Strong jump-traceability and pseudo-jump inversion

Rod Downey and Noam Greenberg

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The jump operator

For every set $X \in 2^\omega$, X', the Turing jump of X is (some consistent choice of) a universal X-computably enumerable set.

Consistent choice (uniformity) means that there is a Turing machine M with oracle tape, such that for all X, $X' = \operatorname{dom} M^X$.

The jump is strictly increasing, Turing-wise: for all X, $X <_T X'$.

Pseudo-jump operators

We generalise these properties of the jump operator.

Definition (Jockusch and Shore)

A function $J\colon 2^\omega \to 2^\omega$ is a pesudo-jump operator if:

1. There is a Turing machine M with oracle tape such that for all X, $J(X) = \operatorname{dom} M^X$; and
2. For all X, $X \leqslant_T J(X)$.

A pseudo-jump operator is increasing if for all X, $X <_T J(X)$.

Pseudo-jump inversion

Theorem (Jockusch and Shore)

Let J be an increasing pseudo-jump operator. There is a c.e. set W such that $J(W) \equiv_T \emptyset'$.

This gives, for example, an incomplete high c.e. degree.

Cone avoidance

Question (Downey, Jockusch and LaForte)

Let J be an increasing pseudo-jump operator, and let A be a non-computable c.e. set. Is there a c.e. set W such that $J(W) \equiv_T \emptyset'$ and $A \not\leq_T W$?

Downey, Jockusch and LaForte showed that there is a pseudo-jump operator which is increasing on the c.e. sets, for which inversion + cone avoidance fails.

Minimal pairs

A related question:

Question

If J is an increasing pseudo-jump operator, are there c.e. sets W_0 and W_1 such that $J(W_0) \equiv_T J(W_1) \equiv_T \emptyset'$ and $\deg_T(W_0) \wedge \deg_T(W_1) = \mathbf{0}$?

Partial relativisation of K-triviality

Definition

For $A, B \in 2^\omega$, $A \leqslant_{\mathrm{LR}} B$ if every B-random set is A-random.

A set B is LR-hard if $\emptyset' \leqslant_{\mathrm{LR}} B$.

Theorem (Kjos-Hanssen, Miller, Solomon)

A set B is LR-*hard if and only if almost every set is computably dominated by B.*

LR-hard c.e. sets

For $A, B \in 2^\omega$, B is K-trivial relative to A if and only if $B \oplus A \leqslant_{\mathrm{LR}} A$.

Hence for all Δ^0_2 sets A, A is LR-hard if and only if $\emptyset'$ is K-trivial relative to A.

The standard cost function construction of a non-computable K-trivial set can be relativised, giving an increasing pseudo-jump operator J_K, such that for all X, $J_K(X)$ is K-trivial relative to X.

Thus if W is c.e. and $J_K(W) \equiv_{\mathrm{T}} \emptyset'$, then W is LR-hard.

A possible counterexample?

Theorem (Nies)

There is an incomplete c.e. set which computes all K-trivial sets.

The analogue would give us a non-computable lower bound for all LR-hard c.e. sets. The related question is:

Question

Is there a minimal pair of c.e., LR-*hard sets?*

Note that Barmpalias has constructed a cappable LR-hard c.e. set.

SJT reducibility

Nies has partially relativised strong jump-traceability.

Definition
For $A, B \in 2^\omega$, $A \leqslant_{\mathrm{SJT}} B$ if for all (computable) order functions h, every A-partial computable function has a B-c.e. h-trace.

A set B is SJT-hard if $\emptyset' \leqslant_{\mathrm{SJT}} B$.

The construction of Figueira, Nies and Stephan can be relativised, giving an increasing pseudo-jump operator J_{SJT} such that for all X, $J_{\mathrm{SJT}}(X)$ is strongly jump-traceable relative to X, and so $J_{\mathrm{SJT}}(X) \leqslant_{\mathrm{SJT}} X$.

No minimal pair

Theorem

There is no minimal pair of SJT*-hard c.e. sets.*

Corollary

J_{SJT} *cannot be inverted to a minimal pair.*

Note: Ng has constructed a cappable, SJT-hard c.e. set.

A proper upper cone

Theorem (In preparation)

There is a non-computable c.e. set A such that all c.e., SJT*-hard sets compute A.*

Corollary

J_{SJT} cannot be inverted together with upper cone avoidance.

This gives us an ideal of c.e. degrees: all those that are computable from all SJT-hard c.e. sets.

Question

What kind of sets are in this ideal? Can they be cuppable? Are they all K-trivial?

Toward bridging SJT and LR: levels of jump-traceability

SJT is the limit of a hierarchy of traceability, indexed by order functions.

Let h be a computable order function. For $A, B \in 2^\omega$, $A \leqslant_{h-\mathrm{JT}} B$ if every A-partial computable function has a B-c.e. h-trace. This is not transitive. We make this definition for the sake of the following two cases:

- A is h-jump-traceable if $A \leqslant_{h-\mathrm{JT}} \emptyset$.
- B is $h-\mathrm{JT}$-hard if $\emptyset' \leqslant_{h-\mathrm{JT}} B$.

Theorem

There is a computable order h such that there are no minimal pairs of c.e., $h-\mathrm{JT}$*-hard sets.*

$h-$JT and LR

Theorem

There is a computable order h such that every h-jump-traceable set is K-trivial.

Corollary

There is a computable order h such that every $h-\mathrm{JT}$ *hard* Δ_2^0 *set is* LR-*hard.*

We might settle the LR-hard question if we find such an order h and show that there are no minimal pairs of $h-\mathrm{JT}$-hard sets.

Related questions

1. For which orders h does h-jump traceability imply K-triviality? For which orders h are there only countably many h-jump-traceable sets? Miller and Yu showed that if $h(n) \geqslant (1+\epsilon)n$ then there are uncountably many h-jump-traceable sets.
2. For which orders h is every K-trivial set h-jump-traceable? Current knowledge: if $\sum 1/h(n)$ is finite. What about the identity function?
3. Does $\leqslant_{\mathrm{SJT}}$ imply $\leqslant_{\mathrm{LR}}$?