Admissible uncountable computable model theory

Noam Greenberg

Victoria University of Wellington

16th August 2013

Why uncountable?

- “because it’s there”: study familiar uncountable objects using the tools of computability. One can essentially ask of anything, “how complicated is this?”
- To understand something, generalise. New light is shed on countable computability by comparing it with its uncountable siblings.

So why admissibility?

- because it's there (already well studied)
- most closely resembles countable computability (techniques / intuition)
- one method fits all sizes

Drawbacks:

- requires set-theoretic assumptions for smooth development
- only one size at a time

So how does it go?

Various approaches yield the same notion of computability:

- A Turing machine with an uncountable tape which is allowed to run with ordinal time.
- Equational deduction calculus.
- Definability (descriptive complexity).

With time one develops an understanding similar to the Church-Turing thesis.

Countable computability

Two ideas:

- Σ_1 (computably enumerable) is a good basic concept. The rest follows.
- A natural way to formalise mathematical practice is by using set theory.

Let $\mathcal{H}_\omega = (H_\omega; \in, \text{all parameters})$.

Definition

A subset of H_ω is c.e. if it is $\Sigma_1(\mathcal{H}_\omega)$.

(this coincides with traditional definitions for subsets of ω).

Computable sets and functions

Definition

- A subset of H_ω is computable if it is c.e. and co-c.e.
- A partial function $f: H_\omega \to H_\omega$ is partial computable if its graph is c.e.
- A partial computable function is (total) computable if its domain is computable.

Proposition

A set $A \subseteq H_\omega$ is computable if and only if its characteristic function 1_A is computable.

Bounded quantification

Proposition

If $A \subseteq H_\omega$ is c.e. and $a \in H_\omega$, then

$$\{x \in H_\omega \ : \ \forall y \in a \ \ [(x, y) \in A]\}$$

is c.e.

Recursion

This is a main tool.

Proposition

Let $I\colon H_\omega \to H_\omega$ *be computable. There is a unique function* $f\colon \omega \to H_\omega$ *such that for all* n, $f(n) = I(f\restriction_n)$. *This function* f *is computable.*

But I like only numbers

Does it matter if we use ω or H_ω?

Proposition

- *If A and B are computable subsets of H_ω then there is a computable bijection between A and B.*
- *ω and H_ω are computable sets.*

Why ‘computable enumerability’?

Proposition

The following are equivalent for a non-empty subset A of H_ω:

- *A is the domain of a partial computable function.*
- *A is the range of a computable function.*
- *A is c.e.*

The halting problem

Proposition

The set

$$\{(\psi, a) : \psi \text{ is a } \Sigma_1 \text{ formula, } a \in H_\omega \text{ and } H_\omega \models \psi(a)\}$$

is c.e.

Proof.
$H_\omega \models \psi(a)$ if and only if there is a transitive set $M \in H_\omega$ such that $M \models \psi(a)$. The latter is computable (for *all* formulas ψ). □

Note that the collection of Σ_1 formulas is computable. We let W_n be the c.e. subset of H_ω defined by the n^{th} Σ_1 formula. The halting problem (the universal c.e. set) is thus

$$\bigoplus_n W_n = \{(a, n) : a \in W_n\}.$$

Another example

Similarly we can effectively enumerate partial computable functions $\langle \varphi_n \rangle$ (really indexed by Σ_1 formulas).

Proposition

If $f(x, y)$ is a partial computable function, then there is a (total) computable function g such that for all a, $\varphi_{g(a)} = f(a, -)$.

Proof.

Let $\psi(x, y, z)$ be a Σ_1 formula defining the graph of f. The set

$$\{\psi(a, y, z) \,:\, a \in H_\omega\}$$

is computable. Then $g(a)$ is (the natural number code, if you like, of) $\psi(a, y, z)$. $\square$

Relatively c.e.

Definition

A c.e. operator is a c.e. set Ψ of pairs (σ, a) where $a \in H_\omega$ and $\sigma \in 2^{<\omega}$. For $A \in 2^\omega$, we let

$$\Psi(A) = \{a : (\sigma, a) \in \Psi \text{ for some } \sigma \prec A\}.$$

Proposition

The following are equivalent for sets $A, B \subseteq H_\omega$:

1. *There is a c.e. operator Ψ such that $B = \Psi(A)$;*
2. *B is $\Sigma_1(\mathcal{H}, A)$.*

We say that B is c.e.^{A}.

Turing reducibility

A Turing operator is a c.e. set $\Psi \subseteq 2^{<\omega} \times 2^{<\omega}$. For $A \in 2^{\omega}$, we let

$$\Psi(A) = \bigcup \{\tau : (\sigma, \tau) \in \Psi \text{ for some } \sigma \prec A\}.$$

Proposition

The following are equivalent for $A, B \subseteq H_\omega$:

1. *There is a Turing operator Ψ such that $B = \Psi(A)$;*
2. *B is c.e.^{A} and co-c.e.^{A}.*

We write $B \leqslant_{\mathrm{T}} A$.

Uncountable computability

... and so on and so forth.

We can make the same definitions for $\kappa > \omega$. Replace H_ω by H_κ.

We assume that κ is regular and that there is a computable bijection between κ and H_κ. Then there are no changes to the theory.

(For $\kappa = \omega_1$ this is equivalent to $\mathbb{R} \subset L$, in which case $H_{\omega_1} = L_{\omega_1}$).

κ-computability

Let $\mathcal{H}_\kappa = (H_\kappa; \in, \text{all parameters})$.

Definition

A subset of H_κ is c.e. if it is $\Sigma_1(\mathcal{H}_\kappa)$.

Computable sets and functions

Definition

- A subset of H_κ is computable if it is c.e. and co-c.e.
- A partial function $f\colon H_\kappa \to H_\kappa$ is partial computable if its graph is c.e.
- A partial computable function is (total) computable if its domain is computable.

Proposition

A set $A \subseteq H_\kappa$ is computable if and only if its characteristic function 1_A is computable.

Bounded quantification

Proposition

If $A \subseteq H_\kappa$ is c.e. and $a \in H_\kappa$, then

$$\{x \in H_\kappa \ : \ \forall y \in a \ \ [(x,y) \in A]\}$$

is c.e.

Recursion

This is a main tool.

Proposition

Let $I\colon H_\kappa \to H_\kappa$ *be computable. There is a unique function* $f\colon \kappa \to H_\kappa$ *such that for all* n*,* $f(n) = I(f\restriction_n)$*. This function* f *is computable.*

But I like only ordinal numbers

Does it matter if we use κ or H_κ? Generally, yes. Under our assumption, no.

Proposition

- *If A and B are computable subsets of H_κ then there is a computable bijection between A and B.*
- *κ and H_κ are computable sets.*

Why ‘computable enumerability’?

Proposition

The following are equivalent for a non-empty subset A of H_κ:

- *A is the domain of a partial computable function.*
- *A is the range of a computable function.*
- *A is c.e.*

The halting problem

Proposition

The set

$$\{(\psi, a) : \psi \text{ is a } \Sigma_1 \text{ formula, } a \in H_\kappa \text{ and } H_\kappa \models \psi(a)\}$$

is c.e.

Proof.
$H_\kappa \models \psi(a)$ if and only if there is a transitive set $M \in H_\kappa$ such that $M \models \psi(a)$. The latter is computable (for *all* formulas ψ). □

Note that the collection of Σ_1 formulas is computable. We let W_n be the c.e. subset of H_κ defined by the n^{th} Σ_1 formula. The halting problem (the universal c.e. set) is thus

$$\bigoplus_n W_n = \{(a, n) : a \in W_n\}.$$

Another example

Similarly we can effectively enumerate partial computable functions $\langle \varphi_n \rangle$ (really indexed by Σ_1 formulas).

Proposition

If $f(x, y)$ is a partial computable function, then there is a (total) computable function g such that for all a, $\varphi_{g(a)} = f(a, -)$.

Proof.

Let $\psi(x, y, z)$ be a Σ_1 formula defining the graph of f. The set

$$\{\psi(a, y, z) \,:\, a \in H_\kappa\}$$

is computable. Then $g(a)$ is (the natural number code, if you like, of) $\psi(a, y, z)$. □

Relatively c.e.

Definition

A c.e. operator is a c.e. set Ψ of pairs (σ, a) where $a \in H_\kappa$ and $\sigma \in 2^{<\kappa}$. For $A \in 2^\kappa$, we let

$$\Psi(A) = \{a : (\sigma, a) \in \Psi \text{ for some } \sigma \prec A\}.$$

Proposition

The following are equivalent for sets $A, B \subseteq H_\kappa$:

1. *There is a c.e. operator Ψ such that $B = \Psi(A)$;*
2. *B is $\Sigma_1(\mathcal{H}, A)$.*

We say that B is c.e.^{A}.

Turing reducibility

A Turing operator is a c.e. set $\Psi \subseteq 2^{<\kappa} \times 2^{<\kappa}$. For $A \in 2^{\kappa}$, we let

$$\Psi(A) = \bigcup \{\tau \,:\, (\sigma, \tau) \in \Psi \text{ for some } \sigma \prec A\}.$$

Proposition

The following are equivalent for $A, B \subseteq H_{\kappa}$:

1. *There is a Turing operator Ψ such that $B = \Psi(A)$;*
2. *B is c.e.^{A} and co-c.e.^{A}.*

We write $B \leqslant_{\mathrm{T}} A$.

Computable model theory

The Turing degree of a structure of size κ is the κ-Turing degree of its atomic (or quantifier-free) diagram.

Examples

($\kappa = \omega_1$, so CH):

- $(\mathbb{R}; +, \cdot, <, 0, 1)$.
- $(\mathbb{C}; +, 0, 1, \exp)$ and in fact, with all entire analytic functions at once.

Saturated models

Recall that under our assumption, $\kappa = 2^{<\kappa}$.

Proposition

Let T be a complete theory, $|T| < \kappa$. The saturated model of T of size κ has a decidable presentation.

Intrinsic relations

Let $\mathcal{M}$ be a structure, and let R be a relation on $\mathcal{M}$. R is relatively intrinsically Σ_α if for any isomorphism $f \colon \mathcal{M} \to \mathcal{N}$, $f[R]$ is $\Sigma_\alpha(\mathcal{N})$.
Following [Ash,Knight,Mannase,Slaman / Chisholm]:

Proposition (Greenberg,Knight for $\alpha = 1$; Carson,Johnson, Knight,Lange,McCoy,Wallbaum for all $\alpha < \kappa$)

A relation R on $\mathcal{M}$ is relatively intrinsically Σ_α if and only if it is $\exists_\alpha$-definable in $\mathcal{M}$.

hmm... what is $\exists_\alpha$?
Answer: use the logic $L_{\kappa^+,\kappa}$ and allow formulas with $< \kappa$ many variables.
Indeed, we can allow relations of arity any $\alpha < \kappa$ in our notion of "structure".

[Diamondstone,Greenberg,Turetsky] This makes sense for many $\alpha > \kappa$ too.

Relative computable categoricity

A structure $\mathcal{M}$ is relatively computably categorical if for any $\mathcal{N} \cong \mathcal{M}$ there is an isomorphism $f : \mathcal{N} \to \mathcal{M}$ computable in $\mathcal{N} \oplus \mathcal{M}$.
Again following [Ash,Knight,Mannase,Slaman / Chisholm], saying that the back-and-forth construction is the only way to make this hold:

Proposition (Greenberg,Knight)

A structure $\mathcal{M}$ is relatively computably categorical if and only if (letting $X = \{x_\alpha : \alpha < \kappa\}$ be a fixed set of variables) there an expansion $\mathcal{M}' = (\mathcal{M}, \bar{c})$ of $\mathcal{M}$ by $< \kappa$ many constants, a computable closed unbounded set $\mathcal{C}$ of $[X]^{<\kappa}$ and a c.e. set of $\exists_1$-formulas $\Psi = \{\psi_a : a \in \mathcal{C}\}$ such that:

- *Ψ defines the orbits of $< \kappa$-tuples in $\mathcal{M}'$; and*
- *If $a_1 \subseteq a_2 \subseteq \ldots$ are elements of $\mathcal{C}$ then $\psi_{\bigcup_{i<\gamma} a_i}$ is equivalent to $\bigwedge_{i<\gamma} \psi_{a_i}$.*

An example

Theorem (Dzgoev,Goncharov/Remmel)

A countable linear ordering is (relatively) computably categorical if and only if it has only finitely many adjacencies.

Theorem (Greenberg,Kach,Lempp,Turetsky, following an idea of Knight's)

A linear ordering $\mathcal{L}$ of size $\aleph_1$ is (relatively computably categorical) if and only if there is a countable subset $C \subset \mathcal{L}$ such that:

- *Every C-interval is either finite or saturated (dense is not enough!);*
- *For all n, the set of C-intervals which have either n elements or are saturated, is c.e.*

An example

The proof uses the Hausdorff analysis of countable linear orderings, and so does not generalise to $\kappa \geqslant \aleph_2$.

Unlike the countable world, there are linear orderings $\mathcal{L}$ of size $\aleph_1$ which are not computably categorical, but for a cone of degrees **d**, every two **d**-computable copies of $\mathcal{L}$ are isomorphic by a **d**-computable isomorphism.

Further work

- [Johnson] Computable categoricity of Zilber fields and other structures
- [Greenberg, Turetsky] A generalisation of Hausdorff's derivative operation and of the Ash-Watnick theorem.
- [Greenberg,Melnikov] Computable categoricity of completely decomposable torsion-free abelian groups.