HIGHNESS PROPERTIES CLOSE TO PA-COMPLETENESS

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Abstract. Suppose we are given a computably enumerable object. We are interested in the strength of oracles that can *compute* an object that approximates this c.e. object. It turns out that in many cases arising from algorithmic randomness or computable analysis, the resulting highness property is either close to, or equivalent to being PA-complete. We examine, for example, dominating a c.e. martingale by an oracle-computable martingale, computing lower bounds for two variants of Kolmogorov complexity, and computing a subtree of positive measure with no dead ends of a given Π^0_1 tree of positive measure. We separate PA-completeness from the latter property, called the *continuous covering property*. We also separate the corresponding principles in reverse mathematics.

Contents

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1. Introduction

Recall that the *PA degrees* are those Turing degrees that can compute a path through every nonempty Π^0_1 subclass of 2^ω, or equivalently, every nonempty, computably bounded Π^0_1 subclass of ω^ω. The PA degrees are so named because they are the degrees of complete consistent extensions of Peano Arithmetic. In practice, it is usually easier to think of them as the degrees of DNC_2 functions, that is, diagonally noncomputable, $\{0, 1\}$-valued functions. It is easy to see that the collection of all such functions is a Π^0_1 class in 2^ω, and not too difficult to see that all such functions have PA degree. The halting problem, $\varnothing'$, can easily compute a DNC_2 function, so it has PA degree. But it is important to understand that PA degrees can be much more computationally feeble, for example, low [14].

The PA degrees have played an interesting supporting role in the study of algorithmic randomness, in part because they allow us to approximate certain objects that play a central role. We start with three illustrative examples: plain Kolmogorov complexity, prefix-free complexity, and the optimal supermartingale. All three are intrinsically c.e. objects that are optimal in their classes, they are all Turing equivalent to $\varnothing'$, and they can all be approximated using PA degrees. For example, every PA degree computes a martingale that majorizes the optimal c.e. supermartingale [9].

This paper is motivated by the following question: in these and related examples, are the PA degrees necessary? We will see that it depends on the type of example. PA degrees are necessary in the case of plain complexity and the optimal supermartingale, but not in the case of prefix-free complexity. Moving beyond these examples, we will explore the highness class $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$, which consists of the oracles relative to which computable randomness implies (unrelativized) Martin-Löf randomness. Every PA complete set is in this class because, as noted, every PA degree computes a martingale that majorizes the optimal c.e. supermartingale. In this case, it remains open whether PA completeness is necessary.

***C*-compression functions.** Let $C\colon 2^{<\omega} \to \omega$ denote plain Kolmogorov complexity. It is easy to see that C is computable from $\varnothing'$, and in fact, that they are Turing equivalent. But if all we want to do is find a lower bound for C that respects the same combinatorial restriction as C itself, then it turns out that a PA degree is sufficient. This was first observed by Nies, Stephan, and Terwijn [21].

Definition 1.1. A *C-compression function* is an injective function $F\colon 2^{<\omega} \to 2^{<\omega}$ such that $|F(\sigma)| \leqslant C(\sigma)$ for all σ.

Intuitively, we think of F as mapping σ to a minimal program for σ. By definition, $f(\sigma) = |F(\sigma)|$ is a lower bound for $C(\sigma)$ and at most 2^n strings have f-complexity n. Alternately, from an f with these properties, it is easy to compute a C-compression function.

The property of being a C-compression function is Π^0_1; if F is not a C-compression function, then this is eventually apparent. Furthermore, there is a constant c such that $C(\sigma) \leqslant |\sigma| + c$, for all σ, so there are only finitely many possibilities for $F(\sigma)$. In particular, the collection of all C-compression functions is a computably bounded Π^0_1 class. Therefore, every PA degree computes a C-compression function. Nies, Stephan, and Terwijn [21] used this fact, along with the low basis theorem, to prove that every 2-random has infinitely many initial segments with maximal C-complexity (up to a constant). Note that in this case, the PA degrees are used as a tool in proving a result that makes no mention of them.

Kjos-Hanssen, Merkle, and Stephan [16, Thm. 4.1] showed that a PA degree is actually necessary to compute a C-compression function. In particular, they proved that for some constant $k \in \omega$, there is a uniform procedure that computes a DNC_k function (i.e., a $\{0, \ldots, k-1\}$-valued DNC function) from a C-compression function. Jockusch [13] showed that DNC_k functions have PA degree. However, he also proved that there is no uniform procedure to compute a DNC_2 function from a DNC_k function for $k > 2$, so this leaves open a question about uniformity. We prove in Propositions 2.2 and 2.4 that the amount of uniformity that is possible depends on the universal (plain) machine that is used to define C. We build a universal machine such that DNC_2 functions can be uniformly computed from a C-compression function, and another universal machine such that this is impossible.

***K*-compression functions.** Let $K\colon 2^{<\omega} \to \omega$ denote prefix-free (Kolmogorov) complexity. Similar to C, one easily verifies that the function K is Turing equivalent to $\varnothing'$.

Definition 1.2. A *K-compression function* is an injective function $F\colon 2^{<\omega} \to 2^{<\omega}$ with prefix-free range such that $|F(\sigma)| \leqslant K(\sigma)$ for all σ.

This definition ensures that $f(\sigma) = |F(\sigma)|$ is a lower bound for $K(\sigma)$ and, because the range is prefix-free, $\sum_{\sigma \in 2^{<\omega}} 2^{-f(\sigma)} \leqslant 1$. Conversely, given any such f, we can compute a corresponding K-compression function F using the Kraft–Chaitin theorem. Note that the property of being a K-compression function is Π^0_1. Furthermore, there is a constant c such that $K(\sigma) \leqslant 2|\sigma| + c$, for all σ, so there are only finitely many possibilities for $F(\sigma)$. Thus, as above, the collection of all K-compression functions is a computably bounded Π^0_1 class, so a PA degree can compute a K-compression function. This was observed by Nies [20, Solution to Exercise 3.6.16] and used by Bienvenu, et al. [2, Proposition 3.4 (with Hirschfeldt)].

In contrast to the previous example, K-compression functions are not necessarily PA complete. We show this in Section 3. This fact will also follow from Theorem 5.5, but the proof in Section 3 serves as a nice warm-up for that result.

What are the degrees of K-compression functions? We do not have a complete answer, but we can say a couple of things. The Π^0_1 class of K-compression functions was studied by Bienvenu and Porter [4, Thm. 7.9], where it was shown that it is *deep* in the sense of their Def. 4.1. This implies that no incomplete Martin-Löf random can compute a K-compression function [4, Thm. 5.3]. In Proposition 4.3, we prove that despite the fact that K-compression functions do not always compute constant bounded DNC functions, they always compute very slow growing DNC functions: for any computable, nondecreasing, unbounded $h\colon \omega \to \omega \setminus \{0,1\}$, every K-compression function computes an h-bounded DNC function. Note that for a sufficiently slow growing h, the Π^0_1 class of h-bounded DNC functions also is deep [4], so this is connected to the previous fact.

Majorizing the optimal supermartingale. A c.e. supermartingale $m\colon 2^{<\omega} \to \mathbb{R}_{\geqslant 0}$ is called *optimal* if for each c.e. supermartingale f there is a constant $c > 0$ such that $cm(\sigma) \geqslant f(\sigma)$ for each string σ (Schnorr; see [8, Def. 5.3.6]). Obviously, all optimal c.e. supermartingales are equal up to multiplicative constants, and so we refer to *the* optimal c.e. supermartingale m. Again, m is Turing equivalent to

$\varnothing'$ and, again, we can bound m (this time, from above) using a PA degree. This was proved by Franklin, Stephan, and Yu [9]; we provide a proof to clarify how we get a bounded Π^0_1 class in this case.

Proposition 1.3 (Franklin, et. al [9]). *Every PA degree computes a martingale that majorizes the optimal c.e. supermartingale.*

Proof. We may assume that $m(\lambda) \leqslant 1$, where λ is the empty string. The martingales that majorize the optimal c.e. supermartingale do not obviously form a computably bounded Π^0_1 class because there are infinitely many possibilities for each value. To address this, we restrict to the values at each level of the martingale. For each n, let $V_n = \{0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \dots, 2^{n+1}\}$. Define

$$P = \left\{M \colon 2^{<\omega} \to \mathbb{Q} \,:\, (\forall \sigma \in 2^{<\omega})\ M(\sigma) \in V_{|\sigma|} \text{ and } M(\sigma) \geqslant m(\sigma)\right\}.$$

Then P is clearly a computably bounded Π^0_1 class. We must show, however, that P is nonempty. First, define a supermartingale $S\colon 2^{<\omega} \to \mathbb{Q}$ as follows. For each $\sigma \in 2^{<\omega}$, take $S(\sigma) \in V_{|\sigma|}$ such that $m(\sigma) + 2^{-|\sigma|} \leqslant S(\sigma) < m(\sigma) + 2^{-|\sigma|+1}$; this is possible by the definition of $V_{|\sigma|}$ and the fact that $m(\lambda) \leqslant 1$. Note that S has the supermartingale property:

$$\begin{aligned} \frac{S(\sigma\hat{\ }0) + S(\sigma\hat{\ }1)}{2} &< \frac{m(\sigma\hat{\ }0) + 2^{-|\sigma\hat{\ }0|+1} + m(\sigma\hat{\ }1) + 2^{-|\sigma\hat{\ }1|+1}}{2} \\ &= \frac{m(\sigma\hat{\ }0) + m(\sigma\hat{\ }1)}{2} + \frac{2^{-|\sigma|} + 2^{-|\sigma|}}{2} \leqslant m(\sigma) + 2^{-|\sigma|} \leqslant S(\sigma). \end{aligned}$$

Now it is straightforward to define a martingale M majorizing S and with values in $\langle V_n\rangle_{n\in\omega}$, in which case $M \in P$. For specificity, recursively define M as follows. Let $M(\lambda) = S(\lambda)$. Assuming that $M(\sigma)$ is defined, let $M(\sigma\hat{\ }0) = S(\sigma\hat{\ }0)$ and $M(\sigma\hat{\ }1) = 2M(\sigma) - M(\sigma\hat{\ }0)$. ☐

As before, we would like to know if the problem of majorizing m is PA complete. It is; in Proposition 2.9, we construct an atomless c.e. martingale M such that only the PA degrees can compute a martingale majorizing M. Since the optimal c.e. supermartingale majorizes M up to a multiplicative constant, only the PA degrees can compute a martingale that majorizes m. The proof has an interesting case breakdown that introduces nonuniformity. We show in Proposition 2.12 that this nonuniformity is, at least, somewhat necessary; for no $k \in \omega$ is there a uniform procedure that computes a DNC_k function from every martingale that majorizes m. Contrast this with the case of C-compression functions. The exact amount of uniformity that is possible is open; see Remark 2.13.

Jordan decomposition. Our next application of PA degrees is not as obviously related to algorithmic randomness as the first three. Brattka, Miller, and Nies [5] used the fact that a PA degree can compute a Jordan decomposition *on the rational numbers* of a computable function of bounded variation. This is connected to the topic of this paper for two reasons. First, it was used in their proof of a result of Demuth [7]: $x \in [0, 1]$ is Martin-Löf random if and only if every computable function $f\colon [0, 1] \to \mathbb{R}$ of bounded variation is differentiable at x. So it is an application of PA degrees to randomness. Second, we will see in Section 2.3 that the problem of finding a Jordan decomposition on the rationals is very closely related to computing a martingale that majorizes an atomless c.e. martingale.

If $f\colon [0,1] \to \mathbb{R}$ and $x \in [0,1]$, then the *variation of f on $[0,x]$* is

$$V_f(x) = \sup_P \sum_{i=1}^{n_P} |f(t_i) - f(t_{i-1})|,$$

where P ranges over finite sequences $0 \leqslant t_0 < t_1 < \cdots < t_{n_P} \leqslant x$. If $V_f(1)$ is finite, we say that f has *bounded variation*. Jordan proved that a function $f\colon [0,1] \to \mathbb{R}$ has bounded variation if and only if there are nondecreasing functions $g, h\colon [0,1] \to \mathbb{R}$ such that $f = g - h$. In particular, we can take $g = V_f$ and $h = f - g$. Moreover, if f is continuous, then V_f is also continuous, so both g and h can be taken to be continuous.

A PA degree is *not* sufficient to find a continuous Jordan decomposition of a computable function of bounded variation.

Theorem 1.4 (Greenberg, Nies, and Slaman (unpublished)). *There is a computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation such that every continuous nondecreasing $g\colon [0,1] \to \mathbb{R}$ for which $g - f$ is also nondecreasing computes $\varnothing'$.*

Proof. Fix disjoint rational intervals I_n in $[0,1]$. In each I_n, fix an increasing sequence of rational numbers q_k^n converging to $q_\omega^n = \max I_n$. Let $I_{n,k} = [q_k^n, q_{k+1}^n]$. Define the function f as follows. If n enters $\varnothing'$ at stage s, make $f \restriction I_{n,s}$ a "saw-tooth" with positive variation 2^{-n} but of height 2^{-s}. Elsewhere make $f = 0$. Note that at stage s we can approximate f to within 2^{-s}, so f is computable.

Suppose that g is a solution. For each n, find an s such that $g(q_\omega^n) - g(q_s^n) < 2^{-n}$. Then $n \in \varnothing'$ if and only if $n \in \varnothing'_s$. □

Brattka, et al. [5] considered a weaker version of Jordan decomposition, one that can be solved using a PA degree. Let $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$. Given a computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation, they considered the problem of finding nondecreasing functions $g, h\colon I_{\mathbb{Q}} \to \mathbb{R}$ such that $f \restriction I_{\mathbb{Q}} = g - h$. We will see that this problem is essentially the same as finding a martingale that majorizes an atomless c.e. martingale. This allows us to conclude that a PA degree is necessary, in general, to find Jordan decompositions on the rationals. Furthermore, the connection to martingales implies that we cannot uniformly compute DNC_k functions, for any k, from the Jordan decompositions of a computable function of bounded variation.

High(CR, MLR). We turn to an application of PA degrees to randomness where the relationship to PA-completeness remains open. We say that an oracle D is *high for computable randomness versus Martin-Löf randomness* if whenever a sequence is computably random relative to D, it is Martin-Löf random. We abbreviate this property as High(CR, MLR) and use the same notation for the collection of such oracles. Franklin, Stephan, and Yu [9] were the first to study highness for pairs of randomness notions, and in particular, were the first to study the class High(CR, MLR).[1] They observed that if D has PA degree, then it is High(CR, MLR). To see this, note that by Proposition 1.3, there is a D-computable martingale M that majorizes the optimal c.e. supermartingale. In particular, M succeeds on every non-ML-random sequence, so no non-ML-random can be computably random relative to D.

[1] Lowness for pairs of randomness notions was introduced earlier by Kjos-Hanssen, Nies, and Stephan [17].

Proposition 1.5 (Franklin, Stephan, and Yu [9])**.** *Every PA-complete oracle is* $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$*.*

Most similar highness classes and lowness classes for pairs of randomness notions are well-understood. In contrast, we cannot answer a very fundamental question about $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$: is every $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ oracle PA-complete? Some things are known. Franklin, et al. [9] showed that every $D \in \mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ computes a Martin-Löf random, and that the class $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ has measure zero. Note that if D is $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$, we know that all D-computable martingales together succeed on the non-ML-random sequences. In fact, we can do better:

Proposition 1.6 (Kastermans, Lempp, and Miller; see Bienvenu and Miller [3, Prop. 20])**.** *If D is* $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$*, then there is a* single *D-computable martingale N that succeeds on every non-ML-random.*

Note that N need not majorize the optimal c.e. supermartingale m; it must only succeed on every sequence on which m succeeds.

The class $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ also appears in unpublished work of Miller, Ng, and Rupprecht. Building on the proposition above, they proved that there is a single D-computable martingale that succeeds on every non-computably random sequence if and only if D is $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ or high (i.e., $D' \geqslant_{\mathrm{T}} \varnothing''$). The same oracles are necessary to compute a single martingale that succeeds on every non-Schnorr random sequence. Finally, D computes a single martingale that succeeds on all non-Kurtz random sequences if and only if D is $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ or has hyperimmune degree.

Covering properties. In attempting to capture computability-theoretic properties weaker than—but close to—PA, we introduce two "covering properties". Both properties say, in different ways, that an oracle can compute a "small" cover of a "small" c.e. object.

For the first, if $\bar{A} = \langle A_n \rangle$ is a sequence of subsets of ω, we let $\mathrm{wt}(\bar{A}) = \sum_n 2^{-n}|A_n|$. In other words, every element of A_n receives weight 2^{-n}. We say that $\bar{B}$ *covers* $\bar{A}$ if $A_n \subseteq B_n$ for all n. An oracle D has the *discrete covering property* if every uniformly c.e. sequence of finite weight is covered by some D-computable sequence of finite weight. In Section 4, we prove that an oracle D has the discrete covering property if and only if it computes a K-compression function. We also prove that such an oracle computes slow growing DNC functions: for any order function $h\colon \omega \to \omega \smallsetminus \{0, 1\}$, there is an h-bounded DNC function computable from D.

For our second covering property, if $U \subseteq 2^\omega$ is open, then let S_U be the set of strings σ such that $[\sigma] \subseteq U$. We say that an oracle D has the *continuous covering property* if for every Σ^0_1 class $U \subseteq 2^\omega$ of measure less than 1, there is an open superset $V \supseteq U$ such that $\lambda(V) < 1$ and S_V is computable from D. Equivalently, by focusing on the complements of the open sets, D has the continuous covering property if for any computable tree T such that $\lambda([T]) > 0$, there is a D-computable tree S with no dead ends such that $S \subseteq T$ and $\lambda([S]) > 0$. If we further require that every nonempty, relatively clopen subtree of S has positive measure, we get a useful variant: the *strong continuous covering property*.

In Section 5, we prove that the (strong) continuous covering property is implied by $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ and that it implies the discrete covering property. We do not know if either implication is strict. However, in Theorem 5.5, we prove that having

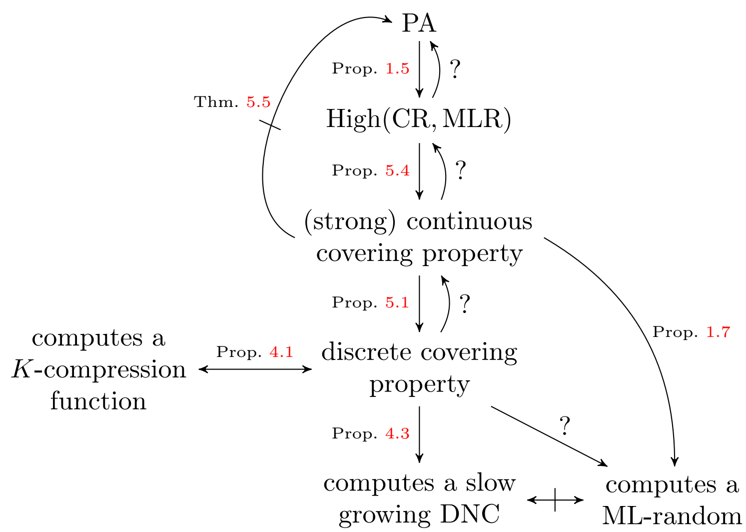

FIGURE 1. The relationship of the covering properties to other computability-theoretic properties.

the strong continuous covering property is strictly weaker than having PA degree; this is the most difficult argument of the paper.

Recall that Franklin, et al. [9] showed that every oracle in $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ computes a Martin-Löf random. This also holds for an oracle D with the continuous covering property. To see this, take a Σ^0_1 class $U \subseteq 2^\omega$ such that $\lambda(U) < 1$ and all non-ML-random sequences are in U. For example, U could be the second component of a universal Martin-Löf test. Let T be a computable tree such that $[T] = 2^\omega \smallsetminus U$. Take a D-computable subtree $S \subseteq T$ with no dead ends such that $\lambda([S]) > 0$. Of course, every infinite path through S in Martin-Löf random, and since S has no dead ends, there are paths computable from $S \leqslant_{\mathrm{T}} D$. We have proved:

Proposition 1.7. *Every oracle with the continuous covering property computes a Martin-Löf random.*

Fig. 1 gives a summary of the results related to the covering properties and indicates several open question. Two references are missing from the diagram. The fact that a Martin-Löf random sequence need not compute a slow growing DNC function was mentioned above; a proof can be found in Bienvenu and Porter [4]. Greenberg and Miller [11] proved that slow growing DNC functions do not always compute a Martin-Löf random. That is, they proved that if $h\colon \omega \to \omega \smallsetminus \{0, 1\}$ is any order function (i.e., a computable, nondecreasing, unbounded function), then there is an h-bounded DNC function that does not compute a Martin-Löf random. (A related but different proof of this fact was given by Khan and Miller [15].)

Strong weak weak Kőnig's lemma. In the final section of the paper, we consider reverse mathematical aspects of some of our results. In particular, we introduce a new principle, *strong weak weak Kőnig's lemma* (SWWKL), which corresponds to the strong continuous covering property. It states:

> If $T \subseteq 2^{<\omega}$ is a tree with positive measure, then there is a nonempty subtree $S \subseteq T$ such that if $\sigma \in S$, then S has positive measure above σ.

We also introduce the apparently weaker principle, *weak strong weak weak Kőnig's lemma* (WSWWKL), which corresponds to the continuous covering property; it only requires that $S \subseteq T$ has positive measure and no dead ends, so unlike SWWKL, it does not even guarantee that S is perfect. Using the work of the previous sections, we prove that, over RCA_0, both SWWKL and WSWWKL are strictly between weak Kőnig's lemma (WKL, the axiom corresponding to the existence of sets of PA degree) and weak weak Kőnig's lemma (WWKL, the axiom corresponding to the existence of Martin-Löf random sequences).

Other uses of PA degrees in algorithmic randomness. We started the introduction by declaring that PA degrees play an interesting supporting role in algorithmic randomness. We have presented some evidence for this claim, but the reader should not be led to believe that we have exhausted the subject. Far from it. Without making a complete survey, let us finish the introduction by mentioning a few other examples.

- Stephan [25] proved that a Martin-Löf random sequence has PA degree if and only if it computes $\varnothing'$; this gives an easy proof of a result of Kučera [18], that the degrees of Martin-Löf random are not closed upward.
- Barmpalias, Lewis, and Ng [1] proved that every PA degree is the join of two Martin-Löf random degrees. Together with the previous result, this gives many examples of pairs of random degrees that join to a nonrandom degree.
- Stephan and Simpson [24] gave an unexpected characterization of the K-trivial sets (i.e., those with minimal growth of prefix-free Kolmogorov complexity of their initial segments) as the sets computable from every PA degree relative to which Chaitin's Ω remains ML-random.
- Higuchi, Hudelson, Simpson, and Yokoyama [12] proved that strong f-randomness is equivalent to f-randomness relative to a PA degree. Both f-randomness and its strong variant are "partial randomness" notions that have been studied, in various degrees of generality, by several authors.

Notation. For $f, h \in \omega^\omega$ we write $f \leqslant h$ to mean that f is majorized by h. In this case, we say that f is *h-bounded*. Let $h^\omega = \{f \in \omega^\omega : f \leqslant h\}$ be the set of h-bounded functions. We write $h^{<\omega}$ for the set $\{\sigma \in \omega^{<\omega} : (\forall n < |\sigma|)\ \sigma(n) \leqslant h(n)\}$ of h-bounded strings. Let $\mathrm{id}^\omega = \{f \in \omega^\omega : (\forall n)\ f(n) \leqslant n\}$, in other words, the *identity bounded* functions.

We let J denote a fixed universal partial computable function, based on an acceptable listing of the partial computable functions; the usual choice is $J(e) = \varphi_e(e)$. A function f is diagonally non-computable if $J(e) \neq f(e)$ whenever $J(e)\downarrow$.

2. Properties that imply PA degree

In this section, we look at three examples from algorithmic randomness where PA degrees turn out to be necessary. We will see that C-compression functions and martingales that majorize the optimal c.e. supermartingale must have PA degree.

We will also show that there is a computable function of bounded variation $f\colon [0,1] \to \mathbb{R}$ such that every Jordan decomposition of f on the rationals has PA degree. In each case, we examine the amount of uniformity possible.

2.1. ***C*-compression functions.** Kjos-Hanssen, Merkle, and Stephan [16] gave a uniform procedure to compute a DNC_k function from a C-compression function, for some k. Since we will modify their proof below, we reproduce it here.

Proposition 2.1 (Kjos-Hanssen, Merkle, and Stephan [16, Theorem 4.1])**.** *Every C-compression function has PA degree. Moreover, for large enough $k \in \omega$, there is a uniform way to compute a DNC_k function from a C-compression function.*

Proof. Define a partial computable function $\psi\colon 2^{<\omega} \to 2^{<\omega}$ as follows: if $\sigma \in 2^{<\omega}$ has length n and if $J(n)\downarrow= \tau$ (where we view τ as an element of $2^{<\omega}$), then $\psi(\sigma) = \sigma\hat{}\tau$. Note that there is a constant $c \in \omega$ such that $C(\psi(\sigma)) < |\sigma| + c$.

Let $F\colon 2^{<\omega} \to 2^{<\omega}$ be a C-compression function. From such an F, we can uniformly compute a function $f\colon \omega \to 2^{<\omega}$ such that $|f(n)| = n$ and $C(f(n)) \geqslant n$. Consider the F-computable function $g\colon \omega \to 2^{<\omega}$ such that $g(n)$ is the last c bits of $f(n+c)$. We claim that g is DNC_k, where $k = 2^c$. Assume that $g(n) = J(n)$. Let σ be the length n prefix of $f(n+c)$. Then $C(f(n+c)) = C(\psi(\sigma)) < |\sigma| + c = n + c$, which is a contradiction. □

As we said in the introduction, the k in the previous result depends on the universal machine used to define C. By designing our machine for the purpose, we can ensure that there is a uniform procedure to compute a DNC_2 function from a C-compression function; this is the next result. On the other hand, in Proposition 2.4, we give a universal plain machine for which this fails.

Proposition 2.2. *There is a universal plain machine $V\colon 2^{<\omega} \to 2^{<\omega}$ such that there is a uniform way to compute a DNC_2 function from a C_V-compression function.*

Proof. We modify the proof of the previous proposition. Now let $\psi\colon 2^{<\omega} \to 2^{<\omega}$ be the partial computable function defined as follows: if $\sigma \in 2^{<\omega}$ has length $2n+1$ and if $\varphi_n(n)\downarrow= \tau$, then $\psi(\sigma) = \sigma\hat{}\tau$.

We want V to be a universal plain machine such that $C_V(\psi(\sigma)) < |\sigma| + 1$. Let $\widehat{V}$ be a given universal machine and define $V(00\hat{}\rho) = \widehat{V}(\rho)$ for all $\rho \in 2^{<\omega}$. This ensures that V is universal, while leaving $3/4$ of the strings of each length free to be used otherwise. Note that $3/4 \cdot 2^{2n+1} + 3/4 \cdot 2^{2n} > 2^{2n+1}$. So we have room left in the domain of V to ensure that $C_V(\psi(\sigma)) \leqslant |\sigma|$ for every (odd length) $\sigma \in 2^{<\omega}$.

As before, from a C_V-compression function F, we can uniformly compute an $f\colon \omega \to 2^{<\omega}$ such that $|f(n)| = n$ and $C_V(f(n)) \geqslant n$. Consider the F-computable function $g\colon \omega \to 2^{<\omega}$ such that $g(n)$ is the last bit of $f(2n+2)$. We claim that g is DNC_2. If not, then $g(n) = \varphi_n(n)$ for some $n \in \omega$. Let σ be the length $2n+1$ prefix of $f(2n+2)$. Then $C_V(f(2n+2)) = C_V(\psi(\sigma)) \leqslant |\sigma| = 2n+1$, which is a contradiction. □

Toward proving Proposition 2.4, we need the following simple combinatorial lemma. It generalizes the observation that either a graph G (on at least two vertices) or its complement $\overline{G}$ has no isolated vertices. For, if v is isolated in G, then it has edges to every other vertex in $\overline{G}$. Recall that for any set X, the set of subsets of X of size k is written $[X]^k$.

Lemma 2.3. *Let X be an arbitrary set and fix $k \in \omega$. For any colouring $c\colon [X]^k \to k$, there is an $i < k$ such that*

$$(\forall v \in X)(\exists w_1, \dots, w_{k-1} \in X)\ c(\{v, w_1, \dots, w_{k-1}\}) = i.$$

Proof. We prove this lemma by induction on k. Note that it is trivial for $k = 1$. Now assume that it holds for k and consider a colouring $c\colon [X]^{k+1} \to k+1$.

If the lemma holds for $i = k$, we are done. Otherwise, there is a $u \in X$ such that the induced colouring $\hat{c}$ on $[X \smallsetminus \{u\}]^k$ has range in $k = \{0, \dots, k-1\}$. Hence, by induction, there is an $i < k$ such that

$$(\forall v \in X \smallsetminus \{u\})(\exists w_1, \dots, w_{k-1} \in X \smallsetminus \{u\})\ \hat{c}(\{v, w_1, \dots, w_{k-1}\}) = i.$$

But then, for all $v \in X \smallsetminus \{u\}$, we have

$$(\exists w_1, \dots, w_{k-1} \in X \smallsetminus \{u\})\ c(\{v, u, w_1, \dots, w_{k-1}\}) = i. \qquad \square$$

The lemma fails if we increase the number of colours. To see this, let $X = \{0, \dots, k\}$ and define $c\colon [X]^k \to k+1$ by $c(\{0, \dots, i-1, i+1, \dots, k\}) = i$. Then for every $i < k+1$, if $Y \subseteq X$ has size k and $i \in Y$, then $c(Y) \neq i$.

The proof of the following proposition illustrates a technique that will be used in later proofs. We want to diagonalize against a functional Γ on some element of a Π^0_1 class P. However, we are not able to effectively guarantee that any *specific* string σ has an extension in P. Our solution is to use the structure of P together with Lemma 2.3 to diagonalize against Γ on enough strings so that we know that at least one of them is extendible in P.

Proposition 2.4. *For each k, there is a universal plain machine V such that there is no uniform way to compute a DNC_k function from a C_V-compression function.*

In fact, we show the following. For a plain machine V and $r \in \omega$, we say that V *uses at most $1/r$ of the available strings of each length* if, for all n, there are at most $2^n/r$ many strings of length n in the domain of V.

Proposition 2.5. *Let $k \in \omega$. If V is a universal plain machine that uses at most $1/k+1$ of the available strings of each length, then there is no uniform way to compute a DNC_k function from a C_V-compression function.*

Proposition 2.4 follows by taking any universal plain machine $\widehat{V}$, fixing some $c \geqslant \log_2(k+1)$, and letting $V(0^c\sigma) = \widehat{V}(\sigma)$ for all σ.

Proof of Proposition 2.5. Let P be the collection of C_V-compression functions. By fixing a computable numbering of all finite binary strings, we can view P as a Π^0_1 class in Baire space; as mentioned above, P is computably bounded. Fix a computable function $h\colon \omega \to \omega$ such that $P \subseteq h^\omega$. By the assumption on V, there is some $F \in P$ such that the range of F includes at most $1/k+1$ of the strings of each length.

Now assume, for a contradiction, that Γ computes a DNC_k function from every $F \in P$. Without loss of generality, Γ is k-valued. We may also make it total on h^ω by ensuring that it converges on any oracle not in P.

We define a computable process that will output an $i < k$. The result of this process will be $J(e)$, for some e. By the recursion theorem, we may assume that we know e in advance.[2] By compactness, there is an $n \in \omega$ such that $\Gamma(\sigma, e)\downarrow$ for every

[2] More formally, we define a partial computable function ψ; by the Recursion Theorem, there is an e such that $J(e) = \psi(e)$.

$\sigma \in h^n$ (where, as above, h^n is the collection of h-bounded strings of length n). Our goal is to output an $i < k$ such that $\Gamma(\sigma, e) = i$ for some $\sigma \in h^n$ that is extendible to an element of P. Of course, we cannot hope to effectively identify such a σ, but we will see that we *can* effectively find such an i.

Define E to be the collection of strings $\sigma \in h^n$ which are injective; and let

$$\widehat{E} = \{\tau \in E : \tau \text{ maps onto at most } 1/k+1 \text{ of the strings of each length}\}.$$

By our assumption on V, we know that there is a $\tau \in \widehat{E}$ that is extendible to an element of P.

We define a map $Q \mapsto \tau_Q$ from $[\widehat{E}]^k$ to E as follows: given $Q \subseteq \widehat{E}$ of size k, since each $\sigma \in Q$ maps to at most $1/k+1$ of the strings of each length, we can let τ_Q map each $x < n$ to a string $\tau_Q(x)$ with $|\tau_Q(x)| \leqslant |\sigma(x)|$ for all $\sigma \in Q$. That is, there is enough room in the range to permit all of the desired compression while keeping τ_Q injective. Note that the minimality condition also implies that $\tau_Q \in h^n$, and so $\tau_Q \in E$. Further, τ_Q uses at most $k/k+1$ of the strings of each length. This implies that if $\sigma \in Q$ is extendible in P, then so is τ_Q. For suppose that $\sigma \in Q$ is extendible in P. For each $x < n$, $|\tau_Q(x)| \leqslant |\sigma(x)| \leqslant C_V(x)$. Since τ_Q uses at most $k/k+1$ of the strings of each length, and V uses at most $1/k+1$ of the strings of each length, we can extend τ_Q to a function $F \in h^\omega$ such that for all $x \geqslant n$, $|F(x)| \leqslant C_V(x)$ as well.

Now define a colouring $c \colon [\widehat{E}]^k \to k$ as follows: for $Q \in [\widehat{E}]^k$, let $c(Q) = \Gamma(\tau_Q, e)$. Fix $i < k$ as in Lemma 2.3 for the colouring c; this is the output of our computable procedure, i.e., $J(e) = i$. Now fix $\sigma \in \widehat{E}$ extendible to an element of P and any $Q \in [\widehat{E}]^k$ such that $\sigma \in Q$ and $c(\tau_Q) = i$. Then τ_Q is extendible to an element of P, but $\Gamma(\tau_Q, e) = i = J(e)$, which contradicts our choice of Γ. ☐

Remark 2.6. Recall that for sets $P, R \subseteq \omega^\omega$, we write $P \leqslant_s R$ (and say that P is *Medvedev reducible* to R) if there is a Turing functional Γ such that for all $X \in R$, $\Gamma(X)$ is total and $\Gamma(X) \in P$: each element of R computes an element of P, uniformly. In contrast, $P \leqslant_w R$ (P is *Muchnik reducible* to R) if every element of R computes an element of P, but not necessarily uniformly. Jockusch's results mentioned above shows that the classes DNC_k are all Muchnik equivalent (their upward closures in the Turing degrees consist of the PA-complete oracles), but that for all k, $\mathrm{DNC}_{k+1} <_s \mathrm{DNC}_k$. The class DNC_2 is Medvedev-complete for computably bounded Π^0_1 classes.

For a universal plain machine V, let CF_V be the collection of C_V-compression functions. Kjos-Hanssen, Merkle, and Stephan's Proposition 2.1 says that for every universal plain machine V there is some k such that $\mathrm{DNC}_k \leqslant_s \mathrm{CF}_V$; it follows that for all V, each CF_V is Muchnik equivalent to DNC_2. Proposition 2.2 states that for some universal V, $\mathrm{DNC}_2 \leqslant_s \mathrm{CF}_V$; Proposition 2.4 says that for every k there is a V for which $\mathrm{DNC}_k \nleqslant_s \mathrm{CF}_V$.

The proof of Proposition 2.5 can be restricted above any extendible string σ. This allows us to diagonalize against multiple functionals.

Lemma 2.7. *Let $k \in \omega$ and let P be a computably bounded Π^0_1 class. Suppose that for every σ extendible on P, $\mathrm{DNC}_k \nleqslant_s P \cap [\sigma]$. Then there is no finite collection $\Gamma_1, \dots, \Gamma_m$ of functionals such that for all $X \in P$, $\Gamma_i(X) \in \mathrm{DNC}_k$ for some $i \leqslant m$.*

That is, not only do elements of P not compute DNC_k functions uniformly, but no finite collection of functionals is sufficient for $\mathrm{DNC}_k \leqslant_w P$. We remark that the proof of Lemma 2.7 only uses the fact that DNC_k is a Π^0_1 class which is determined pointwise, entry by entry; thus, for example, it also applies to separating classes.

Proof. For brevity, in this proof, for $m \geqslant 1$ and Π^0_1 classes P and Q, write $Q \leqslant_m P$ if there is a collection $\Gamma_1, \dots, \Gamma_m$ of m-many functionals which together reduce Q to P, that is, for all $X \in P$, $\Gamma_i(X) \in Q$ for some $i \leqslant m$.

By induction on m, we show that for every σ that is extendible on P, $\mathrm{DNC}_k \nleqslant_m P \cap [\sigma]$. The case $m = 1$ is the assumption of the proposition.

Let $m > 1$ and suppose that this has been proved for $m - 1$. Let $\Gamma_1, \dots, \Gamma_m$ be a collection of m-many functionals. Define a functional Θ as follows: for all X and e, $\Theta(X, e) = \Gamma_i(X, e)$ for the first i for which we see the convergence (if there is such). Let σ be extendible on P. By assumption, we know that Θ cannot witness that $\mathrm{DNC}_k \leqslant_1 P \cap [\sigma]$. If Θ is not total on $P \cap [\sigma]$ then we are done. Otherwise, there is some $\tau \succcurlyeq \sigma$, extendible on P, and some e, such that $\Theta(\tau, e){\downarrow} = J(e)$. Fix i such that $\Theta(\tau, e) = \Gamma_i(\tau, e)$. Now apply the induction hypothesis to the collection of functionals $\{\Gamma_j : j \leqslant m, j \neq i\}$ and τ to see that this collection cannot witness $\mathrm{DNC}_k \leqslant_{m-1} P \cap [\tau]$; it follows that the original collection $\Gamma_1, \dots, \Gamma_m$ cannot witness $\mathrm{DNC}_k \leqslant_m P \cap [\sigma]$ either. □

As mentioned, the proof of Proposition 2.5 gives the assumption of Lemma 2.7, and so we get:

Proposition 2.8. *For each k, there is a universal plain machine V such that there is no finite collection of functionals $\Gamma_1, \dots, \Gamma_m$ such that if F is a C_V-compression function, then at least one of $\Gamma_1(F), \dots, \Gamma_m(F)$ is a DNC_k function.*

2.2. **Majorizing the optimal c.e. supermartingale.** The case of martingales that majorize the optimal c.e. supermartingale is somewhat different from that of C-compression functions. Although each such martingale has PA degree, the proof has an unusual case breakdown that precludes uniformity. We will see in Proposition 2.12 that this nonuniformity is somewhat necessary: for no k is there a uniform way to compute a DNC_k function from a martingale that majorizes the optimal c.e. supermartingale. This result exploits the fact that such a martingale may have an arbitrarily large initial capital. It is open whether we can salvage uniformity by bounding the initial capital; see Remark 2.13.

Proposition 2.9. *There is an (atomless) c.e. martingale M such that every martingale majorizing M has PA degree.*

Proof. Define a c.e. martingale M as follows. If n enters $\varnothing'$ at stage s, find a string $\sigma \in 2^s$ that looks DNC_2 at stage s, add 2^{-n} much capital to the root and push it up to σ.[3]

Now let N be a martingale that majorizes M.

Case 1. N has a DNC_2 atom.[4] A martingale computes all of its atoms, so in this case, N has PA degree.

Case 2. N has no DNC_2 atoms. Then for each n, there is a stage $f(n) = s$ such that for all strings σ of length s that still look DNC_2 at stage s we have $N(\sigma) \leqslant 2^{s-n}$. By construction, f, which is N-computable, majorizes the settling time function for $\varnothing'$, so N has PA degree. □

[3] In other words, for $\tau \preccurlyeq \sigma$, we let $M_{s+1}(\tau) - M_s(\tau) = 2^{|\tau|-n}$; to preserve the martingale property, for $\tau \succ \sigma$, we let $M_{s+1}(\tau) - M_s(\tau) = 2^{|\sigma|-n}$.

[4] That is, the associated measure $\mu(\sigma) = 2^{-|\sigma|}M(\sigma)$ has a DNC_2 atom.

Of course, the optimal c.e. supermartingale majorizes M, up to a multiplicative constant, so we have the desired result:

Corollary 2.10. *Every martingale that majorizes the optimal c.e. supermartingale has PA degree.*

Remark 2.11. The statements of Proposition 2.9 and Corollary 2.10 are imprecise. The reason is that objects such as martingales (and below, real-valued functions on the rationals) do not necessarily have Turing degree. Rather, they have a *continuous degree* (see Miller [19]). Continuous reducibility uses the notion of a *name* of an object. For example, a name of a martingale N is a function taking a string σ and a positive rational number $\varepsilon > 0$ to a rational number q such that $|M(\sigma) - q| < \varepsilon$. If x and y are objects that have continuous degree (in general, points in computable metric spaces), then $x \leqslant_r y$ if every name for y computes a name for x. This extends Turing reducibility. The proof of Proposition 2.9 shows that if N is a martingale dominating M then the continuous degree of N lies above a PA-complete Turing degree.

But Proposition 2.9 and Corollary 2.10 could also be interpreted in a weaker way—as statements about Turing degrees. For example, Corollary 2.10 could be understood as saying that if a set X can compute a martingale N dominating m, then X is PA-complete. In other words, every name for N computes a DNC_2 function but it is not required that every name for N computes the *same* DNC_2 function. While the stronger interpretation holds above, the distinction is important when we consider uniformity, in particular Weihrauch reducibility, below.

Proposition 2.12. *It is not possible to uniformly compute a* DNC_k *function from a martingale majorizing the optimal c.e. supermartingale.*

More precisely, it is not possible to uniformly compute a DNC_k function from a name for such a martingale, even if the reduction procedure does not promise to compute the same DNC_k function from all names for the same martingale. In other words, DNC_k is not Medvedev below the collection of names for martingales majorizing m.

Proof. We give a variant of the proof of Proposition 2.5. Fix $k \in \omega$. Let Γ be a functional. We may assume that the initial capital of the optimal supermartingale is bounded by 1. Let P be the collection of all martingales with initial capital $\leqslant k$. We can code P as a computably bounded Π^0_1 class in Baire space as follows: for each n, the value $f_M(n)$ of the function coding M codes, for each binary string σ of length at most n, one of the dyadic closed intervals $[i/2^n, (i+1)/2^n]$ (for integer i between 0 and $k2^{n+|\sigma|}$) containing the value of $M(\sigma)$. Let P_n be the collection of strings of length n coding initial segments of martingales in this way. We may assume that every element of P_n is extendible in P.

Let Γ be a k-valued functional, and suppose that $\Gamma(M)$ is total for all $M \in P$. As above, the recursion theorem gives us an e for which we can define $J(e)$. By effective compactness, there is some n such that $\Gamma(\tau, e)\downarrow$ for every string $\sigma \in P_n$. Let $c \geqslant \log_2 k$; we let $\widehat{E}$ be the collection of $\sigma \in P_{n+c}$ that code a martingale with initial capital $\leqslant 1$; we know that some $\sigma \in \widehat{E}$ is extendible on P. For $Q \in [\widehat{E}]^k$ we let $\tau_Q \in P_n$ be the (string coding) the sum of the martingales (with codes) in Q. The proof then follows the rest of the proof of Proposition 2.5. $\square$

Remark 2.13. The proof of Proposition 2.12 does not give the assumption of Lemma 2.7, as adding martingales implies adding their initial capital. We thus have the following unresolved questions for any $k \geqslant 2$:

- Is there a finite collection of functionals Γ_i such that for every martingale M majorizing m, $\Gamma_i(M) \in \mathrm{DNC}_k$ for some i?
- Is there a uniform way to compute a DNC_k function from a martingale M majorizing m whose initial capital is bounded by 1?

2.3. **Jordan decomposition on the rationals.** Given a computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation, we want to find nondecreasing functions $g, h\colon I_{\mathbb{Q}} \to \mathbb{R}$ such that $f \upharpoonright I_{\mathbb{Q}} = g - h$, where $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$. Brattka et al. [5] observed that this can be done with a PA degree. As part of proving that a PA degree is necessary, we will show that finding a Jordan decomposition of f on the rationals is equivalent to finding a martingale that majorizes a related atomless c.e. martingale.

A natural formalisation of this equivalence uses *Weihrauch reducibility.* The objects compared by this reducibility are binary relations, which can be thought of as pairs of "instances" and "solutions". For instance, in this section we consider the problem of finding the positive part of a Jordan decomposition on the rationals:

- $\mathrm{JD}_{\mathbb{Q}}$ is the problem whose instances are continuous functions f on $[0,1]$ of bounded variation, and solutions are Jordan decompositions (g,h) of $f \upharpoonright I_{\mathbb{Q}}$.

If A and B are Weihrauch problems, then we say that A is *Weihrauch reducible* to B (and write $A \leqslant_W B$) if there are two computable mappings $\psi_{\mathtt{inst}}$ and $\psi_{\mathtt{sol}}$ satisfying: for every name a for an instance of A, $\psi_{\mathtt{inst}}(a)$ is a name for an instance of B, such that whenever c is a name for a B-solution of the instance named by $\psi_{\mathtt{inst}}(a)$, $\psi_{\mathtt{sol}}(a,c)$ is a name for an A-solution of the instance named by a. If $\psi_{\mathtt{sol}}$ does not make use of a, then the reduction is called *strong.* Note that the functions $\psi_{\mathtt{inst}}$ and $\psi_{\mathtt{sol}}$ are not required to induce functions on the instances and solutions themselves; two names of the same A-instance may be mapped by $\psi_{\mathtt{inst}}$ to names of distinct B-instances, and the same holds for the solutions. When we define reductions, though, in order to make things readable, we blur the distinction between names and the objects they name.

To show the PA-completeness of the Jordan decomposition problem, we will prove the equivalence of the problem $\mathrm{JD}_{\mathbb{Q}}$ with a martingale domination Weihrauch problem. Let us define a *lower semicontinuous presentation* of a martingale M to be a sequence $\langle M_s \rangle$ of rational-valued martingales such that $M_s \leqslant M_{s+1}$ and $M = \lim_s M_s$. If $\langle M_s \rangle$ is computable then we also call it a *c.e. presentation* of M. We define the following Weihrauch problem.

- AMD is the problem whose instances are lower semicontinuous presentations $\langle M_s \rangle$ of atomless martingales M; AMD-solutions for $\langle M_s \rangle$ are martingales majorizing M (not necessarily atomless).

We will show:

Proposition 2.14. *The problems* $\mathrm{JD}_{\mathbb{Q}}$ *and* AMD *are Weihrauch equivalent.*

From this we can deduce the following:

Corollary 2.15. *Suppose that X is an oracle such that for every computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation, X can compute a Jordan decomposition (g,h) of $f \upharpoonright I_{\mathbb{Q}}$. Then X is PA-complete. In fact, there is a single computable* *function $f\colon [0,1] \to \mathbb{R}$ of bounded variation such that any X computing a Jordan decomposition of $f \upharpoonright I_{\mathbb{Q}}$ is PA-complete.*

Proof. Let $(\psi_{\mathtt{inst}}, \psi_{\mathtt{sol}})$ be a Weihrauch reduction of AMD to $\mathrm{JD}_{\mathbb{Q}}$. Let $\langle M_s \rangle$ be a c.e. presentation of the atomless martingale M given by Proposition 2.9. Let $f = \psi_{\mathtt{inst}}(\langle M_s \rangle)$. Since $\psi_{\mathtt{inst}}$ is a computable mapping and $\langle M_s \rangle$ is computable, so is f. Suppose that X computes a Jordan decomposition (g, h) of $f \upharpoonright I_{\mathbb{Q}}$. Then $N = \psi_{\mathtt{sol}}(\langle M_s \rangle, (g,h))$ is also X-computable; since N majorizes M, X is PA-complete. □

Proposition 2.14 also allows us to transfer our non-uniformity result.

Corollary 2.16. *For any computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation and any k, there is no uniform way to compute a DNC_k function from (a name of) any Jordan decomposition (g,h) of $f \upharpoonright I_{\mathbb{Q}}$.*

Proof. Suppose that Γ is a Turing functional mapping (names of) pairs of real-valued functions (g,h) on $I_{\mathbb{Q}}$ to k-valued functions on ω. Let $f\colon [0,1] \to \mathbb{R}$ be computable of bounded variation; we need to show that there is some Jordan decomposition (g,h) of f such that $\Gamma(g,h) \notin \mathrm{DNC}_k$.

Let $(\varphi_{\mathtt{inst}}, \varphi_{\mathtt{sol}})$ be a Weihrauch reduction of $\mathrm{JD}_{\mathbb{Q}}$ to AMD; let $\langle M_s \rangle = \varphi_{\mathtt{inst}}(f)$, and let $M = \lim_s M_s$. Note that $\langle M_s \rangle$ is computable, so M is c.e. Recalling that m is the optimal c.e. supermartingale, fix a $d > 0$ such that $dm \geqslant M$.

We define a functional Θ by letting $\Theta(N) = \Gamma(\psi_{\mathtt{sol}}(f, dN))$. By Proposition 2.12, there is a martingale N majorizing m such that $\Theta(N) \notin \mathrm{DNC}_k$. Then $(g,h) = \psi_{\mathtt{sol}}(f, dN)$ is a Jordan decomposition of $f \upharpoonright I_{\mathbb{Q}}$ such that $\Gamma(g,h) \notin \mathrm{DNC}_k$. □

Remark 2.17. The proof of Corollary 2.16 shows that we can compute a Jordan decomposition on $I_{\mathbb{Q}}$ of a computable function f of bounded variation, uniformly given a martingale N majorizing m and a computable index for (a name of) f. This is because the constant d can be computed given a computable index for f.

It remains to prove Proposition 2.14.

The first step is to translate the problem to the dyadic rationals, $\mathbb{Q}_2$. Let $I_{\mathbb{Q}_2} = [0,1] \cap \mathbb{Q}_2$. We define the following Weihrauch problem:

- $\mathrm{JD}_{\mathbb{Q}_2}$: instances are continuous functions $f\colon [0,1] \to \mathbb{R}$ of bounded variation; solutions for f are Jordan decompositions of $f \upharpoonright I_{\mathbb{Q}_2}$.

Lemma 2.18. *The problems $\mathrm{JD}_{\mathbb{Q}}$ and $\mathrm{JD}_{\mathbb{Q}_2}$ are strong Weihrauch equivalent.*

Proof. Let $b\colon [0,1] \to [0,1]$ be a computable, order-preserving bijection such that $b[I_{\mathbb{Q}}] = I_{\mathbb{Q}_2}$; we get this by extending a computable, order preserving bijection between $I_{\mathbb{Q}}$ and $I_{\mathbb{Q}_2}$. Note that b^{-1} is also computable.

To reduce $\mathrm{JD}_{\mathbb{Q}}$ to $\mathrm{JD}_{\mathbb{Q}_2}$, map an instance f to $f \circ b$; note that if f has bounded variation, then so does $f \circ b$, in fact $V_{f \circ b}(1) = V_f(1)$. On the solution side, map a pair (g,h) of functions defined on $I_{\mathbb{Q}_2}$ to the pair $(g \circ b^{-1}, h \circ b^{-1})$.

To reduce $\mathrm{JD}_{\mathbb{Q}_2}$ to $\mathrm{JD}_{\mathbb{Q}}$, map an instance f to itself. On the solution side, map a pair (g,h) of functions defined on $I_{\mathbb{Q}}$ to the pair $(g \upharpoonright I_{\mathbb{Q}_2}, h \upharpoonright I_{\mathbb{Q}_2})$. □

Just for notational simplicity later, define the following Weihrauch problem:

- $\mathrm{PJD}_{\mathbb{Q}_2}$: instances are continuous functions $f\colon [0,1] \to \mathbb{R}$ of bounded variation; solutions for f are functions $g\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ such that $(g, g - f \upharpoonright I_{\mathbb{Q}_2})$ is a Jordan decomposition of $f \upharpoonright I_{\mathbb{Q}_2}$.

It is clear that $\mathrm{PJD}_{\mathbb{Q}_2}$ is Weihrauch equivalent to $\mathrm{JD}_{\mathbb{Q}_2}$; The reduction of $\mathrm{JD}_{\mathbb{Q}_2}$ to $\mathrm{PJD}_{\mathbb{Q}_2}$ is not strong.

The variation V_f of a function $f\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ is defined as usual, except that the partitions have binary rationals as endpoints. If $f\colon [0,1] \to \mathbb{R}$ is continuous of bounded variation then so is $\bar{f} = f \upharpoonright I_{\mathbb{Q}_2}$, and $V_{\bar{f}} = V_f \upharpoonright I_{\mathbb{Q}_2}$.

We will transform functions on $I_{\mathbb{Q}_2}$ of bounded variation into signed measures on $[0,1)$. To do this, we associate binary strings with dyadic rational numbers—the endpoints of the associated intervals—in the natural way. For the empty string λ, we let $l_\lambda = 0$ and $r_\lambda = 1$; for any finite binary string σ, we let $l_{\sigma\hat{}0} = l_\sigma$, $r_{\sigma\hat{}1} = r_\sigma$, and $r_{\sigma\hat{}0} = l_{\sigma\hat{}1} = {}^{(l_\sigma + r_\sigma)}/_2$. We write $[\sigma)$ for the half-open interval $[l_\sigma, r_\sigma)$.

For a function $f\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ of bounded variation, there is a (unique) signed measure μ^f on $[0,1)$ defined by

$$\mu^f([\sigma)) = f(r_\sigma) - f(l_\sigma).$$

The map $f \mapsto \mu^f$ is computable. Observe that f (defined on $I_{\mathbb{Q}_2}$) is non-decreasing if and only if μ^f is non-negative. If $f\colon [0,1] \to \mathbb{R}$ is continuous then we write μ^f for $\mu^{f\upharpoonright I_{\mathbb{Q}_2}}$. For brevity, for a signed measure μ and $\sigma \in 2^{<\omega}$, we write $\mu(\sigma)$ for $\mu([\sigma))$.

Observation 2.19. Let $f\colon [0,1] \to \mathbb{R}$ be continuous of bounded variation. A function $g\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ is a $\mathrm{PJD}_{\mathbb{Q}_2}$-solution for f if and only if $\mu^g \geqslant 0$ and $\mu^g \geqslant \mu^f$. This is because $\mu^{g - f\upharpoonright I_{\mathbb{Q}_2}} = \mu^g - \mu^{f\upharpoonright I_{\mathbb{Q}_2}} = \mu^g - \mu^f$.

The operation $f \mapsto \mu^f$ has an inverse of sorts: for any (finite) signed measure μ on $[0,1)$ we define $f_\mu\colon [0,1] \to \mathbb{R}$ by letting

$$f_\mu(x) = \mu([0,x)).$$

This is known as the *cumulative distribution function* of μ. The function f_μ is not necessarily μ-computable (rather it is μ-left-c.e.); however $f_\mu \upharpoonright I_{\mathbb{Q}_2}$ is μ-computable (uniformly), because for $q \in I_{\mathbb{Q}_2}$ positive we have

$$f_\mu(q) = \sum_{\tau \leqslant \sigma, |\tau| = |\sigma|} \mu(\tau)$$

for any σ such that $q = r_\sigma$. If $g\colon [0,1] \to \mathbb{R}$ is continuous of bounded variation, then $f_{\mu^g} = g - g(0)$. A measure μ is atomless if and only if f_μ is continuous.

The Hahn decomposition of a signed measure μ produces the *variation* V_μ of μ (often denoted by $|\mu|$); it is the least measure ν satisfying $\nu(A) \geqslant |\mu(A)|$ for all Borel A. The measure V_μ is μ-left c.e., uniformly: there is a computable mapping taking μ to a lower semicontinuous presentation of V_μ. This is because for all σ,

$$V_\mu(\sigma) = \sup_{k \geqslant |\sigma|} \sum \{ |\mu(\tau)| \,:\, \tau \succcurlyeq \sigma \;\&\; |\tau| = k \}.$$

For a continuous f on $[0,1]$ of bounded variation we have $V_{\mu^f} = \mu^{V_f}$.

Finally, we replace martingales by measures in the familiar way: a martingale M corresponds to the measure defined by $\mu(\sigma) = 2^{-|\sigma|} M(\sigma)$. We thus assume that instances and solutions of AMD are measures rather than martingales. We are ready to prove one direction of Proposition 2.14:

Proposition 2.20. $\mathrm{PJD}_{\mathbb{Q}_2}$ *is strong Weihrauch reducible to* AMD.

Proof. On the instance side, we map a continuous function $f\colon [0,1] \to \mathbb{R}$ of bounded variation to a lower semicontinuous presentation of $V_{\mu^f} = \mu^{V_f}$; we observed that this can be done computably. Since f is continuous, so is V_f, so μ^{V_f} is atomless.

On the solution side, map a measure ν to $f_\nu \restriction I_{\mathbb{Q}_2}$.

To show this works, suppose that $\nu \geqslant \mu^{V_f}$; then $\nu \geqslant \mu^f$, as $\mu^{V_f} \geqslant \mu^f$. Also $\nu \geqslant 0$. By Observation 2.19, $g = f_\nu$ is a $\mathrm{PJD}_{\mathbb{Q}_2}$-solution for f. □

In the other direction, we need two facts.

Lemma 2.21. *Let (g,h) be a Jordan decomposition of a function $f\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ of bounded variation. Then $\mu^g + \mu^h \geqslant \mu^{V_f}$.*

Proof. The minimality property of V_{μ^f} means that it suffices to show that for all σ, $\max\{\mu^g(\sigma), \mu^h(\sigma)\} \geqslant |\mu^f(\sigma)|$. If $\mu^f(\sigma) \geqslant 0$, then $\mu^g(\sigma) \geqslant \mu^f(\sigma)$ as $\mu^g \geqslant \mu^f$ (by Observation 2.19). If $\mu^f(\sigma) < 0$ then $\mu^h(\sigma) = \mu^g(\sigma) - \mu^f(\sigma) = \mu^g(\sigma) + |\mu^f(\sigma)| \geqslant |\mu^f(\sigma)|$ because $\mu^g \geqslant 0$. □

The second fact is taken from the proof of Theorem 3.5 of [10] by Freer et al. (joint with Rute). That theorem states that any continuous non-decreasing interval-c.e. function $f\colon [0,1] \to \mathbb{R}$ is of the form V_g for some computable function g. (To say that f is interval-c.e. means that the real $f(y) - f(x)$ is left-c.e., uniformly in rationals $x < y$.) Note that if μ is an atomless left-c.e. measure, then f_μ is interval c.e. So we can restate the result slightly as follows:

Proposition 2.22 ([10])**.** *There is a computable mapping taking any lower semicontinuous presentation $\langle \nu_s \rangle$ of an atomless measure ν to a continuous function $g\colon [0,1] \to \mathbb{R}$ of bounded variation such that $\nu = V_{\mu^g}$.*

Sketch of proof. We define a signed measure η and let $g = f_\eta$ (so $\eta = \mu^g$). The rough idea is as follows. By stage s, we have defined $\eta(\sigma)$ for all σ of length $\leqslant \ell_s$ for some $\ell_s \in \omega$, with $|\eta(\sigma)| \leqslant \nu_s(\sigma)$ for all such σ. At stage s we define $\eta(\tau)$ for longer strings τ (preserving $|\eta(\tau)| \leqslant \nu_s(\tau)$), by letting $|\eta(\tau\hat{\ }i)| = \nu_s(\tau\hat{\ }i)$ for the i for which the latter is the smaller between $\nu_s(\tau\hat{\ }0)$ and $\nu_s(\tau\hat{\ }1)$; but we keep the sign of $\eta(\tau\hat{\ }i)$ the same as that of $\eta(\tau)$. As we go along, at every level $n \geqslant \ell_s$, at most 2^{ℓ_s} many strings τ of length n have $|\eta(\tau)| \neq \nu_s(\tau)$. As ν_s is atomless (because ν is), eventually the discrepancy between $\nu_s(\sigma)$ and $\sum_{\tau > \sigma \ \&\ |\tau| = n} |\eta(\tau)|$ is small for each σ of length ℓ_s, which is when we halt stage s and declare the next value ℓ_{s+1}. To get $g = f_\eta$ computable from $\langle \nu_s \rangle$, we need to ensure that ℓ_s is sufficiently long so that $\nu_s(\sigma) \leqslant 2^{-s}$ for all σ of length ℓ_s, which again is possible because ν_s is atomless.[5] □

The following now completes the proof of Proposition 2.14:

Proposition 2.23. AMD *is strong Weihrauch reducible to* $\mathrm{JD}_{\mathbb{Q}_2}$.

Proof. On the instance side, using Proposition 2.22, map a lower semicontinuous presentation $\langle \nu_s \rangle$ of an atomless measure function ν to a continuous f such that $\nu = V_{\mu^f} = \mu^{V_f}$. On the solution side, map $(g,h)\colon I_{\mathbb{Q}_2} \to \mathbb{R}$ to $\mu^g + \mu^h$. Lemma 2.21 says that this works. □

[5] The proof as written in [10] uses martingales instead of measures; to translate to the notation of that paper, $\nu(\sigma) = 2^{-|\sigma|}M(\sigma)$ and $\eta(\sigma) = 2^{-|\sigma|}L(\sigma)$. Lemma 3.3 of [10] constructs a computable signed measure η such that $\nu = V_\eta$; In Theorem 3.5, the construction is modifed to get $\eta = \mu^g$ with g computable, starting with a function f such that $\nu = \mu^f$.

3. A K-compression function without PA degree

We provide a proof that there is a K-compression function that does not have PA degree. This should be considered a warm-up for the somewhat more involved proof of Theorem 5.5, which by Propositions 4.1 and 5.1 implies the present result.

As this is a warm-up, we introduce notation which may appear cumbersome at present, but will be useful later. In the current argument, we work in the space id^ω, which, recall, is the space of identity-bounded functions. We also let $\mathrm{id}^{\leqslant\omega} = \mathrm{id}^\omega \cup \mathrm{id}^{<\omega}$ be the collection of identity-bounded sequences, finite and infinite. For $\sigma \in \mathrm{id}^{<\omega}$, we let

$$[\sigma] = \left\{ f \in \mathrm{id}^{\leqslant\omega} \,:\, \sigma \preccurlyeq f \right\}.$$

The sets $[\sigma] \cap \mathrm{id}^\omega$ are the basic clopen subsets of id^ω, and generate the topology on that space, which is the topology inherited from Baire space.

For convenience, we treat prefix-free complexity K as a function on ω (via the length-lexicographical ordering of binary strings). The weight of a function $f \in \omega^{\leqslant\omega}$ is

$$\mathrm{wt}(f) = \sum_{n \in \mathrm{dom}\, f} 2^{-f(n)}.$$

We say that $f \in \omega^\omega$ has *finite weight* if $\mathrm{wt}(f) < \infty$. For a set $A \subseteq \mathrm{id}^{\leqslant\omega}$ and real number r, we let

$$A_{\leqslant r} = \{ f \in A \,:\, \mathrm{wt}(f) \leqslant r \},$$

and we similarly define $A_{<r}$ and $A_{>r}$. If $P \subseteq \mathrm{id}^\omega$ is a Π^0_1 class, then for any rational number q, $P_{\leqslant q}$ is a Π^0_1 class as well. This is not usually true for $P_{<q}$ (let alone $P_{>q}$). Note that the space id^ω, by definition, is computably bounded, and so Π^0_1 subclasses of id^ω are effectively compact: from a cover of such a set generated by a c.e. collection of basic clopen sets, we can effectively find a finite sub-cover. Also, every PA degree computes an element of each nonempty such set.

Let $P^K = \{ f \in \mathrm{id}^\omega \colon f \leqslant K \}$. Note that P^K is a Π^0_1 class.

Lemma 3.1. *P^K contains a finite weight function; indeed, $P^K_{\leqslant 3} \neq \varnothing$.*

Proof. This follows from the fact that $\mathrm{wt}(K) = \sum_{n\in\omega} 2^{-K(n)} < 1$. Let $K^*(n) = \min\{K(n), n\}$. Then $K^* \in P^K$ and $\mathrm{wt}(K^*) \leqslant \mathrm{wt}(K) + \sum_{n\in\omega} 2^{-n} < 3$. □

This proves that every PA degree computes a K-bounded function of finite weight (a fact we already saw in the introduction). Note that as $\mathrm{wt}(K) < 1$, a K-bounded function of finite weight can be, by finite alteration, changed to a K-bounded function with weight bounded by 1, so such functions have the same Turing degrees as K-compression functions.

Our goal is to prove that being of PA degree is *not necessary* to compute a K-bounded function of finite weight.

Theorem 3.2. *There is a K-bounded function $f \colon \omega \to \omega$ of finite weight that does not have PA degree.*

Proof. We build f using a forcing argument. The *forcing conditions* are triples of the form (σ, P, q) where:

- $\sigma \in \mathrm{id}^{<\omega}$;
- $P \subseteq P^K \cap [\sigma]$ is a Π^0_1 class such that:
  - if $h \in P$, $g \leqslant h$, and $g \in [\sigma]$, then $g \in P$;
- $q \in \mathbb{Q}$ and $P_{\leqslant q} \neq \varnothing$.

The condition (σ, P, q) should be thought of as saying that $f \in P_{\leqslant q}$. We say that (τ, R, s) *extends* (σ, P, q) if $\sigma \preccurlyeq \tau$, $R \subseteq P$, and $s \leqslant q$. Note that $(\langle\rangle, P^K, 3)$ is a condition, so the set of conditions is nonempty.

For a filter G of forcing conditions, we let

$$f_G = \bigcup \{\sigma : (\sigma, P, q) \in G \text{ for some } P \text{ and } q\}.$$

Then $f_G \in \mathrm{id}^{\leqslant\omega}$. If (σ, P, q) is a condition, then we can find τ properly extending σ such that $(\tau, P \cap [\tau], q)$ is also a condition (take τ to be an initial segment of a function witnessing that $P_{\leqslant q}$ is nonempty). This shows that if G is only mildly generic, then f_G is defined on all of ω.

Lemma 3.3. *Suppose that $(\sigma, P, q) \in G$. Then $f_G \in P_{\leqslant q}$.*

Proof. Let $\tau \prec f_G$. Then there is a condition $(\tau, Q, s) \in G$. By extending this condition (and possibly τ), we may assume that (τ, Q, s) extends the condition (σ, P, q). Since $Q_{\leqslant s}$ is nonempty and $Q_{\leqslant s} \subseteq [\tau] \cap P_{\leqslant q}$, we have that $[\tau] \cap P_{\leqslant q}$ is nonempty. This is true for all $\tau \prec f_G$. Since $P_{\leqslant q}$ is closed, we have $f_G \in P_{\leqslant q}$. $\square$

By definition, $P \subseteq P^K$ for any condition (σ, P, q), so f_G is K-bounded. Lemma 3.3 also implies that $\mathrm{wt}(f_G)$ is finite.

There is not much difference between $P_{\leqslant q}$ and $P_{<q}$.

Lemma 3.4. *Let (σ, P, q) be a condition. Then $P_{<q}$ is nonempty.*

Proof. Suppose not. Then $P_{\leqslant q}$ is nonempty and every element of $P_{\leqslant q}$ has weight exactly q, i.e., $P_{\leqslant q} = P_{=q}$. This gives us an algorithm for computing $\varnothing'$. Note that if m enters $\varnothing'$ at stage s, then $K(s) \leqslant^+ m$. Hence it suffices, given any $m \in \omega$ to find an $n \in \omega$ such that $K(x) \geqslant m$ for all $x \geqslant n$.

To do so, let T be a computable subtree of $\mathrm{id}^{<\omega}$ such that $[T] = P_{=q}$. For $r < q$, recall that $\mathrm{id}^{<\omega}_{>r}$ is the collection of finite τ such that $\mathrm{wt}(\tau) > r$.[6] Let T_n be the set of strings on T of length n. Since $P_{=q} \subseteq \mathrm{id}^{\omega}_{>r}$ and id^{ω} is compact, for every $r < q$ there is an $n \in \omega$ such that $T_n \subseteq \mathrm{id}^{<\omega}_{>r}$; such n can be of course found effectively from r. If $T_n \subseteq \mathrm{id}^{<\omega}_{>q-2^{-m}}$, then $K(x) > m$ for all $x \geqslant n$. For we know that there is some $\sigma \in T_n$ which is extendible ($[\sigma] \cap P_{=q} \neq \varnothing$); if $h \in [\sigma] \cap P_{=q}$, $x \geqslant |\sigma|$, and $\mathrm{wt}(h) - \mathrm{wt}(\sigma) < 2^{-m}$ then $h(x) > m$. Since (σ, P, q) is a condition, we know that $h \leqslant K$. $\square$

Remark 3.5. By the foregoing fact, if (ρ, R, t) is a condition, then there is a $t' < t$ such that (ρ, R, t') is a condition as well. Thus, by genericity, if $(\sigma, P, q) \in G$, then there is some $q' < q$ such that $(\sigma, P, q') \in G$. By Lemma 3.3, $f_G \in P_{\leqslant q'}$, and so $f_G \in P_{<q}$.

The main work is to show that f_G does not have PA degree. This will follow from genericity (and Lemma 3.3), once we show that for any Turing functional Γ, the collection of conditions

$$D_\Gamma = \{(\sigma, P, q) : (\forall h \in P_{\leqslant q})\, \Gamma(h) \notin \mathrm{DNC}_2\}$$

is dense in our forcing partial order.

[6]Note that it is possible that $\mathrm{wt}(\tau) < r$ but every infinite extension of τ in id^{ω} has weight $> r$; indeed, $[\tau] \subseteq \mathrm{id}^{\omega}_{>r}$ if and only if $\mathrm{wt}(\tau) + 2^{-|\tau|+1} > r$. Nonetheless, $\mathrm{id}^{\omega}_{>r}$ is the open set generated by $\mathrm{id}^{<\omega}_{>r}$.

First, we extend to a condition that gives us some "breathing room". We let

$$F = \left\{(\sigma, P, q) \,:\, P_{<\mathrm{wt}(\sigma)+\varepsilon} \neq \varnothing \text{ where } \varepsilon = (q - \mathrm{wt}(\sigma))/3\right\}.$$

Lemma 3.6. *The collection F of conditions is dense.*

Proof. Let (τ, Q, p) be a condition. By Lemma 3.4, let $h \in Q_{<p}$. Pick ε small enough so that $\mathrm{wt}(h) + 3\varepsilon < p$. Take $\sigma \prec h$ extending τ such that $\mathrm{wt}(h) - \mathrm{wt}(\sigma) < \varepsilon$. Then $(\sigma, Q \cap [\sigma], \mathrm{wt}(\sigma) + 3\varepsilon)$ is an extension of (τ, Q, p) in F. □

It thus suffices to show that every condition in F has an extension in D_Γ. Note that if $(\sigma, P, q) \in F$ with $\varepsilon = (q - \mathrm{wt}(\sigma))/3$, then $(\sigma, P, \mathrm{wt}(\sigma) + \varepsilon)$ is also a condition; however we will find an extension of (σ, P, q) in D_Γ, rather than of $(\sigma, P, \mathrm{wt}(\sigma) + \varepsilon)$.

Fix some $(\sigma^*, P^*, q) \in F$; let $r = \mathrm{wt}(\sigma^*)$ and $\varepsilon = (q - r)/3$.

As we did in the proof of Proposition 2.5, we define a partial computable process which may either output 0 or 1 (or diverge). The output of this process will be $J(e)$ for some e, and by the recursion theorem, we may assume we know e in the definition of this process. Consider the Π^0_1 class Q obtained from P^* by removing not only all the strings τ of weight below $r + 2\varepsilon$ for which $\Gamma(\tau, e)\downarrow$, but also all strings majorizing such strings τ:

$$Q = \{h \in P^* \,:\, (\forall \tau \leqslant h)\ \tau \notin C\},$$

where

$$C = \{\tau \in (\mathrm{id}^{<\omega} \cap [\sigma^*])_{<r+2\varepsilon} \,:\, \Gamma(\tau, e)\downarrow\}.$$

The point is that if $h \in Q$, $g \leqslant h$, and $\sigma^* \prec g$, then $g \in Q$.[7]

If $Q_{\leqslant r+2\varepsilon} \neq \varnothing$, then our partial computable process does not terminate. Suppose now that $Q_{\leqslant r+2\varepsilon} = \varnothing$. This is eventually effectively recognised, as $Q_{\leqslant r+2\varepsilon}$ is a Π^0_1 class effectively obtained from e. In this case the process terminates; we will determine its output $i \in \{0, 1\}$ through Lemma 3.8. We use the following auxiliary fact:

Lemma 3.7. *If $Q_{\leqslant r+2\varepsilon} = \varnothing$, then we can effectively find an $n \in \omega$ and a set $E \subseteq \mathrm{id}^{=n}$ such that:*

1. *Every $\sigma \in E$ extends σ^* and $\mathrm{wt}(\sigma) < r + 2\varepsilon$ (that is, $E \subseteq \mathrm{id}^{=n} \cap [\sigma^*]_{<r+2\varepsilon}$);*
2. *For every $\sigma \in E$, there is a $\tau \leqslant \sigma$ (in particular $|\tau| \leqslant |\sigma|$) in C.*
3. *If $\sigma \in E$, $\sigma' \in [\sigma^*]_{<r+2\varepsilon}$, and $\sigma' \leqslant \sigma$, then $\sigma' \in E$.*
4. *There is a $\sigma \in E$ such that $[\sigma] \cap P^*_{<r+\varepsilon} \neq \varnothing$.*

Proof. Let S be a computable tree such that $[S] = P^*$; we may assume that if $\sigma \in S$, $\sigma' \leqslant \sigma$, and $\sigma' \in [\sigma^*]$, then $\sigma' \in S$; this is because if σ is extendible in P^* (meaning $[\sigma] \cap P^* \neq \varnothing$) then so is σ'. Let $S_Q = \{\sigma \in S \,:\, (\forall \tau \leqslant \sigma)\ \tau \notin C\}$. Then $[S_Q] = Q$; since $Q_{\leqslant r+2\varepsilon}$ is empty, by compactness, we can find some n such that every sequence of length n in S_Q has weight $> r + 2\varepsilon$. We then let E be the collection of sequences of length n in $S_{<r+2\varepsilon}$. Properties (1)–(3) follow from the definition of S_Q. Property (4) holds because $P^*_{<r+\varepsilon}$ is nonempty; some σ of length n has an extension in P^* of weight $< r + \varepsilon$, and necessarily, $\sigma \notin S_Q$. □

[7] Note, however, that Q is not quite the same as the class obtained by removing all h for which there is a $g \leqslant h$ in $P^*_{\leqslant r+2\varepsilon}$ such that $\Gamma(g, e)\downarrow$. The class Q is smaller, since a witness τ may not be extendible to an h-majorized g of weight at most $r + 2\varepsilon$.

Having obtained E, we let

$$\widehat{E} = E_{<r+\varepsilon} = \{\sigma \in E : \mathrm{wt}(\sigma) < r + \varepsilon\}.$$

Condition (4) says that $\widehat{E}$ is nonempty, indeed some $\sigma \in \widehat{E}$ is extendible in $P^*_{<r+\varepsilon}$. Now an important point is that if $\sigma, \sigma' \in \widehat{E}$, then the pointwise minimum $\min(\sigma, \sigma')$ is in E, as both σ and σ' extend σ^* and so $\mathrm{wt}(\sigma) - \mathrm{wt}(\sigma^*) < \varepsilon$, and similarly for σ'. This allows us to show the following. For $i \in \{0,1\}$, let $C_i = \{\tau \in C : \Gamma(\tau, e) = i\}$; we assume that Γ maps into $\{0,1\}$-valued functions, so $C = C_0 \cup C_1$.

Lemma 3.8. *There is some $i \in \{0,1\}$ such that for every $\sigma \in \widehat{E}$ there is a $\tau \leqslant \sigma$ in C_i. We declare this i the output of our partial computable process.*

Proof. For any pair σ, σ' of strings from $\widehat{E}$, find some $\tau \leqslant \min(\sigma, \sigma')$ in C; let $c(\{\sigma, \sigma'\}) = \Gamma(\tau, e)$. By Lemma 2.3, there is a colour $i \in \{0,1\}$ such that for all $\sigma \in \widehat{E}$, there is a $\sigma' \in \widehat{E}$ such that $c(\{\sigma, \sigma'\}) = i$. (This is easy; if it fails for 0, then a single σ' witnesses it for 1.) This colour i is as required. $\square$

We now describe the extension of $(\sigma^*, P^*, r + 3\varepsilon)$ in D_Γ. There are two cases. If $Q_{\leqslant r+2\varepsilon}$ is nonempty, then $(\sigma^*, Q, r + 2\varepsilon)$ is a condition, and $\Gamma(h, e)\uparrow$ for all $h \in Q_{\leqslant r+2\varepsilon}$. We assume, then, that $Q_{\leqslant r+2\varepsilon}$ is empty. Let i be the outcome of the partial computable process described above (which has an output in this case). Let $\sigma \in \widehat{E}$ be extendible in $P^*_{<r+\varepsilon}$; fix an $h \in P^*_{<r+\varepsilon}$ with $\sigma \prec h$. Let $\tau \leqslant \sigma$ be in C_i.

Let $R = P^* \cap [\tau]$. We claim that $R_{\leqslant r+3\varepsilon}$ is nonempty. For we can let $g = \tau\hat{\ }h \upharpoonright [|\tau|, \infty)$. Note that $g \leqslant h$, so $g \in P^*$. And

$$\mathrm{wt}(g) = \mathrm{wt}(\tau) + (\mathrm{wt}(h) - \mathrm{wt}(h \upharpoonright |\tau|)) \leqslant \mathrm{wt}(\tau) + (\mathrm{wt}(h) - \mathrm{wt}(\sigma^*)) < (r + 2\varepsilon) + \varepsilon.$$

Thus $(\tau, R, r + 3\varepsilon)$ is a condition extending $(\sigma^*, P^*, r + 3\varepsilon)$. Every $h \in R$ extends τ, so $\Gamma(h, e) = i = J(e)$. Therefore, $\Gamma(h) \notin \mathrm{DNC}_2$. $\square$

4. The discrete covering property

In this section, we show that having the discrete covering property is equivalent to computing a K-compression function, and that such oracles compute slow growing DNC functions. Recall that we defined the discrete covering property in terms of sequences of subsets of ω. For the first proof in this section, it is convenient to work with sequences $\bar{A} = \langle A_n \rangle$ of subsets of $2^{<\omega}$, which is a clearly a harmless translation. Similarly, in the second proof, we work with sequences of subsets of $\omega^{<\omega}$.

Proposition 4.1. *An oracle D computes a K-compression function if and only if it has the discrete covering property.*

Proof. The equivalence is straightforward. First, assume that D has the discrete covering property. Let $A_n = \{\sigma : K(\sigma) \leqslant n\}$, so $\bar{A} = \langle A_n \rangle$ is a uniformly c.e. sequence such that $\mathrm{wt}(\bar{A}) < 2$. Thus there is a D-computable sequence $\bar{B} = \langle B_n \rangle$ of finite weight that covers $\bar{A}$. Define a D-computable function f as follow: let $f(\sigma)$ be the least n such that $\sigma \in B_n$. This ensures that $f(\sigma) \leqslant K(\sigma)$ and $\mathrm{wt}(f) < \mathrm{wt}(B_n) < \infty$. Some finite alteration of f is a K-bounded function with weight bounded by 1, and an application of the Kraft–Chaitin theorem gives us a D-computable K-compression function.

For the other direction, assume that $F\colon 2^{<\omega} \to 2^{<\omega}$ is a K-compression function computable from D. Let $\bar{A}$ be a uniformly c.e. sequence of finite weight. Then there is a $c \in \omega$ such that $\sigma \in A_n$ implies that $K(\sigma) \leqslant n + c$. Hence $\sigma \in A_n$ implies that $|F(\sigma)| \leqslant n + c$. Define $B_n = \{\sigma : |F(\sigma)| \leqslant n + c\}$, so $\bar{B}$ is a D-uniformly computable sequence that covers $\bar{A}$. Also,

$$\mathrm{wt}(\bar{B}) \leqslant 2^{c+1} \sum_{\sigma \in 2^{<\omega}} 2^{-|F(\sigma)|} \leqslant 2^{c+1}.$$

Therefore, D has the discrete covering property. □

Remark 4.2. Note that by this proof, the sequence $\bar{A} = \langle A_n \rangle$ given by $A_n = \{\sigma : K(\sigma) \leqslant n\}$ is universal: if D computes a cover for $\bar{A}$, it has the discrete covering property.

Recall that an *order function* is a computable, nondecreasing, unbounded function on ω.

Proposition 4.3. *Let $h \colon \omega \to \omega \smallsetminus \{0,1\}$ be any order function. Suppose an oracle D has the discrete covering property. Then D computes an h-bounded DNC function.*

Proof. Fix an increasing computable function g such that

$$\lim_{m \to \infty} \frac{h(g(m))}{2^m} = \infty. \tag{4.1}$$

We build a uniformly c.e. sequence $\bar{A} = \langle A_n \rangle$ of subsets of $h^{<\omega}$ as follows. If k enters $\varnothing'$ at stage s, then find a $\tau \in h^{<\omega}$ of length $g(s)$ such that

$$(\forall n < g(s))\ J_s(n) \downarrow < h(n) \implies \tau(n) = J_s(n).$$

In other words, except that it must remain h-bounded, τ is trying to be an extension of J at stage s. For each $m \leqslant s$, put $\tau \upharpoonright g(m)$ into A_{k+m+2}. We act for each k at most once, so

$$\mathrm{wt}(\bar{A}) < \sum_{k \in \omega} \sum_{m \in \omega} 2^{-k-m-2} = \sum_{k \in \omega} 2^{-k-1} = 1.$$

Let $\bar{B}$ be a D-computable sequence of subsets of $h^{<\omega}$ such that $\mathrm{wt}(\bar{B})$ is finite. By removing finitely many elements, we may assume that $\mathrm{wt}(\bar{B}) \leqslant 1$.

There are two cases, much like in the proof of Proposition 2.9. First, assume that there is a k such that for all m, there is a $\tau \in B_{k+m+2}$ of length $g(m)$ such that

$$(\forall n < g(m))\ J(n) \downarrow < h(n) \implies \tau(n) = J(n).$$

Define a D-computable function $f \colon \omega \to \omega$ such that if $n \in [g(m-1), g(m))$, then $f(n)$ is different from $\tau(n)$ for every $\tau \in B_{k+m+2}$ of length $g(m)$. Our assumption guarantees that if $f(n) < h(n)$, then $f(n) \neq J(n)$. Note that there are at most 2^{k+m+2} elements of B_{k+m+2}, hence we can ensure that $f(n) \leqslant 2^{k+m+2}$ for all such n. But by (4.1), we have $h(g(m-1)) > 2^{k+m+2}$ for all sufficiently large m. Therefore, $f(n) < h(n)$ for all sufficiently large n. By taking a finite modification of f, we get a D-computable h-bounded DNC function.

If the first case fails, then for every k there is an m and a t such that

$$(\forall \tau \in B_{k+m+2} \cap \omega^{g(m)})(\exists n < g(m))[\ J_t(n) \downarrow < h(n) \text{ but } \tau(n) \neq J_t(n)\ ].$$

Note that we can find such an m and t effectively from $\bar{B}$, hence from D. Let $s(k) = \max\{m, t\}$, so s is a D-computable function. By the construction of $\bar{A}$, it cannot be the case that k enters $\varnothing'$ after stage $s(k)$. Therefore, $\varnothing' \leqslant_{\mathrm{T}} D$, so D computes a DNC_2 function (which is certainly h-bounded). □

5. The (strong) continuous covering property

Recall from the introduction that by definition, an oracle D has the continuous covering property if for every Π^0_1 class P of positive measure, there is a D-computable tree $T \subseteq 2^{<\omega}$ with no dead ends such that $\lambda([T]) > 0$ and $[T] \subseteq P$. The main result of this section, and arguably of the paper, is that the continuous covering property does not imply PA-completeness. We also relate the continuous covering property to the discrete covering property and to High(CR, MLR). Everything we prove about the continuous covering property actually holds for an apparently stronger notion; see Definition 5.2 below.

Proposition 5.1. *Every oracle that has the continuous covering property also has the discrete covering property.*

Proof. Let $\langle C_{n,k} \rangle$ be a computable array of independent clopen subsets of 2^ω (each given canonically) such that $\lambda(C_{n,k}) = 2^{-n}$ for all k. For an sequence $\bar{A} = \langle A_n \rangle$ of subsets of ω such that $\mathrm{wt}(\bar{A}) < \infty$. Consider the Σ^0_1 class

$$U = \bigcup \{C_{n,k} \,:\, k \in A_n\}\,.$$

Note that

$$\lambda(U) = 1 - \prod_{n\in\omega} \prod_{k\in A_n} (1 - 2^{-n}).$$

But $\prod_{n\in\omega} \prod_{k\in A_n} (1 - 2^{-n}) > 0$ if and only if $\mathrm{wt}(\bar{A}) = \sum_{n\in\omega} \sum_{k\in A_n} 2^{-n} < \infty$. Therefore, $\lambda(U) < 1$.

If D has the continuous covering property, then there is an open set $V \supseteq U$ such that $\lambda(V) < 1$ and $S_V = \{\sigma \in 2^{<\omega} \,:\, [\sigma] \subseteq V\}$ is D-computable. Let $B_n = \{k \,:\, C_{n,k} \subseteq V\}$. Then $\bar{B} = \langle B_n \rangle$ is a D-computable cover of $\bar{A}$. All that remains is to prove that $\bar{B}$ has finite weight. But

$$1 - \prod_{n\in\omega} \prod_{k\in B_n} (1 - 2^{-n}) \leqslant \lambda(V) < 1,$$

so $\prod_{n\in\omega} \prod_{k\in B_n} (1-2^{-n}) > 0$. This implies that $\mathrm{wt}(\bar{B}) = \sum_{n\in\omega} \sum_{k\in B_n} 2^{-n} < \infty$. $\square$

Note that nothing prevents $[T]$ in the definition of the continuous covering property from having intervals in which it is nonempty but has measure zero (or even from having isolated paths). It is convenient to work with an apparently stronger notion in which such intervals are explicitly forbidden.

Definition 5.2. We say that D has the *strong continuous covering property* if for every Π^0_1 class P of positive measure, there is a D-computable tree $T \subseteq 2^{<\omega}$ such that $T \neq \varnothing$, $[T] \subseteq P$, and for all $\sigma \in T$, $\lambda([T] \cap [\sigma]) > 0$.

One reason that the *strong* continuous covering property is convenient is that we can show that there is a "universal" Π^0_1 class for this property (compare with Remark 4.2). Let U be the first component of the standard universal Martin-Löf test, i.e., the test obtained by combining all Martin-Löf tests. So $\lambda(U) < 1$ and for any Martin-Löf test $\langle V_n \rangle$ there is an n such that $V_n \subseteq U$. Let $\mathbf{P}$ be the complement of U, so it is a positive measure Π^0_1 class. The following lemma states that having the strong continuous covering property for $\mathbf{P}$ is enough to ensure the strong continuous covering property in general.

For any $W \subseteq 2^\omega$ and $\sigma \in 2^{<\omega}$, let $W|\sigma = \{X \in 2^\omega \,:\, \sigma\hat{\ }X \in W\}$.

Lemma 5.3. *Suppose T is a nonempty tree such that $[T] \subseteq \mathbf{P}$, and for all $\sigma \in T$, $\lambda([T] \cap [\sigma]) > 0$. Then T has the strong continuous covering property.*

Proof. We work in the dual setting, with Σ^0_1 classes. Let T be a tree as described in the statement of the lemma. Let W be the open class generated by $2^{<\omega} \setminus T$. So W is T-computable, $\lambda(W) < 1$, and $U \subseteq W$, where $U = 2^\omega \setminus \mathbf{P}$ is the first component of the standard universal Martin-Löf test, as above. Also, we have

$$2^{<\omega} \setminus T = \{\sigma \in 2^{<\omega} : [\sigma] \subseteq W\} = \{\sigma \in 2^{<\omega} : \lambda(W|\sigma) = 1\}.$$

Let V be a Σ^0_1 class with $\lambda(V) < 1$. Let $S \subseteq 2^{<\omega}$ be a prefix-free c.e. set of strings such that $V = [S]$. Define S^n recursively, as usual: let $S^0 = \{\langle\rangle\}$ and define S^{n+1} to be $\{\sigma\hat{}\tau : \sigma \in S^n \text{ and } \tau \in S\}$. It is straightforward to check that $\lambda([S^n]) = (\lambda(V))^n$, so an effective subsequence of $\langle [S^n] \rangle$ forms a Martin-Löf test. Therefore, there is an n such that $[S^n] \subseteq W$. Let n be the least such; $n > 0$ since $\lambda(W) < 1$. Let σ be a string witnessing that $[S^{n-1}] \nsubseteq W$, i.e., $\sigma \in S^{n-1}$ and $[\sigma] \nsubseteq W$. Now consider $W|\sigma$. We have that $[\sigma] \nsubseteq W$ implies that $\lambda(W|\sigma) < 1$, and $[S^n] \subseteq W$ implies that $V = [S] \subseteq W|\sigma$. Note that

$$\{\tau : [\tau] \subseteq W|\sigma\} = \{\tau : [\sigma\hat{}\tau] \subseteq W\}$$

is T-computable. Finally, if $\lambda((W|\sigma)|\tau) = \lambda(W|\sigma\hat{}\tau) = 1$, then $[\sigma\hat{}\tau] \subseteq W$, hence $[\tau] \subseteq W|\sigma$. Since V was an arbitrary Σ^0_1 class with $\lambda(V) < 1$, we have proved that $\deg_{\mathrm{T}}(T)$ has the strong continuous covering property. □

In the next proposition, it will be convenient to work with *exactly computable martingales*, which are rational-valued martingales that are computable as functions from $2^{<\omega}$ to $\mathbb{Q} \cap [0, \infty)$. Schnorr [22] proved that for every computable martingale V, there is an exactly computable martingale B such that $V(\sigma) \leqslant B(\sigma) \leqslant V(\sigma) + 2$ for each $\sigma \in 2^{<\omega}$. In particular, V and B succeed on the same sequences. (See also [20, 7.3.8]; note that there, B is called computable, which should be interpreted as "exactly computable" in the sense above.) We will use the relativized form of this fact.

Proposition 5.4. *Every oracle D in* $\mathrm{High}(\mathrm{CR}, \mathrm{MLR})$ *has the strong continuous covering property.*

Proof. By Proposition 1.6, there is a D-computable martingale N that succeeds on all non-ML-random sequences. By the note above, we may assume that N is exactly computable. We may also assume that $N(\langle\rangle) < 1$. Let Q be the set of minimal strings σ with $N(\sigma) \geqslant 1$ and let $V = [Q]$ be the $\Sigma^0_1[D]$ class generated by Q. Note that if $(\forall \tau \preccurlyeq \sigma)\ N(\tau) < 1$, then $[\sigma] \nsubseteq V$, and in fact $\lambda(V|\sigma) \leqslant N(\sigma) < 1$. Thus

$$\{\sigma \in 2^{<\omega} : [\sigma] \subseteq V\} = \{\sigma \in 2^{<\omega} : (\exists \tau \preccurlyeq \sigma)\ N(\tau) \geqslant 1\},$$

which is D-computable.

Let $U = 2^{<\omega} \setminus \mathbf{P}$ be the first component of the standard universal Martin-Löf test and fix a prefix-free set $S \subseteq 2^{<\omega}$ such that $U = [S]$. We attempt to build a sequence of strings $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \cdots$ as follows. Let $\sigma_0 = \langle\rangle$. If σ_n has been defined, it must be the case that $[\sigma_n] \nsubseteq V$. If possible, pick a $\tau \in S$ such that $[\sigma_n\hat{}\tau] \nsubseteq V$ and let $\sigma_{n+1} = \sigma_n\hat{}\tau$.

If σ_n exists for every n, then let $X = \bigcup_{n\in\omega} \sigma_n$. Note that $X \notin V$, so N does not succeed on X; in fact, it never reaches 1. On the other hand, $X \in \bigcap_{n\in\omega} [S^n]$, so it is not Martin-Löf random. This contradicts the choice of N.

Therefore, there is an n such that σ_n is defined, but σ_{n+1} is not. So $[\sigma_n] \nsubseteq V$, but for every $\tau \in S$, we have $[\sigma_n\hat{\ }\tau] \subseteq V$. This means that $U \subseteq V|\sigma_n$. We also have that $\lambda(V|\sigma_n) < 1$. Finally, if $[\rho] \nsubseteq V|\sigma_n$, then it must be the case that $\lambda((V|\sigma_n)|\rho) = \lambda(V|\sigma_n\hat{\ }\rho) \leqslant N(\sigma_n\hat{\ }\rho) < 1$. Therefore, Lemma 5.3 tells us that D has the strong continuous covering property. □

Theorem 5.5. *There is an oracle D with the strong continuous covering property that does not have PA degree.*

Proof. The proof of this theorem is an elaboration on the proof of Theorem 3.2. We will build a tree T that does not have PA degree but which satisfies the hypothesis of Lemma 5.3. The tree T is built by forcing. In the previous proof, forcing conditions specified a finite initial segment σ of the K-compression function we built, a Π^0_1 class of possible extensions of σ, and a rational number q with the promise that the function that we eventually build will have weight at most q. In the current construction, a forcing condition will specify: a finite initial segment of T (which we code by its set of leaves u); a Π^0_1 class of possible extensions of u to trees $S \subseteq \mathbf{P}$; and for each leaf $\sigma \in u$, a rational number q_σ with the promise that the measure of $T \cap [\sigma]$ is at least q_σ. The structure of the proof is the same as before, but the combinatorial lemmas are more elaborate. We start with some terminology and notation.

By a tree (*arbre* in French) we mean a subset of $2^{<\omega}$ closed under taking initial segments. Note that there is a 1-1 correspondence between closed subsets of 2^ω and subtrees of $2^{<\omega}$ with no dead ends. Let $\mathcal{A}$ denote the set of nonempty trees with no dead ends. Coding strings by numbers, $\mathcal{A}$ itself is an effectively closed subset of Cantor space. We will work with Π^0_1 classes of trees with no dead ends, namely, Π^0_1 subclasses of $\mathcal{A}$. To keep notational complexity in check, below, we ignore the difference between $[T]$ and T (for $T \in \mathcal{A}$) and write T for both. Note that for $S, T \in \mathcal{A}$, we have $S \subseteq T$ iff $[S] \subseteq [T]$. The operation of intersection is well-defined: for $S, T \in \mathcal{A}$, we let $S \cap T$ be the unique element R of $\mathcal{A}$ such that $[R] = [S] \cap [T]$, unless $[S] \cap [T]$ is empty, a case which we will avoid.

Infinite trees are built up of finite ones. Let $\mathcal{A}_{<\omega}$ be the collection of all nonempty finite subtrees of $2^{<\omega}$. For $T \in \mathcal{A}$ and $\vartheta \in \mathcal{A}_{<\omega}$, we say that T *extends* ϑ (and sometimes write $\vartheta \prec T$) if $\vartheta \subset T$ and every $\sigma \in T$ is comparable with a leaf of ϑ. For each $\vartheta \in \mathcal{A}_{<\omega}$, we let $[\vartheta]$ be the collection of $T \in \mathcal{A}$ that extend ϑ. This is a clopen subset of $\mathcal{A}$, and the collection of these sets generates the topology on $\mathcal{A}$. We often restrict ourselves to trees of a fixed height; for $n \in \omega$, let $\mathcal{A}_n$ be the set of finite trees all of whose leaves have length n. For $\vartheta \in \mathcal{A}_{<\omega}$ and n greater than the height of ϑ, we let $[\vartheta]_n$ be the set of trees $\varpi \in \mathcal{A}_n$ which extend ϑ, again in the sense that each $\tau \in \varpi$ extends some $\sigma \in \vartheta$ and each $\sigma \in \vartheta$ is extended by some $\tau \in \varpi$; we write $\vartheta \leqslant \varpi$. Note that for $\vartheta, \varpi \in \mathcal{A}_{<\omega}$, $\vartheta \leqslant \varpi$ if and only if $[\varpi] \subseteq [\vartheta]$.

We also implicitly use the bijection between $\mathcal{A}_{<\omega}$ and the collection of finite antichains of strings (a tree is mapped to its leaves). For example, for a finite antichain of strings u we let $[u]$ be $[\vartheta]$ where u is the set of leaves of ϑ. A tree $\vartheta \in \mathcal{A}_{<\omega}$ and its set of leaves are both identified with the clopen subset of 2^ω determined by ϑ. Thus for example, for a finite antichain u of strings we let $\lambda(u) = \sum_{\sigma \in u} 2^{-|\sigma|}$. Similarly, for $T \in \mathcal{A}$ and $\tau \in 2^{<\omega}$ we let $T \cap \tau = \{\sigma \in T : \sigma \not\perp \tau\}$.

Fix the Π^0_1 class $\mathbf{P}$ from Lemma 5.3. Our forcing conditions are triples $(u, P, \bar{q})$ such that:

- u is a nonempty finite antichain of strings;
- $P \subseteq \mathcal{A}$ is a Π^0_1 subclass of $[u]$ such that:
  - for all $T \in P$ we have $T \subseteq \mathbf{P}$;
  - if $T \in P$, $S \in [u]$ and $S \subseteq T$ then $S \in P$.
- $\bar{q} = \langle q_\sigma \rangle_{\sigma \in u}$ is a sequence of positive rational numbers such that each q_σ is smaller than $2^{-|\sigma|}$, and

$$P_{\geqslant \bar{q}} = \{T \in P \,:\, (\forall \sigma \in u)\ \lambda(T \cap \sigma) \geqslant q_\sigma\}$$

  is nonempty.

If we let P be the set of trees $T \in \mathcal{A}$ such that $T \subseteq \mathbf{P}$ and q be any rational number smaller than $\lambda(\mathbf{P})$, then $(\{\langle\rangle\}, P, \langle q \rangle)$ is a condition. So the set of conditions is nonempty. A condition $(v, R, \bar{r})$ *extends* a condition $(u, P, \bar{q})$ if:

(1) $u \preccurlyeq v$;
(2) $R \subseteq P$; and
(3) for all $\sigma \in u$, we have $q_\sigma \leqslant \sum\{r_\tau \,:\, \tau \in v \ \&\ \tau \succcurlyeq \sigma\}$.

Note that if $u \preccurlyeq v$, then condition (3) is equivalent to $[v]_{\geqslant \bar{r}} \subseteq [u]_{\geqslant \bar{q}}$. In particular, we see that if a condition $(v, R, \bar{r})$ extends a condition $(u, P, \bar{q})$, then $R_{\geqslant \bar{r}} \subseteq P_{\geqslant \bar{q}}$.

Our first lemma is directly analogous to Lemma 3.4.

Lemma 5.6. *Let $(u, P, \bar{q})$ be a condition. Then*

$$P_{> \bar{q}} = \{T \in P \,:\, (\forall \sigma \in u)\ \lambda(T \cap \sigma) > q_\sigma\}$$

is nonempty.

Proof. Suppose not. Let v be a $\subseteq$-maximal subset of u for which there is some $T \in P_{\geqslant \bar{q}}$ with $\lambda(T \cap \sigma) > q_\sigma$ for all $\sigma \in v$, and let T witness this. Let $\varepsilon > 0$ be rational smaller than $\lambda(T \cap \sigma) - q_\sigma$ for all $\sigma \in v$, and let $q'_\sigma = q_\sigma + \varepsilon$ for $\sigma \in v$ and $q'_\sigma = q_\sigma$ for $\sigma \in u - v$. Thus, $P_{\geqslant \bar{q}'}$ is nonempty, and for all $S \in P_{\geqslant \bar{q}'}$, for all $\sigma \in u - v$ we have $\lambda(S \cap \sigma) = q_\sigma$. Choose any $\sigma \in u - v$, and let $Q = \{S \cap \sigma \,:\, S \in P_{\geqslant \bar{q}'}\}$. Let $q = q_\sigma$. So Q is a nonempty Π^0_1 subclass of $\mathcal{A}$ and for all $T \in Q$, $T \subseteq \mathbf{P}$ and $\lambda(T) = q$.

Let $V_n = \varnothing$ if $n \notin \varnothing'$, and otherwise let $V_n = \{\sigma\hat{\ }0^n \,:\, |\sigma| = s\}$ where s is the stage at which n enters $\varnothing'$. Since $\lambda(V_n) \leqslant 2^{-n}$ and $\langle V_n \rangle$ is uniformly c.e., for all sufficiently large n we have $V_n \cap \mathbf{P} = \varnothing$.

Let $m \in \omega$. By compactness, we can effectively find a $t \in \omega$ and a $C \subseteq \mathcal{A}_t$ such that $Q \subseteq \bigcup_{\vartheta \in C} [\vartheta]$ and such that $q/\lambda(\vartheta) > 1 - 2^{-m}$ for all $\vartheta \in C$. We then claim that provided that m is large enough, $m \in \varnothing'$ if and only if $m \in \varnothing'_t$. For fix some $\vartheta \in C$ such that $[\vartheta] \cap Q \neq \varnothing$, and fix some $T \in [\vartheta] \cap Q$. If m enters $\varnothing'$ at stage $s > t$ then for every leaf σ of ϑ, $\lambda(T|\sigma) \leqslant \lambda(\mathbf{P}|\sigma) \leqslant 1 - 2^{-m}$ and so $q = \lambda(T) \leqslant (1 - 2^{-m})\lambda(\vartheta)$ which is not the case. This algorithm for computing $\varnothing'$ gives the desired contradiction. □

We will often use Lemma 5.6 in conjunction with the following:

Lemma 5.7. *Let $(u, P, \bar{q})$ be a condition; let $v \succcurlyeq u$, and suppose that $S > v$ and $S \in P_{> \bar{q}}$. Then there is some $\bar{p} = \langle p_\tau \rangle_{\tau \in v}$ such that $(v, P \cap [v], \bar{p})$ is a condition extending $(u, P, \bar{q})$.*

Proof. For $\sigma \in u$, let $v_\sigma = \{\tau \in v \,:\, \tau \succcurlyeq \sigma\}$. Choose rational p_τ for $\tau \in v$ so that $p_\tau \leqslant \lambda(S \cap \tau)$, and for all $\sigma \in u$, $\sum_{\tau \in v_\sigma} p_\tau \geqslant q_\sigma$; this is possible because

$S \cap \sigma = \bigcup_{\tau \in v_\sigma} S \cap \tau$ and so $\sum_{\tau \in v_\sigma} \lambda(S \cap \tau) = \lambda(S \cap \sigma) > q_\sigma$. Then $S \in [v] \cap P_{\geqslant \bar{p}}$ so $(v, P \cap [v], \bar{p})$ is indeed a condition as required. □

For a filter G of forcing conditions, we let T_G be the downward closure of

$$\bigcup \{u : (u, P, \bar{q}) \in G \text{ for some } P \text{ and } \bar{q}\}.$$

We assume from now that G is fairly generic.

Lemma 5.8. *$T_G \in \mathcal{A}$.*

Proof. It suffices to show that for any condition $(u, P, \bar{q})$, for all large n, there is an extension $(v, Q, \bar{p})$ of $(u, P, \bar{q})$ such that every $\sigma \in v$ has length n.

Let $(u, P, \bar{q})$ be a condition, and let $n > |\sigma|$ for all $\sigma \in u$. By Lemma 5.6, let $S \in P_{>\bar{q}}$. Let $v = S^{=n}$ be the collection of strings on S of length n. Since $u \prec S$, we have $u \prec v$, and of course $v \prec S$; by Lemma 5.7, there is some $\bar{p}$ such that $(v, P \cap [v], \bar{p})$ is a condition extending $(u, P, \bar{q})$. □

Lemma 5.9. *Let $(u, P, \bar{q}) \in G$. Then $T_G \in P_{\geqslant \bar{q}}$.*

Proof. Let $\vartheta \prec T_G$; we can find some $(v, Q, \bar{p}) \in G$ extending $(u, P, \bar{q})$ such that $\vartheta \preccurlyeq v$. Since $Q_{\geqslant \bar{p}} \subseteq [v] \cap P_{\geqslant \bar{q}}$, it follows that $[\vartheta] \cap P_{\geqslant \bar{q}}$ is nonempty. Since $P_{\geqslant \bar{q}}$ is closed, the lemma follows. □

Let $\Gamma\colon \mathcal{A} \to 2^\omega$ be a Turing functional. Let D_Γ be the set of conditions $(u, P, \bar{q})$ such that $\Gamma(T) \notin \mathrm{DNC}_2$ for all $T \in P_{\geqslant \bar{q}}$. We show that D_Γ is dense.

First we prepare. The following is analogous to Lemma 3.6. We define the collection F of conditions that give us sufficient breathing room. Let $(v, P, \bar{q})$ be a condition; for $\sigma \in v$, let $\varepsilon_\sigma = (2^{-|\sigma|} - q_\sigma)/3$. We set $r_\sigma = 2^{-|\sigma|}$, so that we can write $\bar{q} = \bar{r} - 3\bar{\varepsilon}$. The condition $(v, P, \bar{q})$ is in F if $P_{>\bar{r}-\bar{\varepsilon}}$ is nonempty.

Lemma 5.10. *The collection F of conditions is dense.*

Proof. Let $(u, P, \bar{q})$ be a condition. By Lemma 5.6, let $T \in P_{>\bar{q}}$. Fix some $\sigma \in u$. Take a positive rational number δ_σ such that $6\delta_\sigma < \lambda(T \cap \sigma) - q_\sigma$. Find some finite antichain v_σ of extensions of σ such that $v_\sigma \prec T \cap \sigma$ and further $\lambda(v_\sigma) - \lambda(T \cap \sigma) < \delta_\sigma$ (where we again identify v_σ with the clopen subset of Cantor space it determines).

For $\tau \in v_\sigma$, let $\eta_\tau = r_\tau - \lambda(T \cap \tau)$; so

$$\sum_{\tau \in v_\sigma} \eta_\tau = \lambda(v_\sigma) - \lambda(T \cap v_\sigma) < \delta_\sigma, \tag{5.1}$$

using the fact that $T \cap \sigma = T \cap v_\sigma$.

Let $u^*_\sigma = \{\tau \in v_\sigma : r_\tau - 3\eta_\tau > 0\}$. We aim to show that:

$$\sum_{\tau \in u^*_\sigma} (r_\tau - 3\eta_\tau) > q_\sigma. \tag{5.2}$$

If this is the case, then each u^*_σ is nonempty; letting $u^* = \bigcup_{\sigma \in u} u^*_\sigma$, we would have $u \preccurlyeq u^*$. We can then choose, for each $\tau \in u^*$, a rational ε_τ just slightly larger than η_τ, so that we still have $\varepsilon_\tau < r_\tau/3$ and $\sum_{\tau \in u^*_\sigma}(r_\tau - 3\varepsilon_\tau) > q_\sigma$ for each $\sigma \in u$. Then $(u^*, P \cap [u^*], \bar{r} - 3\bar{\varepsilon})$ would be a condition extending $(u, P, \bar{q})$; it would be a condition in F, since $T^* = T \cap u^*$ witnesses that $(P \cap [u^*])_{>\bar{r}-\bar{\varepsilon}}$ is nonempty: $T^* \subseteq T$ and so is in P, and for $\tau \in u^*$ we have $\lambda(T^* \cap \tau) = \lambda(T \cap \tau) = r_\tau - \eta_\tau > r_\tau - \varepsilon_\tau$.

Fix $\sigma \in u$. Toward showing (5.2), we note that by (5.1), as $\sum_{\tau \in u^*_\sigma} \eta_\tau \leqslant \sum_{\tau \in v_\sigma} \eta_\tau$, we have

$$\sum_{\tau \in u^*_\sigma} (r_\tau - 3\eta_\tau) > \lambda(u^*_\sigma) - 3\delta_\sigma,$$

so it suffices to show that

$$\lambda(u^*_\sigma) \geqslant q_\sigma + 3\delta_\sigma. \tag{5.3}$$

Let $w_\sigma = v_\sigma \setminus u^*_\sigma = \{\tau \in v_\sigma : \lambda(T|\tau) \leqslant 2/3\}$. Then

$$\lambda(u^*_\sigma) + \lambda(w_\sigma) = \lambda(v_\sigma) \geqslant \lambda(T \cap \sigma) > q_\sigma + 6\delta_\sigma;$$

so it suffices to show that

$$\lambda(w_\sigma) \leqslant 3\delta_\sigma. \tag{5.4}$$

By definition, $\lambda(T \cap w_\sigma) \leqslant (2/3) \cdot \lambda(w_\sigma)$. Now

$$\begin{aligned} \lambda(u^*_\sigma) + (2/3) \cdot \lambda(w_\sigma) + \delta_\sigma &\geqslant \lambda(T \cap u^*_\sigma) + \lambda(T \cap w_\sigma) + \delta_\sigma \\ &= \lambda(T \cap \sigma) + \delta_\sigma \geqslant \lambda(v_\sigma) = \lambda(u^*_\sigma) + \lambda(w_\sigma); \end{aligned}$$

subtracting $\lambda(u^*_\sigma)$ gives the desired result (5.4). ☐

Now fixing a condition $(u^*, P^*, \bar{r} - 3\bar{\varepsilon})$ in F, we find an extension in D_Γ. The main property we use is that for $S, T \in [u^*]_{>\bar{r}-\bar{\varepsilon}}$, for all $\tau \in u^*$ we have $\lambda(S \cap T \cap \tau) > r_\tau - 2\varepsilon_\tau$, so $S \cap T \in [u^*]_{>\bar{r}-2\bar{\varepsilon}}$. We can now run the proof from above.

We define a partial computable process which may output 0 or 1; by the recursion theorem, we obtain an e such that this output is $J(e)$. Let

$$C = \{\varsigma \in \mathcal{A}_{<\omega} : \varsigma \in [u^*]_{>\bar{r}-2\bar{\varepsilon}} \ \& \ \Gamma(\varsigma, e)\downarrow\},$$

and let

$$Q = \{T \in P^* : (\forall \vartheta \subset T)\ \vartheta \notin C\}.$$

Here by $\vartheta \subset T$ we do mean the sets of strings, not the associated closed sets; and we do not require that $T \succ \vartheta$.

Note that for all $T \in P^*$, since $T \subseteq \mathbf{P}$, for all $\tau \in T$ we must have $\lambda(T \cap \tau) < 2^{-|\tau|}$. Hence if $T \in [u^*]_{\geqslant \bar{p}}$ for some $\bar{p}$, then for all $\vartheta \geqslant u^*$ with $\vartheta \prec T$, we must have $\vartheta \in [u^*]_{>\bar{p}}$. Hence, if $Q_{\geqslant \bar{r}-2\bar{\varepsilon}}$ is nonempty, then $(u^*, Q, \bar{r} - 2\bar{\varepsilon})$ is an extension of $(u^*, P^*, \bar{r} - 3\bar{\varepsilon})$ in D_Γ. We suppose then that $Q_{\geqslant \bar{r}-2\bar{\varepsilon}}$ is empty.

As in Lemma 3.7, by compactness, we can find an $n \in \omega$ and a set $E \subseteq \mathcal{A}_n$ such that:

1. $E \subseteq [u^*]_{>\bar{r}-2\bar{\varepsilon}}$;
2. For every $\vartheta \in E$ there is some $\varrho \subseteq \vartheta$ in C;
3. For every $\vartheta \in E$ and $\varrho \subseteq \vartheta$ in $[u^*]_{>\bar{r}-2\bar{\varepsilon}}$ we have $\varrho \in E$;
4. There is some $\vartheta \in E$ such that $[\vartheta] \cap P^*_{>\bar{r}-\bar{\varepsilon}}$ is nonempty.

The proof is the same; we let E be the set of $\varsigma \in \mathcal{A}_n \cap [u^*]_{>\bar{r}-2\bar{\varepsilon}}$ on a computable tree determining P^*, for some n such that every $\vartheta \in \mathcal{A}_n$ on a tree determining Q is in $[u^*]_{\leqslant \bar{r}-2\bar{\varepsilon}}$.

We let $\widehat{E} = E \cap [u^*]_{>\bar{r}-\bar{\varepsilon}}$. As observed above, if $\vartheta, \varrho \in \widehat{E}$ then $\vartheta \cap \varrho \in [u^*]_{>\bar{r}-2\bar{\varepsilon}}$ and so $\vartheta \cap \varrho \in E$. As in the proof of Lemma 3.8, this shows that there is some $i \in \{0, 1\}$ such that for every $\vartheta \in \widehat{E}$ there is some $\varsigma \subseteq \vartheta$ in $C_i = \{\varsigma \in C : \Gamma(\varsigma, e) = i\}$. As above, this i is the output of our computable process, so $J(e) = i$.

Let $\vartheta \in \widehat{E}$ be such that $[\vartheta] \cap P^*_{>\bar{r}-\bar{\varepsilon}}$ is nonempty; fix some T in that set. Find $\varsigma \subseteq \vartheta$ in C_i. Let $S = T \cap \varsigma$. Note that $\varsigma \prec S$ because $\varsigma \subseteq \vartheta$. For all $\sigma \in u^*$, as

$\lambda(T \cap \sigma) > r_\sigma - \varepsilon_\sigma$ and $\lambda(\varsigma \cap \sigma) > r_\sigma - 2\varepsilon_\sigma$, we have $\lambda(S \cap \sigma) > r_\sigma - 3\varepsilon_\sigma$. In particular, S is infinite; as $S \subseteq T$, we have $S \in P^*$. Altogether, $S \in P^*_{>\bar{r}-3\bar{\varepsilon}}$. By Lemma 5.7, as $\varsigma \succcurlyeq u^*$ and $S \succcurlyeq \varsigma$, there is a condition $(\varsigma, P^* \cap [\varsigma], \bar{p})$ extending $(u^*, P^*, \bar{r} - 3\bar{\varepsilon})$; this condition is in D_Γ. ☐

6. (Weak) strong weak weak Kőnig's lemma

In this section, we study two (possibly equivalent) reverse mathematical principles strictly between WKL (weak Kőnig's lemma) and WWKL (weak weak Kőnig's lemma). Our principles correspond to the continuous covering property and its strong variant. We assume that the reader has some familiarity with reverse mathematics; see Simpson [23] for an introduction.

We say that a tree $T \subseteq 2^{<\omega}$ has *positive measure* if there is a $\varepsilon > 0$ such that

$$(\forall n)\ \frac{\#\{\sigma \in T : |\sigma| = n\}}{2^n} > \varepsilon.$$

In the introduction, we defined *strong weak weak Kőnig's lemma* (*SWWKL*):

> If $T \subseteq 2^{<\omega}$ is a tree with positive measure, then there is a nonempty subtree $S \subseteq T$ such that if $\sigma \in S$, then S has positive measure above σ.

In particular, note that SWWKL implies that S is a *perfect* subtree of T. Our second principle is the one corresponding to the continuous covering property: *weak strong weak weak Kőnig's lemma* (*WSWWKL*):

> If $T \subseteq 2^{<\omega}$ is a tree with positive measure, then there is a subtree $S \subseteq T$ of positive measure that has no dead ends.

We will prove, over RCA_0, that

$$\mathsf{WKL} \underset{\not\leftarrow}{\rightarrow} \mathsf{SWWKL} \longrightarrow \mathsf{WSWWKL} \underset{\not\leftarrow}{\rightarrow} \mathsf{WWKL}.$$

It is easy to see that SWWKL implies WSWWKL; we do not know whether the reverse implication holds. The remaining implications and non-implications are proved below in Propositions 6.1–6.5.

Proposition 6.1. $\mathsf{RCA}_0 + \mathsf{WKL} \vdash \mathsf{SWWKL}$.

Proof. This is the formalisation in RCA_0 of the fact that every PA degree has the strong continuous covering property. ☐

Proposition 6.2. $\mathsf{RCA}_0 + \mathsf{WSWWKL} \vdash \mathsf{WWKL}$.

Proof. WWKL simply says that if T is a tree with positive measure, then T has an infinite path. Given a tree T with positive measure, let $S \subseteq T$ be the positive measure subtree with no dead ends that is guaranteed by WSWWKL. Since S has positive measure, it must contain the root. Since it has no dead ends, we can construct an infinite path X by always following the leftmost branch in S. Then X is an infinite path in T. ☐

Proposition 6.3. $\mathsf{RCA}_0 + \mathsf{WWKL} \nvdash \mathsf{WSWWKL}$.

Proof. Fix an ω-model $(\omega, \mathcal{S})$ of WWKL such that whenever $X \in \mathcal{S}$, there is an incomplete Martin-Löf random sequence $Y \in \mathcal{S}$ such that $X \leqslant_{\mathrm{T}} Y$. Building such a model is straightforward; for example, we can let $\mathcal{S}$ be the ideal generated by the joins of finitely many columns of some incomplete ML-random sequence. We claim that $(\omega, \mathcal{S})$ is not a model of WSWWKL.

Assume that $(\omega, \mathcal{S})$ actually is a model of WSWWKL. By formalizing Propositions 5.1 and 4.3, it must be the case that for any order function $h\colon \omega \to \omega \smallsetminus \{0, 1\}$, there is an h-bounded DNC function in $\mathcal{S}$. However, as mentioned in the introduction, for a sufficiently slow growing h, only complete Martin-Löf random sequences can compute h-bounded DNC functions. This is a contradiction, so $(\omega, \mathcal{S})$ is not a model of WSWWKL. □

In order to construct a model of SWWKL that is not a model of WKL, we need to strengthen Theorem 5.5.

Theorem 6.4. *Assume that X does not have PA degree. There is an oracle D with the strong continuous covering property relative to X such that $X \oplus D$ does not have PA degree.*

Proof. We modify the proof of Theorem 5.5. We use the same forcing notion, except that we replace $\mathbf{P}$ by $\mathbf{P}^X$. Rather than using the recursion theorem, we argue as follows. Let Γ be an X-computable functional, and let $(u^*, P^*, \bar{r} - 3\bar{\varepsilon})$ be a condition in F. For every e, let $\psi(e)$ be the output of the partial computable process described in the proof of Theorem 5.5 when computing $\Gamma(\varsigma, e)$. The function ψ is X-partial computable. If ψ is not total, and in particular $\psi(e)\uparrow$, then the corresponding extension $(u^*, Q, \bar{r} - 2\bar{\varepsilon})$ is an extension forcing that $\Gamma(h, e)\uparrow$. Otherwise, ψ is an X-computable function. By assumption, $\psi \notin \mathrm{DNC}_2$; so there is some e such that $\psi(e) = J(e)$. The corresponding extension $(\varsigma, P^* \cap [\varsigma], \bar{p})$ then forces that $\Gamma(h, e) = J(e)$. □

Proposition 6.5. $\mathsf{RCA}_0 + \mathsf{SWWKL} \nvDash \mathsf{WKL}$.

Proof. By iterating the previous result, build an ω-model $(\omega, \mathcal{S})$ such that

- $\mathcal{S}$ is a Turing ideal;
- for every $X \in \mathcal{S}$, there is a $D \in \mathcal{S}$ with the strong continuous covering property relative to X;
- $\mathcal{S}$ contains no set of PA degree.

Therefore, $(\omega, \mathcal{S}) \models \mathsf{RCA}_0 + \mathsf{SWWKL} + \neg\mathsf{WKL}$. □

We should mention a connection to recent work of Chong, Li, Wang, and Yang [6], who studied the complexity of computing perfect subsets of sets of positive measure. They report that during a discussion with Wei Wang about their work, Ludovic Patey proved that

$$\mathsf{RCA}_0 + \text{“every closed set of positive measure has a perfect subset”} \nvDash \mathsf{WKL}.$$

Note that the principle SWWKL implies that every closed set of positive measure has a perfect subset *of positive measure*, so Proposition 6.5 improves on the result of Patey. More recently, Barmpalias and Wang have announced that they showed that RCA_0 together with the principle “every closed set of positive measure has a perfect subset of positive measure” does not imply WKL, in other words, that they have also proved Proposition 6.5.

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