Demuth randomness, strong jump-traceability, and lowness

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Two research programmes

1. Kučera’s programme: which incomplete random sets compute which c.e. sets? (Interplay between randomness and classical computability.)

2. Lowness for randomness: which oracles are too weak to detect patterns in random sequences?

Kučera's programme

Theorem (Kučera)
Every ML-random Δ^0_2 set computes a promptly simple c.e. set.

Theorem (Hirschfeldt,Miller)
A ML-random set X computes a non-computable c.e. set if and only if it is not weakly 2-random.

The covering problem: which c.e. sets are computable from incomplete ML-random sets?

Lowness for randomness

The main result in this area is the isolation (by Nies, Hirschfeldt, Stephan, Downey,...) of the ideal of K-trivial sets, those that are low for ML-randomness (as well as for prefix-free complexity K).

Coincidences include the notion of a base for ML-randomness: A is K-trivial if and only if it is computable from an A-ML-random set.

Relation to Kučera's programme: any c.e. set computable from an incomplete ML-random set is K-trivial.

Traceability

Traceability is a notion of weakness, or lack of information. An oracle is traceable if the values of the functions it computes can be effectively guessed with few errors.

Formally,

Definition

1. A trace is a sequence $\langle T_x \rangle_{x<\omega}$ of finite sets;
2. A trace $\langle T_x \rangle$ traces a partial function $\psi \colon \omega \to \omega$ if for all $x \in \operatorname{dom} \psi$, $\psi(x) \in T_x$.

Traceability

Traces are measured by their size and by their complexity.

Definition

An index function for a trace $\langle T_x \rangle$ is a function g such that for all x, $T_x = W_{g(x)}$.

A c.e. trace is a trace which has a computable index function.

Definition

An order function is a computable, non-decreasing, unbounded function $h\colon \omega \to \omega \setminus \{0\}$.

If h is an order function, then an h-trace is a trace $\langle T_x \rangle$ such that for all x, $|T_x| \leqslant h(x)$.

Traceability

A couple of fairly representative notions:

Zambella;Ishmukhametov A Turing degree $\mathbf{a}$ is c.e. traceable if for any order function h every $f \in \mathbf{a}$ has a c.e. h-trace.

Figueira,Nies,Stephan A Turing degree $\mathbf{a}$ is strongly jump-traceable if for any order function h, every $\mathbf{a}$-partial computable function has a c.e. h-trace.

Traceability

Traceability shows up in algorithmic randomness quite often:

Theorem (Terwijn,Zambella; Kjos-Hanssen,Nies,Stephan)
A Turing degree **a** *is c.e. traceable and hyperimmune-free (computably dominated) if and only if every Schnorr random set is* **a***-Schnorr-random.*

So traceability coincides with a notion of lowness.

Strong jump-traceability

Unlike the c.e. traceables, there are only countably many strongly jump-traceable sets.

Theorem (Downey, G)

Every strongly jump-traceable set is K-trivial.

Restricted to the c.e. degrees, they behave particularly nicely:

Theorem (Cholak, Downey, G)

The strongly jump-traceable c.e. degrees form an ideal, strictly contained in the K-trivial degrees.

Strong jump-traceability and Kučera's programme

C.e. strong jump-traceability can be characterised by randomness and by PA completeness.

Theorem (G, Hirschfeldt, Nies)

The following are equivalent for a c.e. degree **a***:*

1. **a** *is computable from every superlow ML-random set.*
2. **a** *is computable from every superlow PA degree.*
3. **a** *is computable from every superhigh ML-random set.*
4. **a** *is computable from every superhigh PA degree.*
5. **a** *is strongly jump-traceable.*

In particular, c.e. strongly jump-traceable degrees are ML-coverable.

ω-computable approximations

Definition

A computable approximation of a function $f : \omega \to \omega$ is a uniformly computable sequence of functions $\langle f_s \rangle$ such that for all n, for almost all s, $f_s(n) = f(n)$.

Shoenfield's limit lemma says that a function has a computable approximation if and only if it is computable relative to the halting problem.

The mind-change function associated with a computable approximation $\langle f_s \rangle$ is

$$m_{\langle f_s \rangle}(n) = \# \{ s : f_{s+1}(n) \neq f_s(n) \} .$$

Definition

A function f is ω-c.a. if it has a computable approximation whose associated mind-change function is bounded by a computable function.

Demuth randomness

Recall that a (statistical) test is a representation of null G_δ set. Formally, it is a sequence $\langle \mathcal{U}_n \rangle_{n<\omega}$ of open sets such that for all n, $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$.
The null set covered by a test $\langle \mathcal{U}_n \rangle$ is

$$\limsup_n \mathcal{U}_n = \{Z \in 2^\omega \ : \ \exists^\infty n \ (Z \in \mathcal{U}_n)\} .$$

Any real outside $\limsup_n \mathcal{U}_n$ is said to pass the test $\langle \mathcal{U}_n \rangle$.

An index function for a test $\langle \mathcal{U}_n \rangle$ is a function f such that for all n, $\mathcal{U}_n = [W_{f(n)}]$.

So for example, a Martin-Löf test is a test that has a computable index function.

Demuth randomness

Definition
A Demuth test is a test that has an ω-c.a. index function.

A real is Demuth random if it passes all Demuth tests.

The motivation for this notion comes from constructive analysis:

Theorem (Demuth)
If X is Demuth random, then every constructive function satisfies the Denjoy alternative at X.

Demuth randomness

Demuth random sets have some nice properties, not shared by ML-randoms or weak 2-randoms:

- A Demuth random set cannot be complete; in fact it is GL_1.
- There are Δ^0_2 Demuth random sets.

And so by Kučera's theorem, some Demuth random set computes a non-computable c.e. set.

Demuth randomness and SJT

Kučera and Nies improved the result of Hirschfeldt, Nies and Stephan, that an incomplete ML-random set can compute only K-trivial c.e. sets, and my result that there is a Δ^0_2 random set which only computes strongly jump-traceable c.e. sets.

Theorem (Kučera,Nies)

Any c.e. set computable from a Demuth random set is strongly jump-traceable.

This raises the covering problem for Demuth randomness: which c.e. sets are computable from Demuth random sets?

Demuth randomness and SJT

Theorem
A c.e. set is strongly jump-traceable if and only if it is computable from a Demuth random set.

Base for Demuth

A set A is a base for Demuth randomness if it is computable from a set Z which is Demuth random relative to A.

Theorem (Nies)

1. *Every base for Demuth randomness is strongly jump-traceable.*
2. *There is a c.e. set which is a base for Demuth randomness.*

Theorem

There is a c.e., strongly jump-traceable set which is not a base for Demuth randomness.

So the collection of c.e. sets which are bases for Demuth randomness is a proper subclass of the strongly jump-traceable sets, about which we know almost nothing. E.g., do they form an ideal? What is the complexity of this class?

Base and lowness

The fact that every K-trivial is a base for ML-randomness follows directly from two other facts:

1. Every K-trivial set is computable from a ML-random set.
2. Every K-trivial is low for ML-randomness.

This approach fails quite badly for c.e. sets and Demuth randomness:

Theorem (Downey,Ng)

Every set which is low for Demuth randomness is hyperimmune-free ($\mathbf{0}$-dominated).

And so cannot be c.e.

But we would like to use this method nonetheless. We already have (1) after all!

Partial relativisation of randomness notions

What makes the previous results on relativisations of Demuth randomness work is the fact that the bounds on the mind-change function is A-computable.

Definition (Cole,Simpson)

Let $A \in 2^\omega$. A function $f : \omega \to \omega$ is A-bounded limit recursive (or BLR(A)) if there is an A-computable approximation of f whose associated mind-change function is bounded by a computable function.

Hence BLR$(\emptyset)$ is the class of ω-c.a. functions.

Partial relativisation of randomness notions

Definition

Let $A \in 2^\omega$. An A-test $\langle \mathcal{U}_n^A \rangle$ is an A-Demuth$_{\mathrm{BLR}}$ test if it has a BLR(A)-index function.
A set Z is A-Demuth$_{\mathrm{BLR}}$ random if it passes all A-Demuth$_{\mathrm{BLR}}$ tests.

A set is Demuth random if and only if it is Demuth$_{\mathrm{BLR}}$ random, and the equivalence persists for hyperimmune-free oracles. For $A \in 2^\omega$, every A-Demuth random set is A-Demuth$_{\mathrm{BLR}}$ random, but this containment may be proper.

Remark (Hölzl,Kräling,Stephan,Wu)

For Demuth randomness, we may assume that all test components are in fact clopen, and canonically so.

Lowness and bases for $\text{Demuth}_{\text{BLR}}$

Theorem (Cole,Simpson)

For $A \in 2^\omega$, $\text{BLR}(A) = \text{BLR}(\emptyset)$ if and only if A is superlow and jump-traceable.

Corollary

Every superlow c.e. set is low for $\text{Demuth}_{\text{BLR}}$ randomness.

Corollary

A c.e. set is a base for $\text{Demuth}_{\text{BLR}}$ randomness if and only if it is strongly jump-traceable.

Lowness for Demuth and for $\mathrm{Demuth}_{\mathrm{BLR}}$

We are left with the task of understanding lowness for Demuth and for $\mathrm{Demuth}_{\mathrm{BLR}}$ randomness.

Proposition

A Turing degree is low for Demuth randomness if and only if it is low for $\mathrm{Demuth}_{\mathrm{BLR}}$ *randomness and is hyperimmune-free.*

Theorem

There is a Π^0_1 *class containing no computable elements, all of whose elements are low for* $\mathrm{Demuth}_{\mathrm{BLR}}$ *randomness.*

Corollary

There is a non-computable set which is low for Demuth randomness.

Lowness for Demuth$_{\text{BLR}}$

Definition (Nies)

A trace is ω-c.a. if it has an ω-c.a. index function.

A Turing degree $\mathbf{a}$ is BLR-traceable if for any order function h, every $f \in \text{BLR}(\mathbf{a})$ has an ω-c.a. h-trace.

Theorem (Bienvenu,G,Nies)

A Turing degree is low for Demuth$_{\text{BLR}}$ *randomness if and only if it is* BLR-*traceable.*