On The Elementary Theory of the Metarecursively Enumerable Degrees

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Definition. ϕ-*comprehension* is the statement "for all x, $\{y \in x : \phi(y)\}$ exists". ϕ-*collection* is the statement

$$(\forall x \in u)\,(\exists y)\,\phi(x,y) \rightarrow \\ (\exists v)\,(\forall x \in u)\,(\exists y \in v)\,\phi(x,y)$$

Definition (Kripke(1964), Platek(1966)). A transitive set M is Σ_n-*admissible* if it is closed under the Gödel operations, satisfies
Δ_0-comprehension (i.e. ϕ-comprehension for all ϕ that are $\Delta_0(M)$) and Σ_n-collection. An ordinal α is Σ_n-admissible if L_α is a Σ_n-admissible set.

We say "admissible" for Σ_1-admissible.

Definition.

- $A \subset L_\alpha$ is *α-recursively enumerable* if it is $\Sigma_1(L_\alpha)$.

- $A \subset L_\alpha$ is *α-recursive* if it is $\Delta_1(L_\alpha)$ (i.e. it is r.e. and co-r.e.)

- A partial function $f : L_\alpha \to L_\alpha$ is *partial α-recursive* if it is $\Sigma_1(L_\alpha)$-definable (i.e. its graph is r.e.)

- A is *α-finite* if $A \in L_\alpha$.

Hence:

α is admissible iff the α-recursive image of an α-finite set is α-finite.

Examples

1. ω_1^{CK}, the least non-recursive ordinal. $A \subset \omega$ is ω_1^{CK}-r.e. iff it is Π^1_1, and ω_1^{CK}-finite iff it is hyperarithmetic (Δ^1_1).

2. ω_1^X ($X \subset \omega$), the least ordinal not recursive in X (these are all of the countable admissible ordinals).

3. δ^1_2, the least ordinal not an order type of a Δ^1_2 well-ordering of ω.

4. All cardinals, all cardinals in transitive models of ZF or even KP. For each cardinal κ, H_κ is an admissible set. If κ is regular then κ is Σ_n-admissible for all n.

Familiar Theorems from classical recursion theory are still valid:

- There is a recursive bijection $\alpha \leftrightarrow L_\alpha$.

- Enumeration theorem: there are universal Σ_n sets.

- Recursion: Given a recursive
$I : L_\alpha \to L_\alpha$ there is a unique recursive
$f : \alpha \to L_\alpha$ s.t. for all $\beta < \alpha$,
$f(\beta) = I(f \restriction \beta)$.

- The s-m-n theorem and the recursion theorem.

- A set is r.e. iff it is the domain of some partial recursive function iff it is α-finite or the range of an injective total recursive function.

A *string* is a α-finite partial function $p : \alpha \rightarrow 2$. If p is a string and $A \subset \alpha$, $p < A$ if $p \subset \chi_A$.

Definition. For $A, B \subset \alpha$, $A \leqslant_\alpha B$ if there is an α-r.e. set R (a "functional") such that for all strings,

$$p < A \leftrightarrow (\exists q < B)[(q, p) \in R]$$

$\mathcal{R}_\alpha$ is the structure of $\equiv_\alpha$-degrees of α-r.e. sets with $\leqslant_\alpha$.

Priority arguments are used to establish analogues of classical results about $\mathcal{R}_\omega$:

- A positive solution to Post's problem (there are incomparable α-r.e. degrees): Sacks[1966] for ω_1^{CK} and more, Sacks and Simpson[1972] for all admissible ordinals);

- Splitting (Every non-zero degree is the join of two lower ones): Shore[1975]

- Density: Shore[1976]

- A minimal pair: Lerman and Sacks [1972], Shore [1978] (still open for some α).

Question(Sacks, 1966): Are $\mathcal{R}_\omega$ and $\mathcal{R}_{\omega_1^{CK}}$ elementary equivalent?

Answer(Shore, Slaman, c.1994): No.

Let $(R, <, \vee)$ be an upper semi-lattice, and $\bar{p} = (r, p, q, l)$ be elements of R.

Definition. The *SW set* defined by $\bar{p}$ in R is the set of elements x, minimal below r w.r.t. $q \leqslant x \vee p$.

We define a binary relation on the SW set $G = G_{\bar{p}}$: for $x, y \in G$, let $x \leqslant_{\bar{p}} y$ if $x \leqslant y \vee l$.

Theorem 1 (Slaman, Woodin(?)). *Given any recursive partial order $\prec$, there are $\bar{p} \in \mathcal{R}_\omega$ s.t. $<_{\bar{p}} \cong \prec$.*

Since we can interpret any structure into partial orders, SW sets can be used to code models of Arithmetic. Let $M_{\bar{p}}$ be the model (in the language of arithmetic) coded by $(G_{\bar{p}}, <_{\bar{p}})$. There is a translation taking a formula ϕ in arithmetic to a formula $\tilde{\phi}$ in the language of partial orderings, such that $M_{\bar{p}} \models \phi(\bar{x})$ iff $R \models \tilde{\phi}(\bar{x}, \bar{p})$. Thus we can put first-order conditions on $\bar{p}$ so that $M_{\bar{p}}$ models some finite fragments of arithmetic. One can do better:

Theorem 2 (Nies,Shore,Slaman(1997)). *There is a non-empty formula χ s.t. $\mathcal{R}_\omega \models \chi(\bar{p})$ implies that $M_{\bar{p}}$ is the standard model of arithmetic. There is a formula θ s.t. $\mathcal{R}_\omega \models \chi(\bar{p}) \wedge \chi(\bar{p}')$ implies that $\theta(x, y; \bar{p}, \bar{p}')$ is an isomorphism between $M_{\bar{p}}$ and $M'_{\bar{p}}$.*

As a corollary, by quantifying over all such $\bar{p}$, we get
Theorem 3 (Harrington, Slaman(1984)). *$Th(\mathcal{R}_\omega)$ and $Th(\mathbb{N}; +, \times)$ are recursively isomorphic.*

What about α?
Theorem 4. *Theorems 1 and 2 hold when ω is replaced by ω_1^{CK}.*

In addition, one can code in $M_{\bar{p}}$ any $\Sigma_1(L_{\omega_1^{CK}})$ subset of ω. There is an arithmetic condition $\phi(X)$ such that Kleene's O is the $\subset$-least set satisfying ϕ. We add ϕ to the 'correctness condition' χ and use the comparison maps to find the minimal set (since some models code O). Thus we can pick out the models coding O.

Thus:

Theorem 5. *$O^{(\omega)}$, $Th(\mathcal{R}_{\omega_1^{CK}})$ and $Th(L_{\omega_1^{CK}}, \in)$ are recursively isomorphic.*

This can be extended to some other ordinals. If α is Σ_2-admissible and $\mathrm{cf}_{\Sigma_3}(\alpha) = \omega$ then this coding works. On the other hand, if every $\Sigma_3(L_\alpha)$ function $f : \omega \to \alpha$ is α-finite, one can show that more complicated constructions cannot work, yielding an elementary difference.

Theorem 6. *If α is Σ_2-admissible then $Th(\mathcal{R}_\alpha) \neq Th(\mathcal{R}_\omega)$.*