A new (?) ideal in the c.e. Turing degrees

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The theorem

Theorem

There is a noncomputable c.e. set, computable from all SJT-*complete c.e. sets.*

Δ_2^0 random sets as c.e. sets

As oracles, Δ_2^0 ML-random sets resemble promptly simple c.e. sets.

For example:

Theorem (Kučera)

Every Δ_2^0 random set computes a promptly simple c.e. set.

Corollary

There is no minimal pair of Δ_2^0 random sets.

An ideal

Theorem (Nies, Hirschfeldt,...)

The following are equivalent for $A \in 2^\omega$:

1. $K(A\restriction_n) \leqslant^+ K(n)$;
2. $K(n) \leqslant^+ K^A(n)$;
3. $\mathrm{MLR} \subseteq \mathrm{MLR}^A$.

These sets are called K-trivial. This notion is invariant in the Turing degrees, and induces an ideal in the Turing degrees, generated by its c.e. elements. [Mostly Nies]

The Kučera sets are K-trivial

Theorem (Hirschfeldt,Nies,Stephan)

Every c.e. set computable from an incomplete random set is K-trivial.

Question

Is every K-trivial set computable from an incomplete random set?

Another ideal

An order function is a computable, nondecreasing, unbounded function from ω to $\omega \setminus \{0\}$.

A trace is a uniformly c.e. sequence of finite sets.

A trace $\langle T_x \rangle$ traces a partial function $\psi\colon \omega \to \omega$ if for all $x \in \operatorname{dom} \psi$, $\psi(x) \in T_x$.

A trace $\langle T_x \rangle$ is an h-trace if for all x, $|T_x| \leqslant h(x)$.

A set A is h-jump-traceable if every function partial computable in A has an h-trace.

A set A is strongly jump-traceable if it is h-jump-traceable for every order function h.

Strong jump-traceability is invariant in the Turing degrees.

Every strongly jump-traceable set is K-trivial. [G,Downey]

Restricted to the c.e. degrees, the strongly jump-traceable degrees form an ideal, properly contained in the K-trivial ideal. [Cholak,Downey,G]

SJT and randomness

Kučera's no-minimal-pair theorem can be strengthened if we restrict the class of random sets.

Theorem (Hirschfeldt,G,Nies)

The following are equivalent for a c.e. set A:

1. *A is computable from every superlow random set.*
2. *A is computable from every superhigh random set.*
3. *A is strongly jump-traceable.*

The analogy between random and c.e. oracle seems to fail here:

- [Lachlan?] There is a minimal pair of c.e. superlow sets.
- [Ng; Shore] There is a minimal pair of c.e. superhigh sets.

SJT and randomness II

SJT sets also “solve” the covering question:

Theorem (G,Turetsky;Nies,Kučera)

A c.e. set is strongly jump-traceable if and only if it is computable from a Demuth random set.

Pseudojump inversion

A pseudojump operator is a map of the form $X \mapsto X \oplus W_e^X$ for some e. A pseudojump operator J is increasing if for all $X \in 2^\omega$, $X <_T J(X)$.

Pseudojump inversion says that $\mathbf{0}'$ can have any c.e. property, relative to a base.

Theorem (Jockusch, Shore)

If J is an increasing pseudojump operator, then there is a c.e. set W such that $J(W) \equiv_T \emptyset'$.

pseudojump inversion cone avoidance

Question (Downey, Jockusch and LaForte)

Let J be an increasing pseudo-jump operator, and let A be a non-computable c.e. set. Is there a c.e. set W such that $J(W) \equiv_T \emptyset'$ and $A \not\leq_T W$?

They gave partial results. A strong (weak) form of their question is: can any increasing pseudojump operator be inverted to a minimal pair?

Partial relativisation of K-triviality

Definition

For $A, B \in 2^\omega$, $A \leqslant_{\mathrm{LR}} B$ if $\mathrm{MLR}^B \subseteq \mathrm{MLR}^A$.

A set B is LR-hard if $\emptyset' \leqslant_{\mathrm{LR}} B$.

Theorem (Kjos-Hanssen, Miller, Solomon)

A set B is LR*-hard if and only if almost every set is computably dominated by B.*

Every LR-hard set is superhigh (Simpson).

Psedojump inversion of K-triviality

Theorem (Zambella;Solovay)

There is an increasing pseudojump operator $J_{\mathcal{K}}$ such that for all X, $J_{\mathcal{K}}(X)$ is K-trivial relative to X, and so $J_{\mathcal{K}}(X) \leqslant_{\mathrm{LR}} X$.

So any inversion of $J_{\mathcal{K}}$ is LR-hard.

A possible counterexample?

Theorem (Nies)
There is an incomplete c.e. set which computes all K-trivial sets.

The analogue would give us a non-computable lower bound for all LR-hard c.e. sets. The related question is:

Question
Is there a minimal pair of c.e., LR*-hard sets?*

Note that Barmpalias has constructed a cappable LR-hard c.e. set.

SJT reducibility

Nies has partially relativised strong jump-traceability.

Definition

For $A, B \in 2^\omega$, $A \leqslant_{\mathrm{SJT}} B$ if for all (computable) order functions h, every A-partial computable function has a B-c.e. h-trace.

A set B is SJT-hard if $\emptyset' \leqslant_{\mathrm{SJT}} B$.

Figueira, Nies and Stephan showed that there is an increasing pseudo-jump operator J_{SJT} such that for all X, $J_{\mathrm{SJT}}(X)$ is strongly jump-traceable relative to X, and so $J_{\mathrm{SJT}}(X) \leqslant_{\mathrm{SJT}} X$.

Ng has constructed a cappable, indeed noncuppable, SJT-hard c.e. set.

A proper upper cone

Theorem
There is a non-computable c.e. set, computable from all c.e., SJT-*hard sets.*

Corollary
J_{SJT} *cannot be inverted together with upper cone avoidance.*

This gives us an ideal of c.e. degrees: all those that are computable from all SJT-hard c.e. sets.

Question
What kind of sets are in this ideal? Are they all K-trivial?

By Ng's result, they are all noncuppable.