TWO APPLICATIONS OF ADMISSIBLE COMPUTABILITY

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Abstract. We discuss two applications of admissible computability, namely, to higher randomness, and computability of uncountable structures.

Admissible computability, as formalised by Kripke [62] and Platek [72], is a common generalisation of metarecursion theory (Kreisel and Sacks [61, 60]) and Takeuti's approach to constructibility via recursion on the ordinals [87, 88]. Takeuti showed how to recover constructibility by considering the ordinals first and using a class of partial functions on the ordinals, resembling Kleene's partial recursive functions. Kreisel and Sacks were motivated by Church and Kleene's [20, 19, 57] development of the computable ordinals and the hyperarithmetic sets. The end result was a generalisation of computability to domains beyond the natural numbers, namely some ordinals greater than ω.

The first line of enquiry in the field known at the time as *α-recursion theory* was the attempt to lift to the admissible setting the constructions of classical computability, in particular in the areas of the lattice of c.e. sets and the Turing degrees of c.e. sets. For example, Sacks and Simpson [76] showed that the Friedberg-Muchnik resolution of Post's problem holds for every admissible ordinal, and later Shore extended Sacks's density theorem to all admissible ordinals [81]. On the other hand, some ordinals were shown to have unusual computable structure, for example, for some α, all incomplete c.e. degrees are low [80]. The techniques of α-recursion theory were later used in the study of nonstandard models of arithmetic and their computability, via the resemblance of failure of definable regularity of some singular ordinals and failure of the bounding principle in models of arithmetic. A more recent application of these investigations is Chong, Slaman and Yang's [15] construction of a non-standard model separating the stable and general forms of Ramsey's theorem for pairs, where again a crucial property is that incomplete c.e. degrees are low.

In this chapter we survey a couple of more recent applications of admissible computability, namely to the study of higher randomness and the study of uncountable computable structure theory. The methods of α-recursion theory showed that the generalistion of computability allows us to elucidate the underlying nature of basic notions and constructions of classical computability. This is the main theme of the work that we present. By contrasting classical computability with its generalisations, we can separate between the fundamental and the accidental. What is common to all generalisations, and thus can be considered necessary to computability, and what is special to the natural numbers?

An example is given by the work in [39], which exhibits some of the role that finiteness plays in computability. Kreisel [61] studied the analogy between Π^1_1 sets of numbers and computably enumerable ones, and noted that the correct analogue

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of hyperarithmetic is not computable but *finite* (for example, the image of a hyperarithmetic function is hyperarithmetic). Thus, in admissible computability, it is not only the notion of computability that is generalised, but the notion of finiteness. If α is an admissible ordinal, then α-computable processes are those which take up to α many steps to perform, and each ordinal $\beta < \alpha$ is in this context considered as finite. However in [39] it is shown that some constructions of computability theory rely on the fact that finite ordinals have predecessors. In particular, it is shown that Lachlan's [64] continuous tracing technique is necessary for his embedding of the 1-3-1 lattice into the c.e. degrees, and its success relies on the "true finiteness" of the natural numbers.

Similarly, studying higher randomness allows us to observe how "time tricks" are heavily utilised in classical algorithmic randomness; and studying uncountable linear orderings and free groups shows how to properly generalise classical results such as the Dzgoev-Remmel characterisation of computably categorical linear orderings.

Below, we first give a brief development of admissible computability, and then discuss the two applications mentioned. For more details on admissible computability we refer the reader to the classic [78] and to [8, 18]. For more on α-recursion theory see [14] and [82].

1. Admissible computability

There are several equivalent ways for defining admissible computability. Kripke, following Takeuti and Kleene, used an equation calculus. Platek [72] and later Köpke and Seyfferth [59] gave a more intuitive definition in terms of idealised computers or Turing machines with ordinal-length tape. The approach used most frequently appeals to set theory. The motivating example here is computability as definability in the structure HF, the collection of hereditarily finite sets. The structure $(\mathrm{HF}; \in)$ is effectively bi-interpretable with the standard model $(\mathbb{N}; +, \times)$ of arithmetic: in one direction, the set $\mathbb{N}$ and the (graphs of the) functions $+$ and $\times$ are Δ_1-definable in HF; and via the *Ackermann interpretation* [1], the structure $(\mathrm{HF}; \in)$ is interpretable in $\mathbb{N}$ by a computable relation. Further, the map sending $n \in \mathbb{N}$ to the number coding n in this interpretation is computable, with computable range. It follows that a function $f\colon \mathbb{N} \to \mathbb{N}$ is computable if and only if it is Δ_1-definable in $(\mathrm{HF}; \in)$, and a set $A \subseteq \mathbb{N}$ is computably enumerable if and only if it is Σ_1-definable in $(\mathrm{HF}; \in)$. Here by Σ_1 we refer to the Levy hierarchy of formulas in the language of set theory, which is built up from formulas which only use bounded quantifiers.

We can therefore redevelop the theory of computability by *defining* a set $A \subseteq \mathrm{HF}$ to be c.e. exactly if it is Σ_1-definable in $(\mathrm{HF}; \in)$, and a partial function $f\colon \mathrm{HF} \to \mathrm{HF}$ to be partial computable if its graph is c.e. As a kind of motivation, we consider that once we agree that the relations $\in$ and equality should be considered computable, then bounded quantifiers correspond to a bounded search, which should also be admitted as computable; and so all Δ_0-definable sets should be considered computable. The external existential quantifier in a Σ_1 formula then corresponds to an unbounded search.

Proceeding with our defintions, a we call a set $A \subseteq \mathrm{HF}$ computable if it is Δ_1-definable, that is, if it is c.e. and co-c.e. A partial function $f\colon \mathrm{HF} \to \mathrm{HF}$ is computale if it is partial computable and its domain is computable. We can then proceed to prove the basic facts about computability from these definitions. For example:

Proposition 1.1. *A set $A \subseteq \mathrm{HF}$ is computable if and only if its characteristic function 1_A is computable.*

To see this, assuming that A is computable, we observe that the relation $1_A(x) = y$ is

$$\big((y = 0)\ \&\ (x \notin A)\big) \vee \big((y = 1)\ \&\ (x \in A)\big),$$

which is Δ_1-definable as well. Similarly:

Proposition 1.2. *The composition of partial computable functions is partial computable.*

For if the relations $f(x) = y$ and $g(y) = z$ are both Σ_1-definable, then so is the binary relation

$$(\exists y)\ f(x) = y\ \&\ g(y) = z.$$

Proposition 1.3. *A set $A \subseteq \mathrm{HF}$ is c.e. if and only if it is the range of an injective partial computable function.*

To see this, in the harder direction, let $A \subseteq \mathrm{HF}$ be c.e.; let $R \subseteq \mathrm{HF}^2$ be a Δ_0-definable relation such that $x \in A \iff (\exists y)\, R(x, y)$. There is a computable ordering $<_{\mathrm{HF}}$ of HF of order-type ω. We let C be the collection of pairs (x, y) such that y is $<_{\mathrm{HF}}$-least such that $R(x, y)$ holds; then C is computable, and the map $(x, y) \mapsto x$ restricted to C is injective, computable and its range is A.

We should say a little more on the ordering $<_{\mathrm{HF}}$. It is the image of the natural ordering on $\mathbb{N}$ under the isomorphism between HF and $\mathbb{N}$ given by the Ackermann interpretation. A direct construction of $<_{\mathrm{HF}}$, not appealing to arithmetic, is done by *recursion*. In the context of HF, defining computable functions by recursion is stated as follows:

Proposition 1.4. *Suppose that $I\colon \mathrm{HF} \to \mathrm{HF}$ is computable. Then there is a unique function $g\colon \mathbb{N} \to \mathrm{HF}$ satisfying $g(n) = I(g\restriction n)$ for all $n \in \mathbb{N}$; and this function is computable.*

Since $\mathrm{HF} = V_\omega = \bigcup_n V_n$, we can construct $<_{\mathrm{HF}}$ as the union of linear orderings $<_n$ of V_n such that each $<_{n+1}$ is an end-extension of $<_n$; to do so, all we need to do is to define a computable operation taking a linear ordering $<_X$ of a finite transitive set X and producing an ordering $<_{\mathscr{P}(X)}$ of $\mathscr{P}(X)$ which is an end-extension of $<_X$. One way to do this is to let $<_{\mathscr{P}(X)}$ be the right-lexicogrpahic ordering of $\mathscr{P}(X)$ based on $<_X$.[1] We just need to check that the map taking $<_X$ to $<_{\mathscr{P}(X)}$ is Σ_1-definable in HF, and then use recursion.

We remark that the recursion scheme given by Proposition 1.4 can be extended to define computable functions on other well-founded relations, for example $\in$; that type of recursion shows, for example, that the function taking $x \in \mathrm{HF}$ to its transitive closure is computable. Another use of recursion allows us to formalise first-order logic in HF. Under any reasonable formalisation of formulas, the collection of formulas is computable, and the satisfaction relation between finite structures

[1] $a <_{\mathscr{P}(X)} b$ if $x \in b$ for the $<_X$-greatest element of $a \triangle b$

and formulas is also computable, as all involve "bounded search". Now a Σ_1 sentence ψ (with parameters in HF) holds in HF if and only if there is some transitive $M \in \mathrm{HF}$ such that $M \models \psi$. This is because of absoluteness for Δ_0 predicates. This shows that the global Σ_1 satisfaction relation is c.e. We can then fix a computable numbers $\langle \psi_e \rangle$ of all Σ_1 formulas, and let

$$W_e = \{x \in \mathrm{HF} \,:\, \mathrm{HF} \models \psi_e(x)\}\,;$$

then the list $\langle W_e \rangle$ is a list of all c.e. sets; it is *uniformly* c.e., in that the set $\bigoplus_e W_e$ is c.e. (because the global Σ_1 satisfaction relation is Σ_1-definable). It is also *acceptable*, which means that whenever $\langle A_e \rangle$ are uniformly c.e. sets, then there is a computable function f such that for all e, $A_e = W_{f(e)}$. For if $\theta(x, y)$ is a Σ_1 formula defining $\bigoplus_e A_e$, then the function f takes e to the code of the formula $\theta(e, -)$.[2]

1.1. **Admissible sets.** The definition of an admissible set aims to answer the question: what is the minimal amount of set-theoretic closure required of a set M so that we can mimic the definition of computability above with satisfactory result? That is, if we: consider the elements of M to be "finite"; the ordinals of M to be "numbers"; and interpret "c.e." as Σ_1-definable in M — would this give us a reasonable theory of computability?

Some minimal amount of closure is certainly required. Take, for example, the structure $M = V_{\omega+\omega}$. The ordinals of this structure are $\omega+\omega$. Under any reasonable definition, ordinal addition should be a computable operation. However, $\omega \in M$ but $\omega + \omega \notin M$. That is, the sum of two "finite numbers" is "infinite", which should not be the case. So for a reasonable theory of computability, it should be the case that the ordinals of M are closed under addition. We could make a longer and longer list of similar operations (multiplication, exponentiation,...) but it is not clear where to stop. Rather, we (i) require some very basic amount of closure, so that definability of M makes any kind of sense; and (ii) then, anticipating the definition of M-computable functions, we require the image of a finite object under an M-computable function to be finite, or at least bounded. Here are the formal details.

Definition 1.5. A nonempty transitive set M is *amenable* if:

- For all $x, y \in M$, $\{x, y\} \in M$, $\bigcup x \in M$, and $x \times y \in M$;
- For every $\Delta_0(M)$ predicate R and every set $a \in M$, $a \cap R \in M$.

The second condition is referred to as Δ_0-*comprehension*.

Definition 1.6. Let M be an amenable set. A set $A \subseteq M$ is M-*computably enumerable* if it is $\Sigma_1(M)$. It is M-*computable* if it is $\Delta_1(M)$, i.e., M-c.e. and M-co-c.e.

A partial function from M to M is M-*partial computable* if its graph is M-c.e. An M-partial computable function is M-*computable* if its domain is M-computable.

Some basic facts about computability hold for all amenable sets, with exactly the same proofs. For example:

- The graph of an M-computable function is M-computable.
- A set is M-computable if and only if its characteristic function is M-computable.

[2] The acceptability property is also known as the "s-m-n theorem".

- The composition of M-partial computable functions is M-partial computable.

However, as was observed above, some amenable sets, such as $V_{\omega+\omega}$, are poor choices for computability purposes. The seond step consists of the following definition:

Definition 1.7. A nonempty transitive set M is *admissible* if it is amenable, and it satisfies Δ_0 *collection*: If $R \subseteq M^2$ is a $\Delta_0(M)$ relation, then for all $a \in M$ such that $a \subseteq \operatorname{dom} R$ there is some $b \in M$ such that for all $x \in a$ there is some $y \in b$ such that $R(x, y)$.

The definition is made to be minimal, so that it is easier to verify that certain sets are admissible; however it implies more:

Proposition 1.8. *Every admissible set satisfies Δ_1 comprehension and Σ_1 collection.*

Recalling our intentions, it is common to refer to the elements of an admissible set M as "M-finite". Thus, Δ_1-comprehension says: the intersection of an M-finite set with an M-computable set is M-finite. And Σ_1-collection means: if R is an M-c.e. relation and $a \subseteq \operatorname{dom} R$ is M-finite, then there is an M-finite $b \subseteq \operatorname{range} M$ which contains R-images for all $x \in a$. In particular, the image of an M-finite set under an M-computable function is contained in an M-finite set, and by comprehension, is in fact M-finite. A key fact used is that M-c.e. relations are closed under bounded quantification: if R is M-c.e., then so is $(\forall x \in y)\, R$.

Example 1.9. If κ is a cardinal, then

$$H_\kappa = \{x : |\operatorname{tc}(x)| < \kappa\}$$

(where $\operatorname{tc}(x)$ is the transitive closure of x) is an admissible set. This is clear if κ is regular, and uses a reflection argument when κ is singular. In particular, $\mathrm{HF} = H_\omega$ is admissible, and HF-computability is classical computability. On the other hand, for every admissible set M, we have $\mathrm{HF} \subseteq M$, and if $M \neq \mathrm{HF}$ then $\mathrm{HF} \in M$.

The key to admissibility is that it is precisely what is required to be able to define M-computable functions by recursion. The analogue of Proposition 1.4 is:

Proposition 1.10. *Let M be an admissible set. Let $\alpha = o(M) = M \cap \mathrm{On}$ (the ordinals of M). Suppose that $I \colon M \to M$ is M-computable. Then there is a unique function $g \colon \alpha \to M$ satisfying:*

- *For all $\beta < \alpha$, $g\restriction\beta$ is M-finite and $g(\beta) = I(g\restriction\beta)$.*

The unique such function g is M-computable.

Given Proposition 1.4 (and its generalisations to other M-computable well-founded relations other than $(\alpha; <)$), we can proceed with the development of computability theory as above, with proofs copied over nearly verbatim. For example, the formalisation of first-order logic proceeds in the same way, giving us a universal M-c.e. set: an M-c.e. set W such that letting, for $x \in M$, $W^{[x]} = \{y : (x, y) \in W\}$ be the x-section of W, the collection $\left\{W^{[x]} : x \in M\right\}$ is the collection of all M-c.e. sets. All standard proofs of the Kleene fixed point ("recursion") theorem hold in all admissible sets, and so on. An admissible set M is closed under basic ordinal arithmetic (addition, multiplication, exponentiation), and these operations are M-computable.

Further, some set-theoretic concepts have admissible effectivisations. A set-theoretic way of viewing admissibility is by saying that the ordinal $\alpha = o(M)$ for an admissible set M is "M-effectively regular"; it may fail to be a regular cardinal (indeed may not be a cardinal at all), but M-computable functions cannot witness this fact: there is no M-computable sequence of order-type $< \alpha$, unbounded in α. Some properties of regular cardinals then carry over to admissible ordinals, once we restrict to M-computable objects. For example:

Lemma 1.11. Let $M \neq \mathrm{HF}$ be admissible, let $\gamma < o(M)$ and let $\langle C_\alpha \rangle_{\alpha<\gamma}$ be a uniformly M-computable sequence of closed and unbounded subsets of $o(M)$. Then $\bigcap_{\alpha<\gamma} C_\alpha$ is M-computable, closed and unbounded in $o(M)$.

Similarly, if $M \neq \mathrm{HF}$ is admissible, and $f\colon o(M) \to o(M)$ is M-computable, then

$$\{\beta < o(M) \,:\, f[\beta] \subseteq \beta\}$$

is closed and unbounded in $o(M)$.

1.2. **Constructibility.** One part of classical computability which we developed above but have not generalised yet is $<_{\mathrm{HF}}$, the computable well-ordering of the "universe". This is because in general, there is no reason to assume such a well-ordering exists. For example, $M = H_{\omega_1}$ may not have any definable well-ordering, as the reals may fail to have such an ordering. To make computability "linear", we restrict ourselves to the constructible universe.

Just as for HF, if M is admissible, then satisfaction for structures inside M is M-computable (this was used to get a universal M-c.e. set). Further applications of the recursion principle (analgoues of Proposition 1.10) shows that if $A \in M$ then $\mathscr{P}_{\mathrm{DEF}}(A)$, the collection of A-definable subsets of A, is also an element of M, and the map $A \mapsto \mathscr{P}_{\mathrm{DEF}}(A)$ is M-computable. Applying recursion once more, we get:

Proposition 1.12. *Let M be admissible.*

(1) For all $\alpha < o(M)$, $L_\alpha \in M$, and the map $\alpha \to L_\alpha$ is M-computable.
(2) $L_{o(M)} = L^M = \bigcup_{\alpha<o(M)} L_\alpha$ is M-c.e.

Further, recall that $<_L$, the well-ordering of L, is defined recursively, with the ordering of $L_{\alpha+1}$ being an end-extension of the ordering of L_α; again, an examination shows that this operation can be defined in a Σ_1 way, and so the restriction of $<_L$ to an admissible set M is its restriction to $L_{o(M)}$, and is M-c.e.; the map taking α to $<_L \restriction L_\alpha$ is M-computable, and the map taking $x \in L_{o(M)}$ to $\{y \in L \,:\, y <_L x\}$ is M-partial computable.

A key fact is that L inherits admissibility:

Proposition 1.13. *Let M be an admissible set. Then $L_{o(M)}$ is admissible as well.*

There are some details to the argument, which concern the development of L inside L, but the crux of the proof is in showing that if $\beta < o(M)$ and $f\colon \beta \to o(M)$ is $L_{o(M)}$-computable, then it is bounded below $o(M)$; and the point is that since L^M is itself M-c.e., any Σ_1-definition within L^M can be translated to a Σ_1-definition in M, replacing the unrestricted quantifiers by quantifiers ranging over L^M. Hence f is also M-computable, and hence bounded. We thus define:

Definition 1.14. An ordinal α is admissible if L_α is an admissible set.

By Proposition 1.13, α is admissible if and only if there is an admissible set M such that $\alpha = o(M)$. We say that a set is α-c.e. if it is L_α-c.e., and α-computable if it is L_α-computable.

Working in initial segments of L, we utilise the following:

Proposition 1.15. *If α is admissible, then there is an α-computable bijection between α and L_α.*

Indeed, if $j(\beta)$ is the β^{th} element of L according to $<_L$, then j restricts to a bijection between α and L_α, as j can be defined by recursion, and so $j[\alpha] \subseteq L_\alpha$ and the map $j \restriction \alpha$ is L_α-computable. Similarly, a recursion on $<_L \restriction L_\alpha$ defines $j^{-1} \restriction L_\alpha$ by recursion inside L_α, and so $j \restriction \alpha$ is the required bijection.[3]

Proposition 1.15 allows us to "linearize" α-computability. For example, we now get a numbering $\langle W_\beta \rangle_{\beta<\alpha}$ of all α-c.e. sets, rather than a numbering indexed only by the elements of L_α. Similarly, when performing priority arguments in α-computability, we can order all requirements in order-type α (rather than just indexed by elements of the admissible set), and so can set a priority ordering between them. We can regard every α-computable process as being recursively defined along α. More informally, we think of such processes as taking α many steps. In general, working in α-computability, with experience, we apply some kind of Church-Turing thesis to α-computable functions. Just as in classical computability, we eventually describe computable processes informally, rather than writing computer programs in detail, in admissible computability, we eventually cease to write down precise Σ_1 formulas defining the functions we are interested in. Instead, we develop an intuition as to what constitutes "legal" α-computable manipulations of α-finite objects (elements of L_α), and get a sense of the "time" that a process takes; if it takes fewer than α steps, then it "halts".

1.3. **The least admissible ordinal (beyond ω).** So far we have given only one kind of example of admissible ordinals, namely the cardinals (Example 1.9). By a reflection argument (collapsing elementary substrctures), we see that there are many admissible ordinals which are not cardinals, indeed many countable ones. We can give a concrete description of the least admissible ordinal beyond ω. Interestingly, this ordinal arises from Church and Kleene's theory of the computable ordinals. An ordinal β is called computable if there is a computable well-ordering of $\mathbb{N}$ of order-type β. The computable ordinals form a countable initial segment of the ordinals, and the least non-computable ordinal is denoted by ω_1^{ck} (Church-Kleene ω_1). Now a computable ordinal $\beta > \omega$ cannot be admissible: if $<_\beta$ is some computable well-ordering of $\mathbb{N}$ of order-type β, then the ordering $<_\beta$ is an element of $L_{\omega+1}$, and so of L_β; if L_β were admissible, then by β-recursion, we would see that the isomorphism from $(\mathbb{N}; <_\beta)$ to $(\beta; <)$ would be β-computable, contradicting admissibility. However:

Proposition 1.16. *ω_1^{ck} is admissible.*

The reason for this is Σ^1_1 bounding, a key aspect of the theory of computable ordinals and hyperarithmetic sets. We give a quick review. Σ^1_1 bounding says that if A is a Σ^1_1 collection of well-orderings of $\mathbb{N}$, then there is a computable bound on

[3] We remark that in fact, for every limit ordinal α, L_α is amenable and there is an L_α-computable bijection between α and L_α, but the definition of this bijection is not uniform in α.

the order-types of all the orderings in A. One way to see this is to note that the collection of computable well-orderings is Π^1_1-complete, and so, by Cantor's diagonal argument, cannot be Σ^1_1; however, if the order-types of the orderings in A are not computably bounded, then we can give a Σ^1_1 definition of the computable well-orderings by asking for an embedding into some element of A. A more constructive approach (which also gives uniformity) is as follows: by a normal form argument, A is the projection of an effectively closed set P in Baire space. We let Q be the closed set of triples (L, f, g), where $(L, f) \in P$ (so $L \in A$) and g is an infinite descending sequence in L. Since every $L \in A$ is a well-ordering, Q is actually empty. This means that the tree S associated with the definition we gave Q is well-founded. For every $L \in A$, the tree of finite descending sequences in L is embeddable into S, and so the rank of S, which is computable, bounds the order-types of all elements of A.

Spector [85] showed, essentially, that if you take an iteration of the Turing jump along any computable well-ordering, then the result depends only on the order-type of the ordering. Thus, for every computable ordinal α, there is a well-defined Turing degree $\mathbf{0}^{(\alpha)}$, which contains all computable iterations of the Turing jump of length α. This is an increasing hierarchy of Turing degrees of length ω_1^{ck}; a set of numbers is defined to be *hyperarithmetic* if its Turing degree lies below some $\mathbf{0}^{(\alpha)}$. Kleene [56] used the Σ^1_1 bounding principle to show that the hyperarithmetic sets coincide with the Δ^1_1 sets, an effective analogue of the coincidence of the Borel sets with the $\boldsymbol{\Delta^1_1}$ ones (due to Suslin).

The next step is the Sepctor-Gandy theorem [84, 35], which analyses the quantifiers ranging over the hyperarithmetic reals.

Theorem 1.17. *A set $A \subseteq \mathbb{N}$ is Π^1_1 if and only if it is of the form "there exists a hyperarithmetic x such that $Q(-, x)$", where Q is Π^0_2.*

One direction is easier. Suppose that Q is an arithmetical predicate. The map taking a (computable index of a) computable well-ordering K to the iteration $\varnothing^{(K)}$ of the Turing jump along K is Π^0_2-definable, and so $\varnothing^{(K)}$ is Δ^1_1 uniformly in K. There is a hyperarithmetic x such that $Q(-, x)$ if and only if there is a computable well-ordering K and a Turing reduction Φ such that $Q(-, \Phi(\varnothing^{(K)}))$ holds. The search for indices for K and Φ is arithmetic; the main complexity is asking whether K is well-founded or not.

In the other direction, we start with an analysis of *pseudo-ordinals*. A (computable) pseudo-ordinal is a computable ordering of $\mathbb{N}$ which is not well-founded, however it has no hyperarithmetic infinite descending sequences. The existence of such objects can be concluded using Σ^1_1 bounding, in its guise as a "overspill" argument. By the easy direction of the Spector-Gandy theorem, the collection of computable well-orderings together with the computable pseudo-ordinals is Σ^1_1; since it contains all computable well-orderings, and it cannot coincide with the collection of computable well-orderings, a pseudo-ordinal must exist.[4] The length of the well-founded part of any pseudo-ordinal must be precisely ω_1^{ck}; it cannot be longer, as then a principal initial segment would give a computable copy of ω_1^{ck}. And it cannot be shorter, because otherwise, an argument using effective transfinite recursion shows that the well-founded part would be hyperarithmetic, and so

[4] A completely different way to obtain pseudo-ordinals is by the Gandy basis theorem, from which we get a countable, ill-founded model of ZFC whose well-founded part has height ω_1^{ck}; that model contains ill-founded "ordinals" which the model believes are computable ordinals.

we would be able to hyperarithmetically define an infinite descending sequence by avoiding the well-founded part. It follows that any iteration of the Turing jump along a pseudo-ordinal must compute all hyperarithmetic sets (and in particular will not be hyperarithmetic). Note that the existence of such an iteration is not automatic, as the pseudo-ordinal is, in fact, ill-founded; however, an overspill argument shows that there is a pseudo-ordinal L^* with a jump hiearchy along L^*.

We can now prove the harder direction of the Spector-Gandy theorem. Let A be Π^1_1. Membership of some n in A can be translated to the question of whether some computable linear ordering K is well-founded or not. K is well-founded if and only for some e in the well-founded part of L^*, the unique interation of the Turing jump along L^* up to e computes an embedding of K into L^* up to e; and this happens if and only if for some $e \in L^*$, a hyperaeithmetic iteration of the jump along L^* up to e computes such an embedding.

In essence, a similar argument can be now used to show that ω_1^{ck} is admissible. After learning some general facts about L_α for limit α, and about $L_{\omega_1^{\mathrm{ck}}}$ in particular, we can show that for admissibility, it suffices to show that every function $f\colon \omega \to \omega_1^{\mathrm{ck}}$ which is Σ_1-definable in $L_{\omega_1^{\mathrm{ck}}}$ (without parameters) is bounded below ω_1^{ck}. By effective transfinite recursion, we show that for all $\alpha < \omega_1^{\mathrm{ck}}$, there is a hyperarithmetic copy of L_α, in a uniform way: essentially, $\mathbf{0}^{(\omega\alpha)}$ computes such a copy. Now consider the copies of various L_α's computed by an iteration of the jump along L^*. At the well-founded levels we get L_α for all $\alpha < \omega_1^{\mathrm{ck}}$. At the ill-founded levels, we get ill-founded models behaving like L_α's, whose well-founded part includes $L_{\omega_1^{\mathrm{ck}}}$. Observing the interpretation of the function $f\colon \omega \to \omega_1^{\mathrm{ck}}$ in these models, we see that in each copy we get a restriction of f to (possibly) a subset of ω, but that in the ill-founded models, by upward absoluteness, we get f itself. That is, the ill-founded models believe that f is total. By "underspill", there must be some well-founded model which believes that f is total, that is, some L_α for $\alpha < \omega_1^{\mathrm{ck}}$ believes that f is total, whence f is bounded by α.

Yet another similar argument gives a set-theoretic interpretation of the Spector-Gandy theorem: a set $A \subseteq \omega^\omega$ is Π^1_1 if and only if there is some Σ_1 formula in the language of set theory such that for all $y \in \omega^\omega$, $y \in A$ if and only if $L_{\omega_1^y}[y] \models \varphi(y)$. Here ω_1^y is ω_1^{ck} relativised to y, that is, the least ordinal which does not have a y-computable copy. The set $L_{\omega_1^y}[y]$ is the smallest admissible set containing y as an element. Restricting to subsets of $\mathbb{N}$, and recalling the definition above of α-c.e. sets for admissible ordinals α, we obtain:

Proposition 1.18. *A set $A \subseteq \mathbb{N}$ is Π^1_1 if and only if it is ω_1^{ck}-c.e.*

Also, a set $A \subseteq \mathbb{N}$ is Δ^1_1 (hyperarithmetic) if and only if it is ω_1^{ck}-finite (an element of $L_{\omega_1^{\mathrm{ck}}}$). As mentioned above, because of the strong analogy between Π^1_1 sets and c.e. sets, which is exemplified by Proposition 1.18, one would be led to believe that Δ^1_1 should be analogous to "computable". Kreisel and Sacks realised that the correct analogue is "finite", and so turned to investigate the complexity of subsets of ω_1^{ck}, which may be ω_1^{ck}-computable and not ω_1^{ck}-finite. Nonetheless, ω_1^{ck}-computability is very useful in the investigation of Π^1_1 sets.

1.4. **Higher computability and effective descriptive set theory.** As an aside, we give an example for how computability can be used to prove theorems of descriptive set theory. This relies on the connection between the "boldface" set-theoretic notions (Borel, $\boldsymbol{\Pi^1_1}$) and their "lightface" computable analogue (hyperarithmetic, Π^1_1), which was mentioned above (Addison [2]). We note that the coincidence of hyperarithmetic and Δ^1_1 holds for sets of reals as well as of numbers; a set $A \subseteq \omega^\omega$ is hyperarithmetic if for some computable α, $y \in A \iff y^{(\alpha)} \in B$ for some computable set B. We show:

Proposition 1.19. *If $B \subseteq \omega^\omega$ is Δ^1_1 and $f\colon B \to \omega^\omega$ is computable and injective, then $f[B]$ is Δ^1_1.*

Proof. Since the map $y \mapsto y^{(\alpha)}$ is Π^0_2-definable, for every $x \in B$, x is a $\Pi^0_2(f(x))$-singleton, which in turn implies that x is $\Delta^1_1(f(x))$, and so by Kleene's theorem, hyperarithmetic relative to $f(x)$. Now $f[B]$ is naturally Σ^1_1 (as B is Σ^1_1); it is also Π^1_1, as $y \in f[B]$ if and only if there is some $x \in B$, hyperarithmetic relative to y, such that $f(x) = y$; we apply the easy direction of the Spector-Gandy theorem. $\square$

Since a function is contiuous if and only if it is computable relative to an oracle, we obtain:

Corollary 1.20. *If $B \subseteq \omega^\omega$ is Borel and $f\colon B \to \omega^\omega$ is continuous and injective, then $f[B]$ is Borel.*

There are several sophisticated uses of computability in descriptive set theory. For instance, the Glimm-Effros dichotomy (Harrington, Kechris and Louveau [48]) can be deduced from an effective version. Most recently, Day and Marks used a variety of effective considerations in their recent resolution of the decomposability conjecture (in preparation). Work leading up to their resolution also used surprising tools, such as the Shore-Slaman join theorem [83] used by Kihara [55].

2. Higher computability and randomness

In this section we give various examples of how the contrast and comparison between ω_1^{ck}-computability and classical computability give us new insights into the nature of computability itself. Some of these examples arise when we consider computability relative to an oracle; others, when we consider the interaction of computability and randomness. A main theme is the failure of "time tricks". Guided by questions of higher algorithmic randomness, we use ω_1^{ck}-computability to study reals, i.e., subsets of ω, rather than subsets of ω_1^{ck}. That is, the objects that we study have "height" ω. On the other hand, the computable processes that we use take up to ω_1^{ck} many "steps". This discrepancy between height and time exposes many instances of classical computability in which the coincidence of height and time in the lower setting is used in arguments. In the higher setting (of Π^1_1 sets and ω_1^{ck}-computability), some of the classical results still hold, but we need to devise new proofs; and some of the classical results fail. This tells us when time tricks are essential.

Algorithmic randomness attempts to answer the question "what does it means for a (finite or infinite) binary sequence to be random?" Using the tools of computability, it gives a hierarchy of randomness notions, based on ever more complicated null sets. For a detailed account, see Downey's chapter in this volume, or [27, 26, 71].

While most null sets considered in algorithmic randomness are around the level of effectively open and effectively G_δ sets, very early in the development of the theory, Martin-Löf [66] and Sacks [77, 78], and later Stern [86], introduced notions of randomness at a "higher setting", i.e., the level of arithmetic, hyperarithmetic and Π^1_1 sets. They defined the notion of Δ^1_1-randomness, which means avoiding all Δ^1_1 null sets; Sacks and Stern also introduced the notion of Π^1_1-randomness, with a similar definition.

A different approach was taken by Hjorth and Nies [50]. They relied on the analogy between Π^1_1 and c.e., and used the main definition of "classical" algorithmic randomness, namely ML-randomness, and simply replaced every instance of "c.e." by Π^1_1. They showed that the resulting notion of Π^1_1-ML randomness[5] shared many of the properties held by the classical notion. For example, it can be charaterised using a Π^1_1 version of Kolmogorov complexity, and a higher analogue of the Kučera-Gács theorem [63, 34] holds.[6] Along with this notion of randomness, they introduced a continuous reducibility which interacts well with the study of randomness.

This last point was taken up and studied in detail in [10]. The Kučera-Gács theorem is just one of many ways that randomness and Turing reducibility interact.[7] When studying the "higher" (Π^1_1) analogue of randomness, we therefore also need to understand what is the correct higher analogue of Turing reducibility. The first guess would be relative hyperarithmetic reducibility. However the main drawback of this reducibility is that it is not given by continuous maps. This is why Hjorth and Nies introduced $\leqslant_{\text{fin-h}}$, which is a continuous version of hyperarithmetic reducibility. For example, this is the reducibility that they use in their version of the Kučera-Gács theorem.

2.1. **Choosing the correct higher continuous reducibility.** In [10], the authors showed that there are inequivalent ways of generalising Turing reducibility to the higher setting, and argued that one more general than $\leqslant_{\text{fin-h}}$ is the correct one for studying higher randomness. The issue revolves around *consistency* of functionals. Let us give some details. A *functional* is a set $\Phi \subseteq 2^{<\omega} \times 2^{<\omega}$. If Φ is a functional and $x \in 2^\omega$, then we let $\Phi(x)$ be the union of all σ such that for some $\tau \prec x$ we have $(\tau, \sigma) \in \Phi$. The motivation is that for any $x, y \in 2^\omega$ we have $y \leqslant_{\mathrm{T}} x$ if and only if there is some c.e. functional Φ such that $\Phi(x) = y$. The pair (τ, σ) being in a functional Φ says that in the oracle machine coded by Φ, for any oracle x extending τ and any $k < |\sigma|$, on input k with oracle x the machine halts and outputs $\sigma(k)$.

For a functional Φ and $x \in 2^\omega$, there may be two reasons that $\Phi(x)$ would not be properly defined. One is partialness; in our formulation, $\Phi(x)$ may be a finite binary string rather than an infinite one. Another is that $\Phi(x)$ may be inconsistent, i.e., not a function: we could have (τ_0, σ_0) and (τ_1, σ_1) both in Φ where τ_0, τ_1 are both prefixes of x, but σ_0 and σ_1 are incomparable. This last point is often ignored in classical computability, because inconsistencies can be fixed: if $y \leqslant_{\mathrm{T}} x$ then in fact

[5] A Π^1_1-ML null set is a G_δ set of the form $\bigcap_n U_n$, where the sets U_n are uniformly Π^1_1 open, and $\lambda(U_n) \leqslant 2^{-n}$; here λ denotes the fair coin measure on 2^ω. Using Π^1_1 open sets is made easy by the fact that a Π^1_1 set is open if and only if it is generated by a Π^1_1 set of string, that is, if it is of the form $\{x \in 2^\omega : (\exists \sigma \in W)\, \sigma \prec x\}$ for some Π^1_1 set $W \subseteq 2^{<\omega}$.

[6] The Kučera-Gács theorem says that every real is computable from some random sequence.

[7] Another very well-known example is van-Lambalgen's effective "2-step iteration" theorem [90], which says that the join $X \oplus Y$ is ML-random if and only if X is ML-random and Y is ML-random relative to X. Yet another example is the Miller-Yu theorem [67], which states that if $X \leqslant_{\mathrm{T}} Y$ are both ML-random, and Y has one of many stronger randomness properties (for example, being 2-random), then X must share this property as well.

there is a consistent c.e. functional Φ such that $\Phi(x) = y$; Φ being consistent simply means that the situation above cannot occur, in other words, that it is consistent on all oracles.

Let us consider how we can remove inconsistencies. Suppose that Ψ is a c.e. functional; we can produce a consistent c.e. functional Φ such that for all x, if $\Psi(x)$ is consistent then $\Phi(x) = \Psi(x)$. How do we do this? we enumerate the "axioms" of Ψ (the pairs of strings in Ψ). Suppose that at stage s we have already enumerated Φ_s (a finite set of axioms), and see that a new axiom (τ, σ) is now enumerated into Ψ. It is possible that (τ, σ) is inconsistent with some axioms already in Φ_s, but it is also possible that the axiom applies to some oracles on which Ψ is consistent. What we do is look at every extension $\bar{\tau}$ of τ of length s, and enumerate the axiom $(\bar{\tau}, \sigma)$ into Φ_{s+1} only if it does not contradict another axiom already in Φ_s.

This was a time trick: at stage s, we used strings of length s, which were "fresh", in that they are longer than all strings that we dealt with so far. Now suppose that we work in the higher setting, with Π^1_1 functionals, which are enumerated in ω_1^{ck} many stages. We would like to mimic the argument, but now at stage $s \geqslant \omega$ we may be in bad shape. Suppose, for example, that at stage $n < \omega$ we see the axiom $0^n 1 \mapsto 0$ in Ψ and copy it over to Φ; all these axioms are pairwise consistent. However at stage ω we see that Ψ maps some 0^k to 1, and in fact it is possible that $\Psi(0^\omega) = 1 \cdots$ is total and consistent. However enumerating $0^m \to 1$ into Φ for any m after stage ω will make Φ inconsistent.

In fact, this argument is turned around in [9] to show that there are $x, y \in 2^\omega$ such that $\Phi(x) = y$ for some Π^1_1 functional Φ, but there is no consistent Π^1_1 functional Ψ such that $\Psi(x) = y$. That is, the time trick is essential in the previous argument. In [10], the authors argued that the relation $\leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}}$, defined by:

Definition 2.1. Let $x, y \in 2^\omega$. We say that x is *higher computable from* x, and write $y \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} x$, if there is some Π^1_1 functional Φ such that $\Phi(x) = y$.

is the correct definition to use. One piece of evidence is the relationship between computability and enumerability. There is only one reasonable definition for the relation "continuously relatively higher-x-c.e.": an *enumeration functional* is a set $W \subseteq 2^{<\omega} \times \omega$. For an enumeration functional W and an oracle $x \in 2^\omega$, we let W^x be the collection of all $n \in \mathbb{N}$ such that for some $\tau \prec x$ we have $(\tau, n) \in W$. Again the point is that for all x and $A \subseteq \mathbb{N}$, the set A is c.e. relative to x if and only if there is a c.e. functional W such that $W^x = A$. With enumeration functionals there are no issues of partialness or consistency, and so we define:

Definition 2.2. Let $x \in 2^\omega$. A set $A \subseteq \mathbb{N}$ is *higher x-c.e.* if $A = W^x$ for some Π^1_1 enumeration functional W.

The standard classical argument shows that that for any x and A, the set A is higher x-computbale (that is, $A \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} x$) if and only if A is both higher x-c.e. and higher co-x-c.e. Another piece of evidence for using $\leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}}$ is that some basic theorems about ML-random sequences, such as van-Lambalgen's theorem, hold for Π^1_1-ML sequences, with Turing replaced with $\leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}}$, rather than the consistent version of higher Turing reducibility.

Hjorth and Nies's definition is stricter than "consistent higher Turing". They defined $y \leqslant_{\text{fin-h}} x$ if there is a Π^1_1 functional Φ such that $\Phi(x) = y$, and such that Φ (as a set of pairs) is a monotone function defined on a subtree of $2^{<\omega}$. That is, not only is Φ consistent, but when we state that Φ maps τ to σ, we have already stated what Φ does on all of its initial segments. This appears to be a significant restriction. In the lower setting, an argument such as above uses a time trick to take a consistent functional and turn it into a c.e. functional of this type. It was therefore a little suprising to learn the following:

Proposition 2.3. *$y \leqslant_{\text{fin-h}} x$ if and only if there is a consistent Π^1_1 functional Φ such that $\Phi(x) = y$.*

That is, the time trick is not as essential to the result. Of course we need to make a new argument, and this argument is non-uniform (and must be so). Roughly, it goes as follows. Suppose that Φ is a consistent functional and that $y = \Phi(x)$. Define a Π^1_1 tree $T \subseteq 2^{<\omega}$: at stage $s < \omega_1^{\text{ck}}$, enumerate into T_s all strings ρ such that for all n, some extension of ρ is mapped by Φ_s to a string of length $\geqslant n$. Since $\Phi(x)$ is total, an admissibility argument show that $x \in [T]$.[8] Now there are two cases. If there is no s such that $x \in [T_s]$, then x collapses ω_1^{ck} in a continuous way: the map taking n to the least s such that $x{\upharpoonright}n \in T_s$ is cofinal in ω_1^{ck} and higher computable from x. Now we can transform Φ to a fin-h functional by copying $\Phi_s(\tau)$ if $\tau \in T_{s+1} \backslash T_s$. Otherwise, $x \in [T_s]$ for some $s < \omega_1^{\text{ck}}$. We can then use Φ_s to give a fin-h functional Ψ with $\Psi(x) = y$, as we can examine Φ_s in a hyperarithmetic way and let Ψ map ρ to the longest string compatible with $\Phi_s(\tau)$ for all extensions τ of ρ.

Another comparison between higher and classical computability is done by examining relative effectively closed sets. For every x, there is a $\Pi^0_1(x)$ class which contains no x-computable points. The usual argument is a time trick, but nonetheless, by a nonuniform argument, we can show that for every x, there is a higher x-effectively closed set containing no $y \leqslant_{\omega_1^{\text{ck}}\text{T}} x$ [9]. We can again show that that the nonuniformity is necessary.[9]

2.2. **A deeper look into ML randomness.** The idea of replacing "c.e." by Π^1_1 can be now relativised using Definition 2.2. We thus define:

- For every x, a higher x-ML null set is a set $\bigcap_n U_n$ such that $\lambda(U_n) \leqslant 2^{-n}$, and the sets U_n are uniformly higher x-c.e. open (generated by higher x-c.e. sets of strings).

Classically, the centrality of ML-randomness is witnessed by its robustness: many equivalent definitions coincide. Which of the implications are necessary, and which coincidental? Consider, for example, discrete measures. A *discrete measure* is a function $\mu\colon \omega \to \mathbb{R}^{\geqslant 0}$ such that $\mu(\omega) = \sum_n \mu(n)$ is finite. After identifying between numbers and finite binary strings, each discrete measure determines a co-null set: the set R_μ of x such that $\mu(x{\upharpoonright}n) \geqslant^\times 2^{-n}$.[10] Classically, a real x is ML-random if and only if for every left-c.e. (lower semicomputable) discrete measure μ, x is in the associated co-null set R_μ. Let us recall how to show this.

[8] For every $\tau \prec x$, for any n we know that Φ maps some extension of τ to some string of length $\geqslant n$ (namely some initial segment of x); the map taking n to the least s at which such an extension appears is ω_1^{ck}-computable, and so bounded.

[9] An interesting contrast is given by work of J. Miller and M. Soskova (in preparartion), who examine relativised randomness and related notions working with enumeration oracles, that is, using enumeration reducibility. One of their results is the construction of a "self-PA" oracle in their context, an oracle for which the property we discussed above fails.

[10] This means: for some constant $\delta > 0$, for all n, $\mu(x{\upharpoonright}n) \geqslant \delta 2^{-n}$.

In one direction, let μ be a left-c.e. discrete measure. We show that there is a ML null set $\bigcap_n U_n$ continaing the complement of R_μ, so that every real "captured" by μ (in the sense that $x \notin R_\mu$) is captured by the ML-null set; this will show that if x is ML-random then it is random for discrete measures (meaning $x \in R_\mu$ for all left-c.e. μ). For each n, we let S_n be the set of strings σ such that $\mu(\sigma) > 2^n 2^{-|\sigma|}$, and let U_n be the open set generated by S_n. The sets U_n are uniformly c.e., and $\lambda(U_n) \leqslant^\times 2^{-n}$, the point being that $\lambda(U_n)$ is bounded by the *weight* $\sum_{\sigma\in S_n} 2^{-|\sigma|}$ of S_n, which in turn is bounded by $2^{-n}\mu(\omega)$.

In the other direction, let $\bigcap_n U_n$ be an ML-null set; we find a left-c.e. discrete measure μ which captures every $x \in \bigcap_n U_n$. To do this, for every n (uniformly) we find a c.e. antichain A_n of strings which generates U_n. We then define $\mu_n(\sigma) = 2^n 2^{-|\sigma|}$ for every $\sigma \in A_n$, and $\mu_n(\sigma) = 0$ otherwise; then $\mu_n(\omega)$ is bounded by the weight of A_n, which is the measure of U_n, and so $\mu = \sum_n \mu_n$ is as required.

Now the first implication is completely natural, and the same argument shows:

Proposition 2.4. *For every $x \in 2^\omega$, every sequence r which is higher x-ML random is random for higher x-c.e. discrete measures, that is, for every higher x-left-c.e. discrete measure μ, $r \in R_\mu$.*

In the other direction, one step is problematic: finding an antichain A_n generating U_n. This relies on a time trick. Let W be a c.e. set of strings generating an effectively open set W. We enumerate an antichain A in stages; at stage s, when we see a string σ enter S, we enumerate into A all extensions of σ of length s which do not extend any string previously enumerated into A. As above, we use the fact that at stage s, all strings in A have length $< s$. We can, in fact, show that this time trick is necessary:

Proposition 2.5 ([9])**.** *There is a higher effectively open set (a set generated by a Π^1_1 set of strings) which is not generated by any Π^1_1 antichain.*

Sketch of proof. We enumerate a higher c.e. set of strings V, ensuring that for all e, the e^{th} higher c.e. set of strings W_e is not an antichain, or it does not generate $[V]^\prec$ (the open set generated by V). Let $\sigma_e = 0^e 1$. For each e, let $A_e = \{\sigma_e\hat{\ }\sigma_n : n \in \mathbb{N}\}$. We let $V_0 = \bigcup_e A_e$. At stage $s < \omega_1^{\text{ck}}$, for every e, we check if both $[A_e]^\prec \subseteq [W_{e,s}]^\prec$ and $\sigma_e\hat{\ }0^\infty \notin [W_{e,s}]^\prec$. If so, then we enumerate σ_e into V_{s+1}.

Fix e, and suppose that $[W_e]^\prec = [V]^\prec$. By compactness of 2^ω, every $\tau \in A_e$ is covered by finitely many strings in W_e, and so there is some s such that $[\tau] \subseteq [W_{e,s}]^\prec$; this is recognised computably. By admissibility of ω_1^{ck}, as A_e is ω_1^{ck}-finite, there is some s such that $[A_e]^\prec \subseteq [W_{e,s}]^\prec$. Let s be least such. If $\sigma_e\hat{\ }\infty \in [W_{e,s}]^\prec$ then by our instructions, $\sigma_e\hat{\ }0^\infty \notin [V]^\prec$, contradicting $[W_e]^\prec = [V]^\prec$. Hence $\sigma_e\hat{\ }0^\infty \notin [W_{e,s}]^\prec$, whence by our actions, $\sigma_e\hat{\ }0^\infty \in [V]^\prec$, implying that $\sigma_e\hat{\ }0^\infty \in [W_e]^\prec$; an initial segment of $\sigma_e\hat{\ }0^\infty$ is enumerated into W_e at some stage $t > s$, and so must be incomparable with some string already in $W_{e,s}$, as $[A_e]^\prec$ is dense along $\sigma_e\hat{\ }0^\infty$. □

Thus, it is not clear that the result holds in all settings, and indeed, it does not; in upcoming work, the author, together with J. Miller and B. Monin, show that there is some oracle x relative to which there is a sequence r, which is random for higher x-c.e. discrete measures, but is not higher x-ML random.

Is the oracle necessary? the answer is positive; indeed, Hjorth and Nies showed the equivalence of higher ML-randomness with the even weaker property of being random for higher prefix-free complexity. To overcome the reliance on time tricks, they used the next lemma below. In the argument above, it was not actually important that the sets A_n were antichains; what we really used are the properties $U_n \subseteq [A_n]^{\prec}$ and $\mathrm{wt}(A_n) \leqslant 2^{-n}$, where again $\mathrm{wt}(A) = \sum_{\tau\in A} 2^{-|\tau|}$; we used an antichain because if A is an antichain then $\mathrm{wt}(A) = \lambda(A)$.[11] And the condition $\mathrm{wt}(A_n) \leqslant 2^{-n}$ is not fundamental either; all we need is $\sum_n \mathrm{wt}(A_n)$ to be finite. The following lemma then suffices.

Lemma 2.6. For every Π^1_1 open set U and every $\varepsilon > 0$ there is a Π^1_1 set of strings V such that $U \subseteq [V]^{\prec}$ and $\mathrm{wt}(V) \leqslant \lambda(U) + \varepsilon$. This is uniform in U and ε.

Sketch of proof. We give a proof slightly different to the one in [50], by introducing a new tool: the *projectum function*. There is an ω_1^{ck}-computable injective function $p\colon \omega_1^{\mathrm{ck}} \to \omega$ (essentially, take α to some index for a computable copy of α). We use p to "distribute mass" along the stages $s < \omega_1^{\mathrm{ck}}$. By stage s, we will have enumerated V_s with $U_s \subseteq [V_s]^{\prec}$. Suppose that $U_{s+1} = U_s \cup [\tau_s]$. Since $[\tau_s]\backslash[V_s]^{\prec}$ is hyperarithmetic, we can effectively (in the sense of $L_{\omega_1^{\mathrm{ck}}}$) find an antichain C_s (indeed a finite one) satisfying $[\tau_s] \subseteq [V_s]^{\prec} \cup [C_s]^{\prec}$ and $\lambda(C_s) \leqslant \lambda([\tau_s]\backslash[V_s]^{\prec}) + \varepsilon 2^{-p(s)}$; we add C_s to V_{s+1}. The sum of all the "extra errors" $\sum \varepsilon 2^{-p(s)}$ is bounded by 2ε, as p is injective. □

Along these lines, Hjorth and Nies showed that there is a universal Π^1_1-ML test (a largest Π^1_1-ML null set); the standard argument applies. However, when relativising, in the lower setting, a time trick is used to construct a *uniform* oracle universal ML-test; oracle effectively open operators U_n such that for every oracle x, $\langle U_n^x\rangle$ is a universal test for x-ML-randomness. In the higher setting, in [10] it is shown that this cannot be done in the higher setting, and in fact, with more work, an oracle x is constructed for which there is no universal higher x-ML test at all.

Further work, however, is required to completely elucidate the relationships between all variants of ML-randomness, for example, those that rely on c.e. martingales (equivalently, left-c.e. continuous measures), prefix-free Kolmogorov complexity, and Schnorr tests. The necessary implications are clear, but constructions of oracles witnessing the failure of other implications appear to be hard.

2.3. **The higher limit lemma.** Recall that Shoenfield's limit lemma states that a function f is $\varnothing'$-computable functions iff it has a *computable approximation*: a uniformly computable sequence $\langle f_s\rangle$ such that $f = \lim_s f_s$ (in the discrete topology). In the higher setting, any complete Π^1_1 (such as Kleene's $\mathcal{O}$, or the set of indices of computable well-orderings) plays the role of $\varnothing'$. For computable approximations, though, we need approximations of length ω_1^{ck} (the limit of a hyperarithmetic ω-sequence of functions is hyperarithmetic, as Δ^1_1 is closed under taking the Turing jump). Indeed, fixing a complete Π^1_1 set O, the following are equivalent for a function f:

- $f \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} O$;[12]
- there is a ω_1^{ck}-computable sequence $\langle f_s\rangle_{s<\omega_1^{\mathrm{ck}}}$ (of functions $f_s\colon \omega \to \omega$, each necessarily being hyperarithmetic) such that $f = \lim_s f_s$, in the sense that for all n, for some $s < \omega_1^{\mathrm{ck}}$, $f_t(n) = f(n)$ for all $t \in [s, \omega_1^{\mathrm{ck}})$.

[11] We write $\lambda(A)$ for $\lambda([A]^{\prec})$.

[12] We remark that $f \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} O$ if and only if $f \leqslant_{\mathrm{T}} O$.

The fact that the approximation has limit stages allows us to define and investigate subclasses of the O-computable functions which have no classical analogue. For example, a *finite-change approximation* is an approximation $\langle f_s \rangle$ such that for all n, there is no infinite increasing sequence $\langle s_k \rangle$ of stages such that for all k, $f_{s_{k+1}}(n) \neq f_{s_k}(n)$.[13] Not all functions $f \leqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} O$ have finite change approximations.

Such subclasses can be used to answer questions about notions of randomness that lie between higher ML randomness and Π^1_1-randomness, prime among them the higher version of weak 2-randomness (or strong 1-randomness): a higher test for weak 2-randomness is a set $\bigcap_n U_n$ which is null, where the sets U_n are uniformly Π^1_1 open. Each such null set is Π^1_1, and so we get the implications:

$$\Pi^1_1\text{-random} \quad \Longrightarrow \quad \text{higher weakly 2-random} \quad \Longrightarrow \quad \text{higher ML random.}$$

Some basic facts about weak 2-randomness rely on time tricks. Consider, for example, the fact [23] that no weakly 2-random sequence can be Δ^0_2: let $\langle x_s \rangle$ be a computable approximation of a Δ^0_2 sequence x. We let U_n be the open set generated by $\{x_s \upharpoonright n \,:\, s > n\}$; note how stages and lengths are compared. Then $\bigcap_n U_n = \{x\}$, which is null, and the sets U_n are uniformly c.e. We know that to some extent the time trick is necessary: Gandy's basis theorem implies that there is a Π^1_1-random sequence $x \leqslant_{\mathrm{T}} O$.

To what extent can O-computable sequences be higher weak 2-random? Chong and Yu [16] showed, using the Lebesgue density theorem, that no higher weakly 2-random sequence can be higher left-c.e. (left-Π^1_1). In [44], the following is shown:

Proposition 2.7. *No sequence which has a finite-change approximation can be higher weakly-2 random.*

Sketch of proof. Let $\langle x_s \rangle$ be a finite-change approximation of $x = x_{\omega_1^{\mathrm{ck}}}$. The property that we use about such approximations is that (after perhaps modifying the limit steps) the set $X = \left\{x_s \,:\, s \leqslant \omega_1^{\mathrm{ck}}\right\}$ is *closed*. Let U_n be the open set generated by $\left\{x_s \upharpoonright n \,:\, s < \omega_1^{\mathrm{ck}}\right\}$. Then every $y \in \bigcap_n U_n$ lies in the closure of X, and hence in X. Since X is countable, it is null. ☐

Proposition 2.7 is used, for example, to show that the two halves of the higher version of Chaitin's Ω are not Π^1_1-random, indeed, they are not higher weakly 2-random. Another property, weaker than having a finite-change approximation, can be used to separate Π^1_1-randomness from higher weak 2-randomness. This relies on Stern's result [86], rediscovered by Chong, Nies and Yu [17], that x is Π^1_1-random if and only if it is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{ck}}$. Thus, it suffices to constuct a higher weakly 2-random sequence x such that $\omega_1^x > \omega_1^{\mathrm{ck}}$. For example, every non-hyperarithmetic x with a finite-change approximation satisfies $\omega_1^x > \omega_1^{\mathrm{ck}}$: the function mapping n to the least s such that $x \upharpoonright n = x_s \upharpoonright n$ is unbounded in ω_1^{ck}. Unfortunately, Proposition 2.7 says that this cannot be used for the desired separation. However a weaker notion is compatible with being higher weakly 2-random: having an approximation $\langle x_s \rangle$ such that for all n, the set $\left\{s < \omega_1^{\mathrm{ck}} \,:\, x_s \upharpoonright n = x \upharpoonright n\right\}$ is closed (and neccessarily unbounded). Again, if x has such an approximation

[13] Note that it is not enough to require that $f_{s+1}(n) \neq f_s(n)$ for only finitely many $s < \omega_1^{\mathrm{ck}}$; to that, we need to add that for all limit $s < \omega_1^{\mathrm{ck}}$, the limit $f_{<s} = \lim_{t<s} f_t$ exists and for all n, $f_s(n) \neq f_{<s}(n)$ for only finitely many limit s. In general, in approximations for which the limits $f_{<s}$ exist for all limit s, by reindexing, we assume that $f_s = f_{<s}$.

and is not hyperarithmetic, then the map $n \mapsto$the least s such that $x_s\restriction n = x\restriction n$ is unbounded in ω_1^{ck} (if $t \leqslant \omega_1^{\mathrm{ck}}$ is a bound, then by closure, $x_t = x$; for all $t < \omega_1^{\mathrm{ck}}$, x_t is hyperarithmetic).

Finally, we remark that subclasses of the O-computable functions can be used to give Demuth-style characterisations of higher weak 2-randomness. This is related to the class $\mathtt{MLR}[\varnothing']$, determined by null sets $\bigcap_n U_n$ for which $\lambda(U_n) \leqslant 2^{-n}$, each U_n is effectively open, but the sequence itself is not necessarily uniformly so; rather, $\varnothing'$ can be used to find a c.e. open index for U_n. In the lower setting, $\mathtt{MLR}[\varnothing']$ is equivalent to weak 2-randomness, but in the higher setting, the class $\mathtt{MLR}[O]$ (modified so the indices are of Π^1_1 open sets) is very strong, strictly stronger than Π^1_1-randomness, as it is incompatible with being O-computable. A time trick similar to the one above is used in the lower setting. In the higher setting, we restrict the kind of functions that give indices.

For example, letting $\langle W_e\rangle_{e<\omega}$ be an effective enumeration of all Π^1_1 open sets, we say that a *finite-change null set* is a null set of the form $\bigcap\langle W_{f(n)}\rangle_{n<\omega}$ which is nested (meaning $W_{f(n+1)} \subseteq W_{f(n)}$), satisfies $\lambda(W_{f(n)}) \leqslant 2^{-n}$, and such that f has a finite-change approximation. Avoiding all finite-change null sets is equivalent to higher weak 2-randomness. In the direction which requires a new argument, curiously, we are informed by the proof of Proposition 2.7. Suppose that $\langle W_{f(n)}\rangle$ determines a finite-change null set, which we want to cover by a higher weak 2-test; let $\langle f_s\rangle$ be a finite-change approximation of f. By fiddling, we may assume that for all s, $\langle W_{f_s(n)}\rangle$ is nested and $\lambda(W_{f_s(n)}) \leqslant 2^{-n}$. Let $U_n = \bigcup_{s<\omega_1^{\mathrm{ck}}} W_{f_s(n)}$; we show that $\bigcap U_n$ is null. For $s \leqslant \omega_1^{\mathrm{ck}}$, let $A_s = \bigcap_n W_{f_s(n)}$, and let $A = \bigcup_{s\leqslant\omega_1^{\mathrm{ck}}} A_s$. Each A_s is null, and ω_1^{ck} is countable, so A is null. And $\bigcap_n U_n \subseteq A$; this uses the fact that $\left\{f_s \,:\, s \leqslant \omega_1^{\mathrm{ck}}\right\}$ is compact.

2.4. **Other work.** There is much more to say about higher randomness. There is extensive work on lowness notions for higher randomness (the higher analogues of K-trivial sets) in [50, 10, 3], work on the Borel complexity of the set of Π^1_1-randoms [68], and a higher analogue of the Miller-Hirschfeldt theorem saying that a ML-random is weak 2 random if and only if it forms a minimal pair with $\varnothing'$ [44]. In a different direction, recently researchers have been studying randomness defined using infinite-time Turing machines [11, 69], in which admissibility plays an important role as well.

3. Uncountable computable structure theory

Effective considerations of rings and fields were made even before the formalisation of computability itself [21, 49, 89]. The modern incarnation of these is the field of computable structure theory (or computable algebra). The aim is to understand the relationship between the algebraic structure and the information stored in that structure; see the surveys [58, 47, 22], the books [6, 31] and the upcoming book by Montalbán.

By the nature of the tools involved, such considerations are restricted to countable structures. It is natural to attempt to study effective properties of uncountable structures as well, and several approaches have been suggested (see [40]). In [42], J. Knight and the author suggested using admissible computability on cardinals for this purpose (recalling Example 1.9 saying that every cardinal is an admissible ordinal). For example, for any cardinal κ, we say that a group G with universe κ is *κ-computable* if its group operation is a κ-computable function, and similarly a linear ordering with universe κ is κ-computable if the ordering relation is κ-computable. We can then attempt to answer the same questions asked in the countable setting: which linear orderings of size κ have κ-computable copies? What does it take to compute isomorphisms between two κ-computable copies of some structure?

This last question gives rise to the concept of a *κ-computably categorical* structure: a structure M such that for any two κ-computable copies of M, there is a κ-computable isomorphism between them. The classical notion of ω-computably categorical structres, which was introduced by Mal'cev [65], has been studied extensively. There are two lines of inquiry. One is general; important results, for example, are the characterisation of relative computable categoricity by syntactic means of an effective family of formulas defining the orbits of the structure [13, 7]; Goncharov's result saying that computable categoricity is equivalent to its relative version in 2-decidable structures [37]; work on uniform computable categoricity [24]; Goncahrov's construction of structures with finite computable dimension (the number of computable copies up to computable isomorphism) [38]; and more recent work on the complexity of being computably categorical [28]. In [43], some work along these lines is carried out in the higher setting.

Another line of inquiry is trying to understand computable categoricity in particular classes of structures. An important example is linear orderings, where the fundamental result is due to Dzgoev (see [36]) and Remmel [74]:

Theorem 3.1. *A computable linear ordering is computably categorical if and only if it has finitely many elements with successors.*

Theorem 3.1 formalises the idea that the only way that a linear ordering can be computably categorical is if Cantor's back-and-forth technqiue can be used to construct isomorphisms between any two of its copies. The point about having finitely many successors is that after matching up finitely many elements (the successor pairs), which can be done non-uniformly, the rest is broken up to finitely many dense pieces, which we can match with a back-and-forth process.

In [41], the authors characterise ω_1-computably categorical linear orderings. As we shall see, this sheds light on the classical result as well. The naive attempt at generalising Theorem 3.1 to the higher setting would be to guess that an ω_1-computable linear ordering is ω_1-computably categorical if and only if it has only countably many successor pairs; recall that in L, $L_\kappa = H_\kappa$, so the ω_1-finite sets are precisely the hereditarily countable ones.[14] This, however, fails in both directions.

In one direction, we observe that $2 \cdot \mathbb{R}$, the linear ordering obtained from the reals by replacing each real by a successor pair, is ω_1-computably categorical. If we are given two copies of $2 \cdot \mathbb{R}$, then non-uniformly, we fix two copies of $2 \cdot \mathbb{Q}$ inside them. Then we can match the rest. The important point is that if A is a countable subset of an ω_1-computable linear ordering K, then for every $x \in K\backslash A$, we can ω_1-effectively find the left and the right cuts that x defines in A, that is, the sets $\{a \in A : a <_K x\}$ and $\{a \in A : x <_K a\}$. So suppose that K and K' are two ω_1-computable copies of $2 \cdot \mathbb{R}$; let A and A' be the images of $2 \cdot \mathbb{Q}$ in K and K', respectively. As stated, an order-preserving bijection $\pi\colon A \to A'$ can be

[14] Throughout, we assume, for simplicity, that $V = L$.

fixed, as it is a countable object. Now given any $p \in K \backslash A$, we compute the A-cut (B_1, B_2) determined by p. We first wait for another point q which determines the same A-cut. Then, we find two points p' and q' which in K' determine the A'-cut $(\pi[B_1], \pi[B_2])$, and match $\{p, q\}$ with $\{p', q'\}$ in an order-preserving way.

What this second example tells us is that the Dzgoev-Remmel result relies, to put it flippantly, on the fact that $\aleph_0$ is a strongly inaccessible cardinal: in more serious language, on the fact that a finite subset of a linear ordering can be split into only finitely many cuts, and so determines only finitely many intervals in the linear ordering. In any case, the analysis so far shows that the phrasing of the result in terms of successor pairs is a little misleading. Rather, what it should say, is that a countable linear orderin K is computably categorical if and only if there is a finite set $A \subset K$ such that for every cut (A_1, A_2) of A, the order-type of the K-interval determined by that cut allows us to effectively match it to its copy.[15] The example $2 \cdot \mathbb{R}$ shows that in the collection of these "effectively matchable" order-types, we must include the finite ones, as well as, in the countable setting, the rationals.

Under this reconsideration, we could guess that an ω_1-computable linear ordering K is ω_1-computably categorical if and only if there is a countable set $A \subset K$ such that every K-interval determined by a cut of A is either finite or dense. However, in the uncountable setting, density is insufficient. For we note that the linear ordering $\mathbb{Q} \cdot \omega_1$ (the result of replacing each point in the linear ordering ω_1 by a copy of the rationals) is dense, but not ω_1-computably categorical. We can construct two ω_1-computable copies K and K' of this linear ordering and at the same time diagonalise against all ω_1-computable attempts at an isomorphism between them. This is done with a priority argument. For each $e < \omega_1$, we ensure that the e^{th} partial ω_1-computable function φ_e is not an isomorphism from K to K'. To do that, we fix an interval C of K of order-type $\mathbb{Q}$; when we see that φ_e has halted on every point of C, we add a new point to K', between points of $\varphi_e[C]$. Note that if φ_e is total, we will witness $\varphi_e \restriction C$ at a countable stage, as C is countable. The requirement then imposes restraint on weaker ones. This only restrains countable pieces of K and K', and so the regularity of ω_1 implies that each requirement can find some unrestrained space.

What underlies this argument is that in the original plan of carrying out the back-and-forth process, density was not really the important property of $\mathbb{Q}$; what is important is that $\mathbb{Q}$ is *saturated*. There is a unique saturated ω_1-linear ordering, denoted by η_1, and we can construct ω_1-computable copies of it. In any case, this shows that perhaps a reformulation of the Dzgoev-Remmel theorem which grasps at its "real essence" is as follows:

Theorem 3.2. *A computable linear ordering K is computably categorical if and only if there is a finite set $A \subset K$ such that every K-interval determined by a cut A is either finite or saturated.*

We almost have a good guess for how to characterise ω_1-computably categorical linear orderings; we just require the parameter set A to be countable. But there is one last issue. Given two linear orderings of finite size n, we can effectively match

[15] Here a cut (A_1, A_2) of A is a partition of A into an initial and final segment (one of which may be empty), and the K-interval determined by this cut is $\{x \in K : A_1 < x < A_2\}$, where as expected $A_1 < x$ means $(\forall a \in A_1)\, a <_K x$, and similarly for $x < A_2$. We sometimes write $(A_1, A_2)_K$ for this K-interval.

between them by waiting for all n points to appear. However, this assumes that *we know* n. If there are infinitely many intervals, we could have some, many, or all sizes appear as intervals. To complete a construction for all intervals together, we need to know which is which. In the lower setting, this effective aspect of the characterisation is missing, because adding a finite amount of information costs nothing. This hidden aspect is revealed in the eventual characterisation:

Theorem 3.3 ([41])**.** *An ω_1-computable linear ordering K is ω_1-computably categorical if and only if there is a countable set $A \subset K$ such that every K-interval determined by a cut of A is either finite or saturated, and further, there is a partial ω_1-computable function which for every cut (A_1, A_2) of A for which the interval $(A_1, A_2)_K$ is finite, maps the cut to the size of that interval.*[16]

Now one may ask why we restricted ouselves to $\kappa = \omega_1$. What about ω_2, and other cardinals? The answer lies in the proof of Theorem 3.3. It is beyond the scope of this survey to give a detailed account. We only mention that the argument relies on the fact that every countable linear ordering has a proper self-embedding (this is how we force an opponent to add points where we want them). Further, it makes use of Hausdorff's separation between scattered and nonscattered countable linear orderings, and so in contrast with the classical case, the construction on an interval will have two pahses: first, we must force the opponent's interval to be nonscattered; then we make it saturated. This special analysis of countable linear orderings does not generalise to uncountable ones; indeed, there are uncountable linear orderings with no proper self-embeddings. To date, there is no charaterisation of ω_2-computably categorical linear orderings.

3.1. **Free abelian groups.** When a structure is not computably categorical, it still makes sense to ask how much information is required to compute isomorphisms between any two computable copies. In the special case of the countable free abelian group $\mathbb{Z}^{(\omega)}$, computing an isomorphism with the standard copy is the same as computing a basis for the group. In the countable case, Downey and Melnikov [25] showed that the free abelian group is Δ^0_2-categorical; equivalently, if we are given a copy of the countable free abelian group (via its groups table), then with one Turing jump we can construct a basis. The construction is recursive: at a finite stage we have a finite piece of the basis, and then we can extend it to a larger finite piece, making sure that the next element of the group is generated.

Of course, this resembles the construction of a basis of a vector space: add a vector not spanned by the ones chosen so far. In the case of groups, the fact that we cannot always divide means that some finite linearly independent sets cannot be part of a basis (think the subset $\{2\}$ of the group $\mathbb{Z}$). At the minimum, they need to span a *pure subgroup*: roughly, one in which every division which occurs in the larger group, already occurs within the subgroup. Pontryagin [73] essentially showed that a finite set generating a pure subgroup can be extended to a basis; in other words, a finitely generated pure subgroup H of a free group G *detaches* in G, in the sense that $G = H \oplus K$ for some subgroup K of G. This allowed Downey and Melnikov to give to construct a basis by recursion.

[16] On cuts defining infinite outputs, the output of this function may be anything. The point being that during the construction of an isomorphism, we may believe that $(A_1, A_2)_K$ has some size n, and match accordingly. If this guess is false, then $(A_1, A_2)_K$ is saturated, so we can extend the matching that we already made to an isomorphism.

What happens in the uncountable setting? The situation for vector spaces is the same as in the countable world: we may need one jump to determine linear independence, but once we have that, we can construct bases by transfinite recursion, adding one element at a time. With free abelian groups, the situation is very different. Fuchs (see [33]) showed that there is a pure subgroup H of $\mathbb{Z}^{(\omega)}$ which does not detach in $\mathbb{Z}^{(\omega)}$. Suppose that this $\mathbb{Z}^{(\omega)}$ sits inside an uncountable free group G (all groups henceforth are abelian). The following situation may occur: we try to build a basis for G by recursion. We keep adding elements, all within H, each generating a pure subgroup. At each finite step, we have a finite set; by Pontryagin, it can be extended to a basis of G. But if we happened to work within H, at stage ω we may have an infinite set which cannot be extended to a basis of G. New complications are introduced at limit steps that do not occur in the countable world.

In fact, the situation is dire. Not only does not a jump or two suffice, no transfinite iteration suffices, and beyond:

Theorem 3.4 ([45])**.** *For every uncountable regular cardinal κ, and every set $X \subseteq \kappa$ which is $\Delta^1_1(L_\kappa)$-definable, there is a κ-computable copy G of the free abelian group $\mathbb{Z}^{(\kappa)}$ which has no X-computable basis.*

The class $\Delta^1_1(L_\kappa)$ is huge, much bigger than any definition of "hyperarithmetic" in the context of κ-computability. So in particular, Theorem 3.4 implies that there is a κ-computable copy of $\mathbb{Z}^{(\kappa)}$ which has no basis which is first-order definable over L_κ. Essentially, no process of recursion can be designed that builds bases of uncountable free groups "from below". A basis needs to be given in its entirety.

The work in [45] continues work in [46], in which Theorem 3.4 was first proved for all successor cardinals κ, in fact for all regular cardinals which are not weakly compact. For such cardinals, it is shown in [46] that the problem of *identifying* free groups is as complicated as possible: it is $\boldsymbol{\Sigma}^1_1(L_\kappa)$-complete.

In general, the structure of uncountable torsion-free abelian groups is "sufficiently thin" so that set-theoretic considerations play a major role in their investigation. This is most famously seen in Shelah's independence result for the Whitehead problem (asking whether every abelian group satisfying $\mathrm{Ext}(G, \mathbb{Z}) = 0$ is free) [79]; for more on this extensive body of work see the book [30]. The results mentioned above heavily rely on set-theoretic methods as well.

For example, the identification of free groups is tied to the problem of telling whether a given subset of a regular cardinal κ is nonstationary. Suppose that G is an abelian group with universe κ. A *filtration* of G is a sequence $\langle G_\alpha \rangle_{\alpha \leqslant \kappa}$ which is increasing ($\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$), continuous ($G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ for all limit ordinals $\beta \leqslant \kappa$), satisfying $G = G_\kappa$ and $|G_\alpha| < \kappa$ for all $\alpha < \kappa$. If $\bar{G} = \langle G_\alpha \rangle$ is a filtration of G, then we let the *detachment set* of $\bar{G}$ to be

$$\mathrm{Div}(\bar{G}) = \{\alpha < \kappa \,:\, (\forall \beta \in (\alpha, \kappa))\ G_\alpha \mid G_\beta\}\,,$$

where $H \mid K$ denotes that H detaches in K as a direct summand.[17] The detachment set depends on the choice of filtration, however any two choices result in detachment sets that are equivalent modulo the nonstationary ideal on κ; any two filtrations agree on a club (closed and unbounded subset of κ). If every G_α for $\alpha < \kappa$ is free, then G is free if and only if $\mathrm{Div}(\bar{G})$ contains a club [29].

[17] When G_β is free this is equivalent to G_β / G_α being free.

Fokina et al. [32] showed that the nonstationary ideal on ω_1 is $\Sigma^1_1(L_{\omega_1})$-complete. For most regular cardinals, we can show that telling which sets contain clubs is computationally equivalent, in the sense of κ-computability, to the problem of identifying which groups are free. In the harder direction, given a set $X \subseteq \kappa$, we want to κ-effectively construct a filtration $\bar{G}$ of a group $G = G(X)$ which is free if and only if X contains a club. In set theory, this is usually done "statically", but when effective considerations are applied, it is useful to think of the constrction as being done recursively. At step α of the construction we will have defined G_γ for all $\gamma \leqslant \alpha$, and need to define $G_{\alpha+1}$, depending on whether $\alpha \in X$ or not. If $\alpha \in X$ then we want to arrange that $\alpha \in \mathrm{Div}(\bar{G})$, by adding copies of $\mathbb{Z}$ as direct summands. If $\alpha \notin X$ then we want to "twist" G_α inside $G_{\alpha+1}$ so that $\alpha + 1$ witnesses that $\alpha \notin \mathrm{Div}(\bar{G})$. In doing so, we need to ensure that we do not destroy past detachments of G_β's for $\beta < \alpha$.

It would seem that we want to ensure that $\mathrm{Div}(\bar{G}) = X$. However, this ignores an important point: we need to ensure that for all limit $\alpha < \kappa$, G_α is free. In other words, we want to ensure that we haven't twisted too much, even when X is sparse. To do that, we restrict our twisting further, to a set $E \subseteq \kappa$ which is stationary in κ, but does not reflect (for all $\alpha < \kappa$, $E \cap \alpha$ is nonstationary in α). In the case that κ is a successor cardinal, such a set E is given by Jensen's elaborate machinery [51] which he used to define a global square sequence in L. When κ is inaccessible, we need to thin E further, to a set which witnesses the describability of κ, when such a set exists.[18] When such a set does not exist, the cardinal κ is weakly compact, and then this very compactness tells us that the problem of determining which group is free is actually relatively simple: a group G of size κ is free if and only if every subgroup of size $< \kappa$ is free, and so in this case the collection of free groups is $\Pi^0_2(L_\kappa)$.

3.2. **Further work.** Ash et al. [4, 5] generalised Watnick's result [91, 75] (independently discovered by Downey (see [22]) and by Ash et al.) by showing that for any computable ordinal α and any $\mathbf{0}^{(2\alpha)}$-computable linear ordering K, there is a computable copy of $\mathbb{Z}^\alpha \cdot K$.[19] This shows that 2α jumps are not only sufficient but required to compute the iteration of the Hausdorff drivative of a linear ordering (the derivative is taken by identifying points which are finitely far apart). Ash's technique is related to Montalbán's "true stage" approximation of the iterations of the jump [70]. Work in progress by Turetsky and the author examines the situation in the uncountable setting, again revealing what is special about the countable case and what is general. For example, the work seems to imply that the fundamental operation is not actually Hausdorff's derivative, but rather, one closer to that of Cantor-Benixson.

Further extensive work on thin trees of uncountable height was undertaken by Johnston [54], who again utilised set-theoretic tools in novel ways. For example, he uses Suslin trees when building Π^0_1 classes (in the uncountable sense); they allow us to work toward an empty class but later recover the construction if we need to change our mind. Further work by Johnson [52, 53] considers fields and

[18] An interesting aspect of the construction in these cases is that given X, we construct a group G which is X-computable, but the filtration $\bar{G}$ will not be X-computable but merely left-κ-c.e.; this kind of distinction does not of course come up in set theory.

[19] When $\alpha \geqslant \omega$ we need to replace 2α by $2\alpha + 1$.

uses abstract elementary classes. On the arithmetical hierarchy in the uncountable setting see [12].

Many questions abound; a particular one is what happens when we drop the assumption $V = L$.

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