Ranks, derivatives, invariants, some reverse math: A study of an elephant

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Once upon a time there was a certain raja who called to his servant and said, ‘Come, good fellow, go and gather together in one place all the men of Savatthi who were born blind... and show them an elephant...’ [1]

1. The men who observed a foot exclaimed, ‘an elephant is like a tree’.
2. The men who were presented with the trunk said, ‘an elephant is like a linear ordering’.
3. Those who only knew the teeth said, ‘an elephant is like a countable topological space’.
4. Those who knew the back: ‘it is like a Boolean algebra’.
5. And those who knew an ear: ‘an elephant is like an Abelian p-group (where p is a prime)’.

To give more flavour to the story, we give some details. All structures have universe a subset of $\mathbb{N}$.

- Well-orderings are sometimes called ordinals.
- Trees are equipped with predicates for level and a predecessor function, and so can be always taken to be collections of strings of natural numbers.
- The topological spaces are *very countable*: countable spaces, equipped with a countable basis for the topology (so this is a two-sorted structure). We usually assume spaces are Hausdorff. We only lose a little if we take metrizable spaces instead.

Each class is separated into the well and the ill.

- A *reduced* group is one which does not embed a divisible group.
- A *superatomic Boolean algebra* is one which does not embed the atomless Boolean algebra.
- A topological space is *scattered* if it does not have a subset which is dense in itself.

These notions are better understood using the Cantor-Bendixon derivative (grooming the elephant).

- If X is a space, then $X' = X \setminus \{\text{isolated points}\}$.
- If T is a tree, then $T' = T \setminus \{\text{leaves}\}$.
- If B is a Boolean algebra, then $B' = B/\langle\text{atoms}\rangle$.
- If G is a group, then $G' = pG$.

Iterating continuously eventually yields an (the) ill-founded object, or an empty one, if the original one was well-founded. The length of the iteration is the *rank* of the original object. The rank, together with associated dimensions, constitute *invariants* for isomorphism or bi-embeddability.

Notions of reductions and similarity:

1. $\mathcal{A} \subset 2^{\mathbb{N}}$ is "effective (or lightface) Wadge" reducible to $\mathcal{B} \subset 2^{\mathbb{N}}$ if there is some computable function $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $f^{-1}\mathcal{B} = \mathcal{A}$.

2. (Calvert - Cummins - J. Knight - S. Miller) A *computable mapping* of structures is a map $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for some Turing functional Φ, $f(A) = B$ iff for all ϕ, $\phi \in D(A)$ iff there is some $a \subset D(A)$ such that $(a, \phi) \in \Phi$.

   A *computable* embedding is a computable map which preserves both $\cong$ and $\ncong$. If a computable map preserves $\cong$ then it also preserves embeddability, $\preccurlyeq$; we are also interested in maps which preserve $\preccurlyeq$ and $\not\preccurlyeq$.

Let α be an ordinal.

1. $T(\alpha)$ is the tree of descending sequences from α.

2. $B(\alpha) = \mathrm{Int}(\omega^\alpha)$.

3. $G(\alpha) = G(T(\alpha))$, where $G(T)$ is the Abelian group generated by T satisfying $p(\sigma^\frown x) = \sigma$.

4. $X(\alpha)$ is $\omega^\alpha + 1$, equipped with the order topology.

All of these are computable embeddings which preserve rank (and so also $\preccurlyeq$, $\not\preccurlyeq$) and are effective Wadge reductions too.

Obesity: an elephant can be fat.

1. The *fat tree* of rank α: for every $\sigma \in T$ and for every $\beta < \mathrm{rk}_T(\sigma)$, σ has infinitely many immediate successors on T with rank β.

2. The *fat group* of rank α is the group of length α, where every Ulm dimension is ω.

Let T be a tree.

- $T \mapsto \mathrm{KB}(T^\infty)$ is a computable map which preserves well- and ill-foundedness, $\cong, \preccurlyeq, \not\preccurlyeq$ (we always have $\mathrm{KB}(T^\infty) \cong \omega^{\mathrm{rk}(T)} + 1$).
- $B(T)$ is the *tree algebra*, the algebra of sets generated by the cones of T.

Let B be a Boolean algebra. $T(B)$ is the the tree of sequences which code embeddings of uniformly finitely branching trees into B. $B \mapsto \mathrm{KB}(T(B)^\infty)$ is a computable embdding which preserves $\preccurlyeq$, $\not\preccurlyeq$, and well- and ill-foundedness.

Simpson writes:

> "... ATR_0 is the weakest set of axioms which permits the development of a decent theory of countable ordinals." [2]

While trying to separate between well- and ill-foundedness requires Π^1_1-comprehension, if we know that the given objects are well-founded, ATR_0 suffices.

Consider the following statement for a class of structures $\mathcal{A}$:

1. $\mathsf{COMP}(\mathcal{A})$: Any two objects in A are comparable.
2. $\mathsf{EQU{=}ISO}(\mathcal{A})$: The collection of $\mathcal{A}$-isomorphism classes, equipped with $\preccurlyeq$, is a partial ordering.
3. $\mathsf{WQO}(\mathcal{A})$: the partial ordering of $\mathcal{A}$-bi-embeddability classes under $\preccurlyeq$ forms a well-quasi ordering.
4. $\mathsf{RK}(\mathcal{A})$: every object in $\mathcal{A}$ is ranked.
5. $\exists\text{-}\mathsf{ISO}(\mathcal{A})$: whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{A}$, then the set $\{(n,m) : A_n \cong A_m\}$ exists.
6. $\exists\text{-}\mathsf{EMB}(\mathcal{A})$: if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{A}$, then the set $\{(n,m) : A_n \preccurlyeq A_m\}$ exists.

Theorem 1 (RCA_0)**.** *For* φ *one of the statements above, and* $\mathcal{A}$ *one of the well-founded classes discussed, if* $\varphi(\mathcal{A})$ *makes sense and is true, then it is equivalent to* ATR_0*.*

Questions:

1. Suppose that there is a reduction from a class $\mathcal{A}$ to the ordinals. Does ATR_0 follow?
2. Are there more aspects of the elephant?
3. Is there a general theory of the elephant?

References

[1] Tipitaka, Sutta Pitaka, Khudaaka Nikaya, Udana 68-69.

[2] Simpson, S., *Subsystems of second order arithmetic*, Springer, 1999.