The Slicing Problem

Conjecture: There exists $c > 0$ (independent of dimension), such that any $K \subset \mathbb{R}^n$ a convex, centrally symmetric body of volume one, has at least one hyperplane section, with

$$\mathrm{Vol}_{n-1}(K \cap H) > \frac{1}{c}.$$

- There are only estimates which depend on the dimension:

  $c_n < c'\sqrt{n}$ (straightforward)

  $c_n < c'\sqrt[4]{n}\log n$ [Bourgain, '88]

  for general convex bodies.

- In some particular cases, the conjecture is known to be true:
  - unconditional bodies.
  - zonoids, duals to zonoids.
  - duals to bodies with finite volume ratio.

    [Bourgain, Milman-Pajor, Ball '88-'89]
  - Schatten class bodies.

    [Dar '96, König, Meyer, Pajor '98]

Integral Formulation

- For any K of volume 1,

$$\frac{1}{Vol(K\cap v^{\perp})}\approx\sqrt{\int_K \langle x,v\rangle^2\,dx}$$

where v is a unit vector [Hensley '80].

- The inertia matrix M_K of K is defined as

$$\langle M_K u,v\rangle=\int_K \langle x,u\rangle\langle x,v\rangle\,dx$$

Equivalent formulation for the slicing problem:

Is $L_K=det(M_K)^{\frac{1}{2n}}$ bounded by some constant, independent of dimension?

L_K is called the "isotropic constant of *K*".
If M_K is a scalar matrix, *K* is called "isotropic".

- K isotropic implies $M_K=L_K^2\,Id$ and

$$nL_K^2=\int_K |x|^2\,dx$$

$$\sqrt{n}L_K\approx\int_K |x|\,dx$$

Mass distribution

- Another equivalent formulation of the problem:
  K isotropic, volume one.
  $\bar{D}$ a Euclidean ball, volume one.

Does there exist C > 0 such that

$$Vol(K \cap c\bar{D}) > \frac{1}{2} Vol(K)?$$

- A relaxation of the problem:

$$d(K,T) = \inf\left\{ab > 0; \frac{1}{b}K \subset L(T) \subset aK; L \text{ is linear}\right\}$$

is the Banach-Mazur distance.

- An isomorphic version of the slicing problem:

Do there exist c_1, $c_2 > 0$ such that for any dimension *n*, for any $K \subset \mathbb{R}^n$, there exists $T \subset \mathbb{R}^n$ with

$$d(K,T) < c_1$$
$$L_T < c_2?$$

Main result

- The isomorphic slicing problem is correct, up to a logarithmic factor.

Theorem:

For any $K \subset \mathbb{R}^n$, there exists $T \subset \mathbb{R}^n$ such that

$$d(K,T) < c_1 \log n$$
$$L_T < c_2$$

- This logarithmic factor comes from Pisier's estimate for the Rademacher projection.

  If K has a non-trivial type, no need for logarithm.

- Proof involves log-concave functions.

  - f is log-concave if **log f** is concave on its support.
  - f is s-concave if $\mathbf{f^{1/s}}$ is concave.

  Any s-concave is also log-concave.

Log concave functions

- L_f - the isotropic constant - may be defined.

- If f is log-concave and even, then

$$\| x \|_f = \left(\int_0^\infty f(rx) r^{n+1} dr \right)^{-1/(n+2)}$$

is a norm. Its unit ball is denoted by K_f.

Properties:

- related mass distribution: $L_f \approx L_{K_f}$
- If f is s-concave, then

$$d(K_f, supp(f)) < c\frac{s}{n}$$

Aim:

Given $K \subset \mathbb{R}^{\mathbf{n}}$, find a function f supported exactly on K such that

- $L_f <$ const
- f is αn-concave, for α not large.

$\Rightarrow$ K_f is the desired body.

Constructing a function on K

- Define M' such that

$$Vol\left(K \cap \frac{1}{M'}D\right) = \frac{1}{2}Vol\left(\frac{1}{M'}D\right)$$

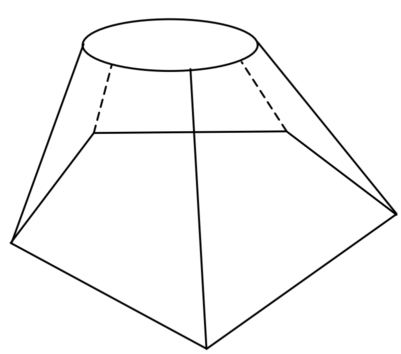
- The function is:

$$f^{\frac{1}{\alpha n}}(x) = \sup\left\{0 < t < 1; x \in t\left[\frac{1}{M'}D \cap K\right] + (1-t)K\right\}$$

and is αn-concave.

Main ingredient of the proof: For relatively small α, most of the mass of f is not far from the origin:

$$\int_K f(x)dx < 2Vol\left(K \cap \frac{1}{M'}D\right)$$



Mixed Volumes

- Sketch of proof:

$$\int_K f(x)dx = \int_0^1 Vol\left(t^{\frac{1}{\alpha n}} \left[K \cap \frac{1}{M'} D \right] + \left(1 - t^{\frac{1}{\alpha n}} \right) K \right) dt$$

- Expand into mixed volumes:

$$Vol\left(s\left[K \cap \frac{1}{M'} D \right] + (1-s) K \right) = \sum_{i=0}^{n} \binom{n}{i} V_i s^i (1-s)^{n-i}$$

- Simplifying, we obtain

$$\int_K f(x)dx \approx V_0 \left[1 + \sum_{i=1}^{n} \left(\frac{1}{c\alpha} \left(\frac{V_i}{V_0} \right)^{1/i} \right)^i \right]$$

- By Alexandrov-Fenchel, $\left(\frac{V_i}{V_0} \right)^{1/i} \leq \frac{V_1}{V_0}$.

  Hence enough to take any $\alpha > 2c\frac{V_1}{V_0}$.

Rademacher Projection

- Need to estimate $\frac{V_1}{V_0}$:

$$V_1 = V\left(K, \frac{1}{M'} D \cap K, \ldots, \frac{1}{M'} D \cap K \right) \le V\left(K, \frac{1}{M'} D, \ldots, \frac{1}{M'} D \right)$$

and therefore,

$$\frac{V_1}{V_0} \le 2M'(K) M^*(K) \le 4M(K) M^*(K)$$

where M*(K) is the mean width of K, and M(K) is the mean width of its polar.

- Pisier's result: There exists a linear image of K, for which $M(K)\,M^*(K) < c \log n$.

<u>Summary</u>:

- $K \subset \mathbb{R}^{\mathbf{n}}$ arbitrary, $\tilde{K}$ a linear image.
- Construct an αn-concave function f on $\tilde{K}$.
- If $\alpha = c \log n$, then $L_f <$ const. Set $T = K_f$.
- $L_T <$ const, $d(K,T) < c \log n$.