Lattice (non)-Embeddings in the α-R.E. Degrees

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Let α be a limit ordinal. A partial function $f : L_\alpha \to L_\alpha$ is *α-recursive* if it is $\Sigma_1(L_\alpha)$. Elements of L_α are called *α-finite*. α is called *admissible* if the α-recursive image of an α-finite set is α-finite.

Examples

1. ω_1^{CK}, the least non-recursive ordinal. $A \subset \omega$ is ω_1^{CK}-r.e. iff it is Π^1_1, and ω_1^{CK}-finite iff it is hyperarithmetic (Δ^1_1).

2. ω_1^X ($X \subset \omega$), the least ordinal not recursive in X (these are all of the countable admissible ordinals).

3. δ^1_2, the least ordinal not an order type of a Δ^1_2 well-ordering of ω.

4. All cardinals; all cardinals in transitive models of ZF or even KP.

We can develop recursive function theory:

- There is an α-recursive bijection $\alpha \leftrightarrow L_\alpha$.
- Enumeration theorem: there are universal Σ_n sets.
- Recursion: Given an α-recursive $I : L_\alpha \to L_\alpha$ there is a unique α-recursive $f : \alpha \to L_\alpha$ such that for all $\beta < \alpha$, $f(\beta) = I(f \upharpoonright \beta)$.
- The s-m-n theorem and the recursion theorem.

And we can define relative computability and degrees:

A *string* is a α-finite partial function $p : \alpha \rightarrow 2$. If p is a string and $A \subset \alpha$, $p < A$ if $p \subset \chi_A$.

Definition. For $A, B \subset \alpha$, $A \leqslant_\alpha B$ if there is an α-r.e. set Φ (a “functional”) such that for all strings,

$$p < A \leftrightarrow (\exists q < B)[(q, p) \in \Phi].$$

$\mathcal{R}_\alpha$ is the structure of $\equiv_\alpha$-degrees of α-r.e. sets with $\leqslant_\alpha$.

Priority arguments are used to establish analogues of classical results about $\mathcal{R}_\omega$:

- A positive solution to Post's problem (there are incomparable α-r.e. degrees): Sacks[1966] for ω_1^{CK} and more, Sacks and Simpson[1972] for all admissible ordinals;

- Splitting (Every non-zero degree is the join of two lower ones): Shore[1975];

- Density: Shore[1976];

- A minimal pair: Lerman and Sacks [1972], Shore [1978] (still open for some α).

M_5 (or the “1-3-1”) is one of the two 5-element non-distributive lattices. It can be embedded in the classical r.e. degrees, with bottom $\mathbf{0}$ (Lachlan [1972]).

Definition. The *recursive cofinality* of a set $A \subset \alpha$ is the least $\beta \leqslant \alpha$ such that there is some cofinal $f : \beta \to \alpha$ which is weakly recursive in A. A is *hyperregular* if $\mathtt{rcf}(A) = \alpha$.

Theorem 1. *Suppose* $\mathtt{rcf}(\mathbf{a}) > \omega$*. Then* M_5 *cannot be embedded in* $\mathcal{R}_\alpha$ *with top* $\mathbf{a}$ *and bottom* $\mathbf{0}$*.*

Thus if there are no incomplete r.e. degrees with recursive cofinality ω, then M_5 is not embeddable in the α-r.e. degrees with bottom $\mathbf{0}$ (and so the α-r.e. degrees and the classical ones are elementarily inequivalent). For example:

- Σ_2-admissible ordinals;
- $\aleph_\omega^L$: Every incomplete r.e. degree is hyperregular;
- Every uncountable α: if $\mathbf{a}$ is an incomplete α-r.e. degree then $\mathtt{rcf}(\mathbf{a})$ is at least the Σ_1-projectum of $\mathbf{a}$ (Shore [1976]).

Using complexity considerations (Greenberg, Shore, Slaman [2003], Greenberg [2004]) for the other cases, we get:

Theorem 2. *For all admissible α, $\mathcal{R}_\alpha \not\equiv \mathcal{R}_\omega$.*

Proof of Theorem 1. Suppose A_0, A_1, A_2 are α-r.e. and not α-recursive, and that for $\{i, j, k\} = \{0, 1, 2\}$,

$$\Psi_i(A_j \oplus A_k) = A_i.$$

Let

$$A = A_0 \oplus A_1 \oplus A_2.$$

Definition. A *computation tree* is a sequence $\langle \beta_n \rangle$ such that $\beta_{n+1} = \max_{i<3} \psi_i(A_j \oplus A_k, \beta_n)$.

Claim. *For all $\beta_0 < \alpha$, the computation tree for β_0 is recognised at some α-finite stage, and thus is α-finite.*

We construct E_0, E_1, E_2 such that $E_i \leqslant_\alpha E_j, E_k$ (by permitting), satisfying

- $P_{a,b,c} : \varphi_a \neq E_0 \vee \varphi_b \neq E_1 \vee \varphi_c \neq E_2$.

How? At s, we search for $\lambda < \alpha$ which is realised (by some φ_a) and such that there is some computation tree $\langle \beta_n \rangle$ with $\sup_n \beta_n < \lambda$, with one of the computations on the tree becoming incorrect at s. λ is permitted by some A_i at s and by another later.

$P_{a,b,c}$ must succeed, for if not, we could compute each A_i by waiting for computation trees to appear, together with some appropriate λ which are realised by φ_a, φ_b and φ_c. $\square$

Theorem 3. *If* $\mathtt{rcf}(\mathbf{a}) > \omega$ *then there is no* M_5 *embedded below* $\mathbf{a}$*.*