Effective properties of uncountable linear orderings

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Church's thesis for ω_1

Throughout, we assume that $\mathbb{R} \subset L$, and so that the continuum hypothesis holds.

Fact (Kripke, Platek, Sacks, Koepke, S. Friedman)

The following are equivalent for a set $A \subseteq \omega_1$:

1. *A is decidable by a Turing machine with tape of length ω_1, for which halting computations are allowed to run for countably many steps.*
2. *Membership in A can be deduced in countable many steps from basic axioms in a reasonable deduction calculus.*
3. *A is Δ_1 definable over L_{ω_1} with parameters.*

Computably enumerable sets, computable functions, etc.

This motivates us to define a set $A \subseteq L_{\omega_1}$ to be computably enumerable if it is Σ_1-definable over L_{ω_1} with parameters.

This yields all familiar notions of recursion theory: computable functions (via their graph), Turing reducibility (via Turing functionals), and so on.

Recursion

The main tool for defining computable functions is recursion:

Fact

If $I\colon L_{\omega_1} \to L_{\omega_1}$ is computable, then there is a unique, computable function $f\colon \omega_1 \to L_{\omega_1}$ such that for all $\alpha < \omega_1$,

$$f(\alpha) = I(f\restriction_\alpha).$$

In practice

Applying Church's thesis,

- A computable process can consider countably many elements simultaneously.
- A computable process can non-uniformly be given countably much information.
- A computable bijection between ω_1 and L_{ω_1} means that we may pass between ω_1 and L_{ω_1} with impunity.

Computable linear orderings

Examples of computable linear orderings:

1. $\mathbb{R}$
2. η_1, the saturated linear ordering.
3. $\mathbb{R} \cdot \omega_1$, the long line.
4. $\mathbb{Q} \cdot \omega_1$, the long line with many holes.
5. $2 \cdot \mathbb{R}$, the double arrow space.

Degrees of linear orderings

The degree spectrum of a linear ordering $\mathcal{L}$ is the collection of Turing degrees which compute a copy of $\mathcal{L}$:

$$\mathrm{DegSpec}(\mathcal{L}) = \left\{\mathbf{d} \,:\, \exists \mathcal{L}' \cong \mathcal{L}\ (\mathcal{L}' \leqslant_T \mathbf{d})\right\}.$$

The degree of $\mathcal{L}$ is $\mathbf{d}$ if $\mathrm{DegSpec}(\mathcal{L}) = \mathcal{D}(\geq \mathbf{d})$.

Theorem (Richter)

If $\mathcal{L}$ is not computable, then there is some $\mathcal{L}' \cong \mathcal{L}$ such that the Turing degrees of $\mathcal{L}$ and $\mathcal{L}'$ form a minimal pair, i.e. $\deg_{\mathrm{T}}(\mathcal{L}) \wedge \deg_{\mathrm{T}}(\mathcal{L}') = \mathbf{0}$.

Hence, if $\mathcal{L}$ has a degree $\mathbf{d}$, then $\mathbf{d} = \mathbf{0}$.

BUT FOR ω_1

THEOREM

Every degree is the degree of some linear ordering.

Jump degrees

The jump-spectrum of a linear ordering $\mathcal{L}$ is

$$\{\mathbf{d}' : \mathbf{d} \in \text{DegSpec}(\mathcal{L})\}.$$

$\mathcal{L}$ has jump-degree $\mathbf{d}$ if the jump spectrum of $\mathcal{L}$ is $\mathcal{D}(\geq \mathbf{d})$. The jump-degree $\mathbf{d}$ is proper if $\mathcal{L}$ doesn't have a degree.

Theorem (Ash, Knight, Richter, ...)

If a linear ordering has jump-degree $\mathbf{d}$ then $\mathbf{d} = \mathbf{0}'$.

Theorem

Every degree $\mathbf{d} \geq \mathbf{0}'$ is the proper jump-degree of some linear ordering.

Double-jump-degrees

Iterating, we get the double-jump spectrum and double-jump degrees.

Theorem (Ash, Knight, Richter, Jockusch, Soare,...)

Every degree $\mathbf{d} \geq \mathbf{0}''$ *is a (proper) double-jump-degree of some linear ordering.*

Theorem

Every degree $\mathbf{d} \geq \mathbf{0}'$ *is a proper double-jump degree of some linear ordering.*

In fact,

Theorem

There is a linear ordering whose degree spectrum is the collection of all non-low degrees.

Computable categoricity

A computable linear ordering $\mathcal{L}$ is computably categorical if for every computable $\mathcal{L}' \cong \mathcal{L}$, there is a computable isomorphism between $\mathcal{L}$ and $\mathcal{L}'$.

Theorem (Dzgoev, Goncharov; Remmel)

A computable linear ordering $\mathcal{L}$ is computably categorical if and only if it has only finitely many adjacencies.

The proof doesn't transfer, because countably much restraint can create uncountably many intervals. So for example,

PROPOSITION

1. *The double arrow space* $2 \cdot \mathbb{R}$ *is computably categorical.*
2. $\mathbb{Q} \cup \mathbb{Q} \cdot (\mathbb{R} \setminus \mathbb{Q})$ *(replacing each irrational number by* $\mathbb{Q}$*) is not computably categorical.*

But,

PROPOSITION

$\mathbb{Q} \cup \eta_1 \cdot (\mathbb{R} \setminus \mathbb{Q})$ *is computably categorical.*

The correct generalisation

Theorem

A linear order is computably categorical if and only if there is a countable set of parameters $\overline{p}$ and a disjoint collection of c.e. sets $\{V_n\}_{n\in\omega-\{0\}}$ such that every $\overline{p}$-interval is either finite or η_1, and every $\overline{p}$-interval of size n is an element of V_n.

ADJACENCIES

Given a linear order L, let $\mathrm{Succ}(L)$ denote the set of successivities (adjacencies) of L, i.e. the set $\{(a, b) \mid a, b \in L, a < b, \ \& \ \forall x \neg(a < x < b)\}$.
The successivity spectrum of $\mathcal{L}$ is the collection

$$\mathrm{SuccSpec}(\mathcal{L}) = \{\deg_{\mathrm{T}}(\mathrm{Succ}(\mathcal{L}')) \ : \ \mathcal{L}' \text{ computable}, \mathcal{L}' \cong \mathcal{L}\}.$$

This is a collection of c.e. Turing degrees.

THEOREM (FROLOV, DOWNEY, LEMPP, WU)

If $\mathcal{L}$ is a computable linear order with infinitely many adjacencies, then SuccSpec($\mathcal{L}$) *is upwards closed in the c.e. degrees.*

THEOREM (DOWNEY, MOSES)

For any c.e. degree **d**, *there is a computable linear ordering $\mathcal{L}$ such that*

$$\text{SuccSpec}(\mathcal{L}) = \mathcal{R}(\geq \mathbf{d}).$$

Proposition

$\text{SuccSpec}(2 \cdot \mathbb{R}) = \{\mathbf{0}\}$.

In fact,

Proposition

For any c.e. degree $\mathbf{d}$*, there is a computable linear order* $\mathcal{L}$ *such that* $\text{SuccSpec}(\mathcal{L}) = \{\mathbf{d}\}$.

$\mathbb{R}$-like linear orderings

A linear ordering $\mathcal{L}$ is $\mathbb{R}$-*like* if there is a countable set of parameters $\overline{p}$ such that any $\overline{p}$-interval is either finite or dense.

Theorem

If $\mathcal{L}$ is a computable linear order which is not $\mathbb{R}$-like, then SuccSpec($\mathcal{L}$) *is upwards closed in the c.e. degrees.*

Let $\mathcal{L}$ be an $\mathbb{R}$-like computable linear ordering, witnessed by $\overline{p}$.

Define $I^{\overline{p}}_{>1}(L)$ to be the set of $\overline{p}$-intervals containing more than 1 element. This is a c.e. set, so fix an injective enumeration $g \colon \omega_1 \to A$.

For $n > 1$, define $I^{\overline{p}}_n(L)$ to be the set of $\alpha < \omega_1$ such that the interval $g(\alpha)$ contains precisely n many elements.

Define $I^{\overline{p}}_n(L)$ to be the set of $\alpha \in \omega_1$ such that the interval $g(\alpha)$ contains infinitely many elements.

Theorem

If L is an $\mathbb{R}$-like computable linear ordering, then for any computable $\mathcal{L}' \cong \mathcal{L}$,

$$I^{\bar{p}}_{\infty}(\mathcal{L}) \leq_{\mathrm{T}} \mathrm{Succ}(\mathcal{L}') \leq_{\mathrm{wtt}} \bigoplus_{n \in \omega - \{0,1\}} I^{\bar{p}}_{n}(\mathcal{L}).$$

Further, the spectrum achieves these upper and lower bounds.

There exists a linear order where not every Π^0_1 degree consistent with the above is in the spectrum. But there is also an $\mathbb{R}$-like linear ordering $\mathcal{L}$ such that $\mathrm{SuccSpec}(\mathcal{L}) = \mathcal{R}$.