Victoria International Conference 2004
Wellington

10 February, 2004

Function Spaces and Metrisability of Manifolds

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MANIFOLD = connected, Hausdorff, locally Euclidean space

For a manifold M the following are equivalent.

1. M is metrisable;
2. M is paracompact;
3. M is strongly paracompact;
4. M is screenable;
5. M is metacompact;
6. M is σ-metacompact;
7. M is paraLindelöf;
8. M is σ-paraLindelöf;
9. M is metaLindelöf;
10. M is nearly metaLindelöf;
11. M is Lindelöf;
12. M is linearly Lindelöf;
13. M is ω_1-Lindelöf;
14. M is ω_1-metaLindelöf;
15. M is nearly linearly ω_1-metaLindelöf;
16. M is almost metaLindelöf;
17. M is hereditarily Lindelöf;
18. M is strongly hereditarily Lindelöf;
19. M is an $\aleph_0$-space;
20. M is cosmic;
21. every open k-cover of M has a countable k-subcover;
22. M is an $\aleph$-space;
23. M has a star-countable k-network;
24. M has a point-countable k-network;
25. M has a k-network which is point-countable on some dense subset of M;
26. M is second countable;
27. M is hemicompact;
28. M is σ-compact;
29. M is Hurewicz;
30. M may be embedded in some euclidean space;
31. M may be embedded properly in some euclidean space;
32. M is completely metrisable;
33. there is a continuous discrete map $f : M \to X$ where X is Hausdorff and second countable;
34. M is Lašnev;
35. M is an M_1-space;
36. M is stratifiable;
37. M is finitistic;
38. M is strongly finitistic;
39. M is star finitistic;
40. there is an open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x, \mathcal{U})$ is homeomorphic to an open subset of $\mathbb{R}^m$;
41. there is a point-star-open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x, \mathcal{U})$ is Lindelöf;
42. there is a point-star-open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x, \mathcal{U})$ is metrisable;
43. the tangent microbundle on M is equivalent to a fibre bundle;
44. M is a normal Moore space;
45. M is a normal θ-refinable space;
46. M is a normal subparacompact space;
47. M is a normal space which has a σ-discrete cover by compact subsets;
48. $M \times M$ is perfectly normal;
49. M is a normal space which has a sequence $\langle \mathcal{U}_n \rangle_{n \in \omega}$ of open covers with $\cap_n \overline{st(x, \mathcal{U}_n)} = \{x\}$ for each $x \in M$;
50. M is perfectly normal and there is a sequence $\langle \mathcal{U}_n \rangle_{n \in \omega}$ of families of open sets such that $\cap_{n \in C(x)} \overline{st(x, \mathcal{U}_n)} = \{x\}$ for each $x \in M$, where
$$C(x) = \{n \in \omega \ / \ \exists U \in \mathcal{U}_n \text{ with } x \in U\};$$
51. M is separable and there is a sequence $\langle \mathcal{C}_n \rangle_{n \in \omega}$ of point-star-open covers such that $\cap_n \overline{st(x, \mathcal{C}_n)} = \{x\}$ for each $x \in M$ and for each $x, y \in M$ and each $n \in \omega$ we have $y \in \overline{st(x, \mathcal{C}_n)}$ if and only if $x \in \overline{st(y, \mathcal{C}_n)}$;
52. M is separable and there is a sequence $\langle \mathcal{C}_n \rangle_{n \in \omega}$ of point-star-open covers such that $\cap_n \overline{st(x, \mathcal{C}_n)} = \{x\}$ for each $x \in M$ and for each $x \in M$ and each $n \in \omega$, $\mathrm{ord}(x, \mathcal{C}_n)$ is finite;
53. M is separable and hereditarily normal and there is a sequence $\langle \mathcal{C}_n \rangle_{n \in \omega}$ of point-star-open covers such that $\cap_n \overline{st(x, \mathcal{C}_n)} = \{x\}$ for each $x \in M$;
54. M is separable and there is a sequence $\langle \mathcal{U}_n \rangle_{n \in \omega}$ of families of open sets such that $\cap_{n \in C(x)} \overline{st(x, \mathcal{U}_n)} = \{x\}$ for each $x \in M$, and $\mathrm{ord}(x, \mathcal{C}_n)$ is countable for each $x \in M$ and each $n \in \omega$;
55. $M \times M$ has a countable sequence $\langle U_n : n \in \omega \rangle$ of open subsets, such that for all $(x, y) \in M \times M - \Delta$, there is $n \in \omega$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$;
56. For every subset $A \subset M$ there is a continuous injection $f : M \to Y$, where Y is a metrisable space, such that $f(A) \cap f(M - A) = \varnothing$;
57. For every subset $A \subset M$ there is a continuous $f : M \to Y$, where Y is a space with a quasi-regular-G_δ-diagonal, such that $f(A) \cap f(M - A) = \varnothing$;
58. M is weakly normal with a G^*_δ-diagonal;
59. M has a quasi-G^*_δ-diagonal and for every closed subset $A \subset M$ there is a countable family $\mathcal{G}$ of open subsets such that, for every $x \in A$ and $y \in X - A$, there is a $G \in \mathcal{G}$ with $x \in G$, $y \notin \overline{G}$;
60. M has a regular G_δ-diagonal;
61. M is submetrisable;
62. M is separable and monotonically normal;
63. $M \times M$ is monotonically normal;
64. M is monotonically normal and of dimension ≥ 2 or $M \approx \mathbb{S}^1$ or $\mathbb{R}$;
65. M is extremely normal;
66. M has property pp;
67. every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed in M;
68. every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is discrete in M;
69. M is a point-countable union of open subspaces each of which is metrisable;
70. M has a point-countable basis;
71. M is separable and M^ω is a countable union of metrisable subspaces;
72. $C_k(M, \mathbb{R})$ is Polish;
73. $C_k(M, \mathbb{R})$ is completely metrisable;
74. $C_k(M, \mathbb{R})$ is second countable;
75. $C_k(M, \mathbb{R})$ is a q-space;
76. $C_k(M, \mathbb{R})$ is Fréchet;
77. $C_k(M, \mathbb{R})$ is countably tight;
78. $C_k(M, \mathbb{R})$ is an $\aleph_0$-space;
79. $C_k(M, \mathbb{R})$ is cosmic;
80. $C_k(M, \mathbb{R})$ is analytic;
81. $C_p(M, \mathbb{R})$ has countable tightness;
82. $C_p(M, \mathbb{R})$ has countable fan tightness;
83. $C_p(M, \mathbb{R})$ is analytic;
84. $C_p(M, \mathbb{R})$ is hereditarily separable;
85. $C_p(M, \mathbb{R})$ (equivalently $C_k(M, \mathbb{R})$) is separable;
86. $[M, \$]$ is first countable;
87. $[M, \$]$ is countably tight;
88. $[M, \$]$ is sequential.

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X a topological space then

$C_k(X)$ = all continuous real-valued functions, compact-open topology

$C_p(X)$ = all continuous real-valued functions, pointwise topology

Sample Preliminary Result: X a q-space:

$C_k(X)$ analytic $\iff C_p(X)$ analytic $\iff X$ σ-compact and metrisable

Analytic means continuous image of a Polish space
(= continuous image of $\mathbb{P}$)

q-space means each point admits a sequence $\langle N_n \rangle$ of neighbourhoods such that $x_n \in N_n$ implies $\langle x_n \rangle$ clusters

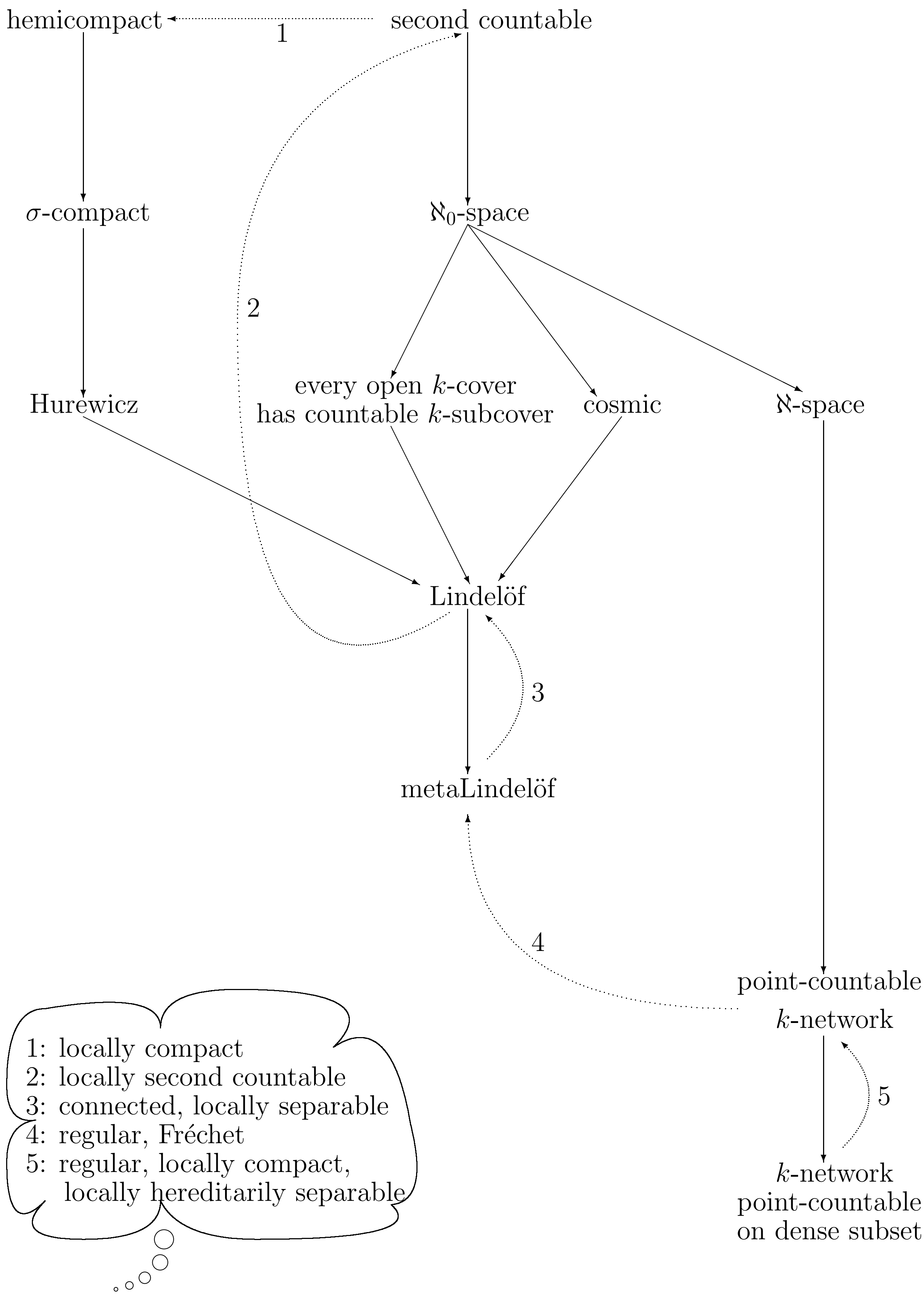
manifold $\Longrightarrow$ first countable $\Longrightarrow$ q-space

A manifold M is metrisable
$\iff M$ is σ-compact

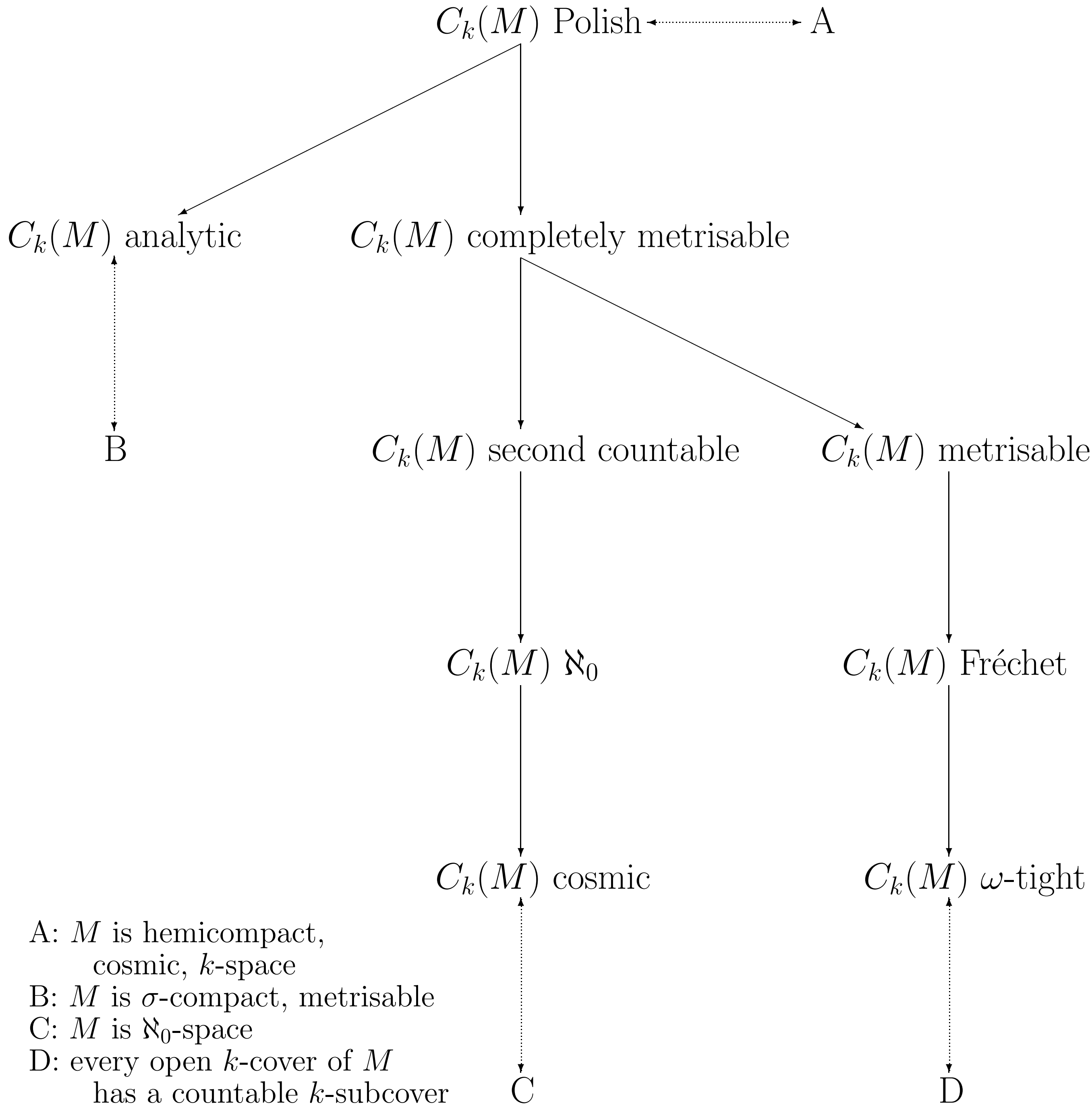
so

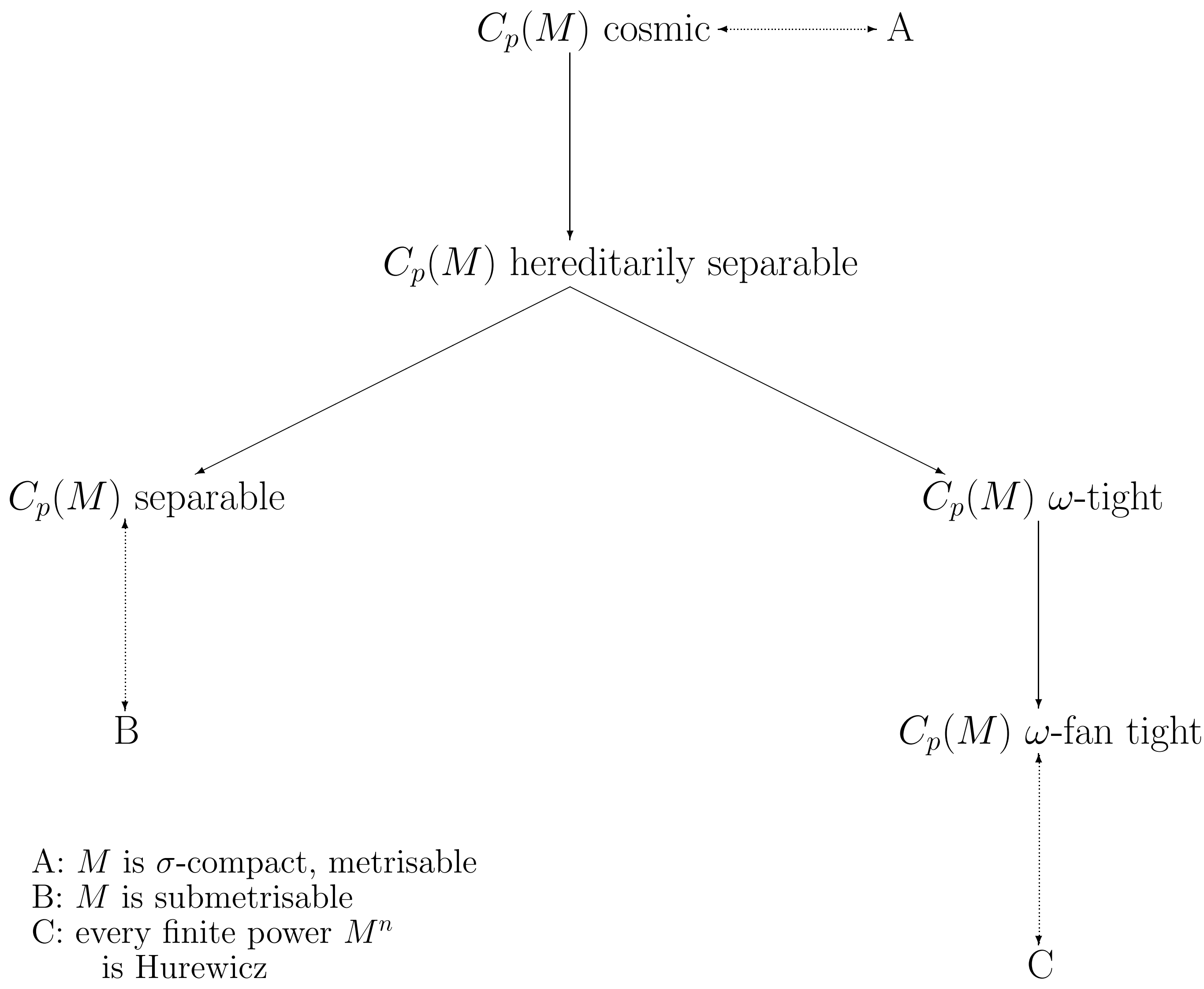
$\iff C_k(M)$ analytic

$\iff C_p(M)$ analytic


hemicompact
second countable
1
σ-compact
ℵ0-space
2
Hurewicz
every open k-cover
has countable k-subcover
cosmic
ℵ-space
Lindelöf
3
metaLindelöf
4
point-countable
k-network
5
k-network
point-countable
on dense subset
1: locally compact
2: locally second countable
3: connected, locally separable
4: regular, Fréchet
5: regular, locally compact,
locally hereditarily separable

M a manifold


$C_k(M)$ Polish
A
$C_k(M)$ analytic
$C_k(M)$ completely metrisable
B
$C_k(M)$ second countable
$C_k(M)$ metrisable
$C_k(M)$ $\aleph_0$
$C_k(M)$ Fréchet
$C_k(M)$ cosmic
$C_k(M)$ ω-tight
A: M is hemicompact, cosmic, k-space
B: M is σ-compact, metrisable
C: M is $\aleph_0$-space
D: every open k-cover of M has a countable k-subcover
C
D


$C_p(M)$ cosmic
A
$C_p(M)$ hereditarily separable
$C_p(M)$ separable
$C_p(M)$ ω-tight
B
$C_p(M)$ ω-fan tight
A: M is σ-compact, metrisable
B: M is submetrisable
C: every finite power M^n is Hurewicz
C

X, a space, is:

- *hemicompact* if $\exists\langle K_n\rangle$, compacta, $\forall K$ compact $\exists n$: $K \subset K_n$;
- *Hurewicz* if $\forall\langle \mathcal{U}_n\rangle$, open covers, $\exists\langle \mathcal{V}_n\rangle$: $\cup_{n\in\omega}\mathcal{V}_n = X$ and $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ $\forall n$;
- *an* $\aleph_0$*-space* if it has a countable *k-network*, i.e. collection $\mathcal{N}$: $\forall K$, compact, $\forall U$, open, with $K \subset U$ $\exists N \in \mathcal{N}$ with $K \subset N \subset U$;
- *an* $\aleph$*-space* if it has a σ-locally finite k-network;
- *cosmic* if it has a countable *network*, i.e. as for k-network but replace K by a point;
- *a k-space* if $A \subset X$ closed whenever $A \cap K$ closed $\forall K$ compact;
- *Fréchet* if $\forall x \in \overline{A}$ $\exists\langle x_n\rangle$ in A converging to x;
- ω*-tight* if $\forall x \in \overline{A}$ $\exists B \subset A$: $x \in \overline{B}$ and B countable;
- ω*-fan tight* if $\forall x \in \cap_{n\in\omega}A_n$ $\exists$ finite $B_n \subset A_n$: $x \in \overline{\cup_{n\in\omega}B_n}$;
- *submetrisable* if the topology has a metrisable subtopology.
- A *k-cover* of X: a collection $\mathcal{S}$ of subsets with each compactum in X a subset of some member of $\mathcal{S}$.