# Generalised High Degrees Have the Complementation Property

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**Definition.** A Turing degree $\mathbf{d}$ has the *complementation property* if every degree $\mathbf{a} < \mathbf{d}$ has a complement in $\mathcal{D}(\leqslant \mathbf{d})$: some $\mathbf{b}$ such that

$$\mathbf{a} \vee \mathbf{b} \ = \ \mathbf{d}$$

and

$$\mathbf{a} \wedge \mathbf{b} \ = \ \mathbf{0}.$$

**Definition.** A Turing degree $\mathbf{d}$ is *generalised high* if

$$\mathbf{d}' \ = \ (\mathbf{d} \vee \mathbf{0}')'.$$

**Theorem 1.** *Every generalised high degree has the complementation property.*

Let $\mathbf{d}$ be generalised high and let $\mathbf{a} < \mathbf{d}$.

The construction of a complement $\mathbf{b}$ for $\mathbf{a}$ below $\mathbf{d}$ is divided into cases, depending on the 'distance' between $\mathbf{a}$ and $\mathbf{d}$.

**Case One**: $\mathbf{a} \in$ **GL**$_2$.

(That is to say, $\mathbf{a}'' = (a \vee \mathbf{0}')'$.)

Complement constructed by Posner [1977].

**Case Two**: $\mathbf{a} \notin$ **GL**$_2$, and for some $\mathbf{a} \leqslant \mathbf{c} < \mathbf{d}$,

$$\mathbf{d} \in \Delta_2(\mathbf{c}).$$

(What is used is: there is some function $d \leqslant_T \mathbf{d}$ which is dominated by no function recursive in $\mathbf{a}$.)

The construction is an elaboration on Slaman and Steel's uniform construction of complements below $\mathbf{0}'$. The fact that $\mathbf{a} \notin$ **GL**$_2$ is used to construct a tree, akin to Slaman and Steel's, recursively in $\mathbf{a}$, using a guess as to how long one should wait for splits.

**Case Three**: None of the above.

Here, independently of $\mathbf{a}$, we construct a minimal degree $\mathbf{b}$ below $\mathbf{d}$. We use a function recursive in $\mathbf{a}$, which 'dominates the construction', to show that $\mathbf{d} \leqslant (\mathbf{a} \vee \mathbf{b})'$.

It then follows that $\mathbf{d} = \mathbf{a} \vee \mathbf{b}$.