Weakly compact Lie groups

C. J. Atkin

12th February, 2004

§1. The problem.

Some 17 years ago I pointed out some facts about the geometry of the "classical groups of compact type" in infinite-dimensional Hilbert space H. Briefly, I have in mind the following groups:

$\boldsymbol{U}(H)$ or simply $\boldsymbol{U}$, the group of all unitary operators in H;
$\boldsymbol{O}(H)$ or $\boldsymbol{O}$, the group of all orthogonal operators;
$\boldsymbol{Sp}(H)$ or $\boldsymbol{Sp}$, the full symplectic group;
$\boldsymbol{UC}(H) := \boldsymbol{U} \cap \boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ denotes the set of Fredholm operators in H; thus $\boldsymbol{UC}(H)$ is the subgroup of $\boldsymbol{U}$ consisting of unitary operators differing from the identity by a compact operator; and, similarly, $\boldsymbol{SpC}(H)$.

The "Fredholm orthogonal" group $\boldsymbol{OC}(H)$ has two components, and it is natural to call the principal component $\boldsymbol{SOC}(H)$, the "special Fredholm orthogonal group". All the other groups are connected.

The groups $\boldsymbol{UC}$ and $\boldsymbol{SOC}$ have non-trivial fundamental group ($\mathbb{Z}$ and $\mathbb{Z}/(2)$ respectively), so there are universal covering groups $\widetilde{\boldsymbol{UC}}$ and $\boldsymbol{SpinC}$.

In each of these cases the topology is induced from the uniform operator-norm; they are infinite-dimensional Lie groups modelled on Banach spaces, and carry a natural left-and right-invariant Finsler structure induced from the operator-norm on the Lie algebra.

My principal observation was this: in *most* of these groups, the exponential map is onto and any point is joined to the identity by a segment of a one-parameter group minimizing the Finsler length — a "minimizing geodesic"; and the Finsler diameter is π (and is attained).

The *exceptions* are

$\boldsymbol{O}$, in which the exponential is not onto and the (attained) diameter is 2π;

$\widetilde{\boldsymbol{UC}}$, which has diameter 2π, and in which, if H is separable, not all distances can be minimized by a rectifiable path (although the exponential is onto). If H is non-separable, there is always a minimizing *geodesic*. The finite-dimensional situation, in which the unitary group has a universal cyclic covering, is no guide here, since there is no determinant defined on $\boldsymbol{UC}$. Indeed, the universal cover of the finite-dimensional unitary group is unbounded;

$\boldsymbol{SpinC}$, which has diameter 2π.

The proofs of these results involved tedious spectral analysis and were not geometrically interesting. There are other results that may be derived, for instance for the "Calkin" quotient groups $\boldsymbol{U/UC}$ etc., but they are equally ad hoc. One may, however, reasonably ask whether similar results are available for other groups, such as closed subgroups of $\boldsymbol{U}$, for which spectral methods may not be applicable. A simple example is a 1-parameter subgroup Γ of $\boldsymbol{U}(L^2(\mathbb{R}))$ which contains a translation S; then $S, S^2, S^3, \ldots$ obviously tend to infinity in Γ, but all shifts are unitarily equivalent. Other examples are furnished by subgroups with Lie algebras in Schatten classes; for instance, one might consider the subgroup $\boldsymbol{UL^1C}$ of $\boldsymbol{U}$ consisting of operators that differ from the identity by something in trace-class. This is a Banach Lie group in a topology finer than the subspace topology.

§2. The results.

Let $\mathfrak{G}$ be a connected Banach Lie group with Lie algebra $\mathfrak{g}$, normed by $\|\|$. It is *adjoint-bounded* if there is some constant K such that $\|\mathrm{Ad}(x)\| \leq K$ for all $x \in \mathfrak{G}$. Here $\mathrm{Ad} : \mathfrak{G} \longrightarrow L(\mathfrak{g})$ is the adjoint representation, and L denotes the Banach algebra of bounded operators in $\mathfrak{g}$ in operator-norm. In that case, $\mathfrak{g}$ may be renormed so that its norm is submultiplicative and all the adjoint operators $\mathrm{Ad}(x)$ are isometric; the induced left-invariant Finsler structure on $\mathfrak{G}$ is then right-invariant, as is the Finsler metric d it defines. I may describe the Finsler structure, norm and distance as "normalized". In the cases listed above these conditions are automatically satisfied.

Lemma 2.1. *If $\mathfrak{G}$ is adjoint-bounded and the Finsler structure is normalized, then the exponential map is distance-nonincreasing,* $d(\exp\xi, \exp\eta) \leq \|\xi - \eta\|$.

Proof. It is only necessary to check that the tangent map of $\exp$ is of norm not exceeding 1. But $T_\zeta \exp = R^e_{\exp\zeta} \circ \Phi(\mathrm{ad}(\zeta))$, where $R^e_{\exp\zeta}$ is right-translation from $\mathfrak{g} = T_e\mathfrak{G}$ to $T_{\exp\zeta}\mathfrak{G}$ and Φ is the holomorphic function

$$\Phi(z) := \frac{e^z - 1}{z} = \prod_{k=1}^{\infty}\left(\frac{\exp(2^{-k}z) + 1}{2}\right)$$

(since $\dfrac{\exp z - 1}{z} = \dfrac{\left(\exp(\frac{1}{2}z) + 1\right)}{2} . \dfrac{\left(\exp(\frac{1}{2}z) - 1\right)}{\frac{1}{2}z}$ etc., and $\dfrac{\exp(2^{-k}z) - 1}{2^{-k}z} \to 1$ as $k \to \infty$). As $R^e_{\exp\zeta}$ is isometric, and $\exp(2^{-k}\mathrm{ad}(\zeta))$ is an isometry for each k, the result follows. □

Now describe $\mathfrak{G}$ as *ergodic* (I don't propose this as a serious name, for obvious reasons, but only as a temporary reference) if each of the adjoint operators $\mathrm{Ad}(x) \in L(\mathfrak{g})$ is mean-ergodic. This condition is clearly satisfied if $\mathfrak{g}$ is reflexive, and therefore for subgroups of $\boldsymbol{U}$ defined by subalgebras in the Schatten class of exponent $p > 1$; it is somewhat less obviously satisfied by closed Lie subgroups of

$\boldsymbol{UC}$ and their covering groups and by $\boldsymbol{UL^1C}$ and similar groups. In fact, I suspect, indeed I am fairly convinced, that there is a useful condition which covers all these cases, to do with weak compactness of the operators $\mathrm{ad}(\xi) \in L(\mathfrak{g})$ for each $\xi \in \mathfrak{g}$ (hence my title), but I have not yet found a satisfactory formulation.

Theorem 2.2. *Let $\mathfrak{G}$ be a connected adjoint-bounded ergodic Banach Lie group, and let $\epsilon > 0$. Then, for any $x \in \mathfrak{G}$, there is a one-parameter subgroup Γ of $\mathfrak{G}$ containing x such that the distance of x from e in Γ does not exceed $(1+\epsilon)d(x,e)$.*

This may seem a little unsatisfactory in comparison with most of the original examples; but recall $\widetilde{\boldsymbol{UC}}$, in which exact minimization of distances is not always possible, and which is ergodic. Indeed, a moment's thought will convince you that even an abelian Banach Lie group need not allow exact minimization.

It is not possible to give the proof in detail, but the only real *idea* that is involved, apart from routine technicalities, is this. Let

$$R := \sup\{r > 0 : \text{the assertion holds for } d(x,e) \leq r\}.$$

There is a constant $\delta > 0$ (specific to $\mathfrak{G}$) such that, over distances less than 2δ, segments of left translates of one-parameter groups approximate distances within a factor of $(1+\epsilon)$. So now suppose that $d(x,e) < R + \delta$.

Conjugation by x preserves distances in $\mathfrak{G}$. Thus every point of the conjugacy class of x is at distance $d(x,e)$ from e. It is a sort of submanifold through x, with tangent space $T_e R_x^e((\mathrm{Ad}(x) - I)\mathfrak{g})$. If $(\mathrm{Ad}(x) - I)\mathfrak{g}$ had dense closure in $\mathfrak{g}$, it would, therefore, be possible to move closer to e from x along a one-parameter subgroup "almost" lying in the conjugacy class. This is absurd. But as $\mathrm{Ad}(x)$ is mean ergodic, this means that $\mathrm{Ad}(x) - I$ is not one-one and there is a projection of norm 1 on its kernel; in turn, it follows from Lemma 1 that travelling in the direction of the kernel will almost-minimize the distance to the ball of radius R. The result follows.

There are several remarks. The first (which was in fact the motivation for seeking a result of the kind above) is that, as one might expect, groups such as $\boldsymbol{UL^pC}$, which have finer topologies than the uniform topology, are *unbounded*, unlike their analogues in the uniform topology. In fact one may deduce a formula for distances in them in terms of the spectral decompositions of their elements; I suppose this could also be done by spectral theory, but far more laboriously.

The second is that the argument applies to Lie subgroups as well, which is not a negligible generalization.

The third is that exact distance-minimization may be deduced in certain circumstances, for instance in Lie subgroups of $\boldsymbol{UL^pC}$ for $p > 1$.