T_\alpha(u_n) - \lambda_{n+1}\Theta'(T_\alpha(u_n))}$ satisfies

$$\lim_{n\to\infty} u_n = u^* \in \arg \inf_{x\in \mathrm{lev}_{\leq 0}\Phi} \Theta(x) \text{ (Unique)}$$

Hybrid Steepest Descent Method (for $T_{sp(\Phi)}$ over K)

Suppose that

(a) $\Phi : \mathcal{H} \to \mathbb{R}$: **Cont. Convex** with $\partial\Phi$: **bdd**,

(b) K: **bdd** closed convex set s.t. $\mathrm{lev}_{\leq 0}\Phi \cap K \neq \emptyset$,

(c) Θ': Lipschitzian& Paramonotone over K,

$$\Downarrow$$

When $\dim(\mathcal{H}) < \infty$

With $(\lambda_n)_{n \geq 1} \subset [0, \infty)$ s.t. (i) $\lim_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n \geq 1} \lambda_n = \infty$,

$$u_{n+1} := P_K T_\alpha(u_n) - \lambda_{n+1}\Theta'(P_K T_\alpha(u_n))$$

satisfies

$$\lim_{n\to\infty} d(u_n, \Gamma) = 0,$$

where $\Gamma := \arg \inf_{K \cap \mathrm{lev}_{\leq 0}\Phi} \Theta(x) \neq \emptyset$.

For related results to this talk,

See for example :

Hybrid Steepest Descent Method and Its Applications

1. I. Yamada: "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," pp.473–504, in Inherently Parallel Algorithm for Feasibility and Optimization and Their Applications, Elsevier 2001.

2. I. Yamada, N. Ogura and N. Shirakawa: "A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems," pp.269-305, in Inverse Problems, Image Analysis, and Medical Imaging, *Contemporary Mathematics,* 313, Amer. Math. Soc., 2002.

3. K. Slavakis, I. Yamada and K. Sakaniwa: "Computation of symmetric positive definite Toeplitz matrices by the Hybrid Steepest Descent Method," *Signal Processing,* vol.83, pp.1135–1140, 2003.

4. H.K. Xu and T.H. Kim: "Convergence of hybrid steepest descent methods for variational inequalities," *Journal of Optimization Theory and Applications,* vol. 119, no. 1, pp.185–201, 2003.

5. I. Yamada and N. Ogura: "Two Generalizations of the Projected Gradient Method for Convexly Constrained Inverse Problems — Hybrid steepest descent method, Adaptive projected subgradient method," *Proceedings of NANIT'03,* RIMS, Kyoto, Dec., 2003.

Thank you very much !!

What is Subgradient ?

Subgradient of Φ at x

Let $\Phi : \mathcal{H} \to \mathbb{R}$: Cont. Convex Function.

$$\Downarrow$$

$$\partial\Phi(\boldsymbol{x}) := \{\boldsymbol{s} \in \mathcal{H} : \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{s} \rangle + \Phi(\boldsymbol{x}) \le \Phi(\boldsymbol{y}), \forall \boldsymbol{y} \in \mathcal{H} \} \neq \emptyset.$$

$\forall \boldsymbol{s} \in \partial\Theta(\boldsymbol{x})$ is called Subgradient of Φ at $\boldsymbol{x}$.

- $\mathbf{0} \in \partial\Phi(\boldsymbol{x}) \Leftrightarrow \Phi(\boldsymbol{x}) = \min_{\boldsymbol{y} \in \mathcal{H}} \Phi(\boldsymbol{y})$.
- $\partial\Phi(\boldsymbol{x}) = \{\nabla\Phi(\boldsymbol{x})\}$ if Φ:G-differentiable at $\boldsymbol{x}$.

$\Rightarrow$ **generalization of Gradient**.

Subgradient: a generalization of Gradient