RANKED STRUCTURES AND ARITHMETIC TRANSFINITE RECURSION

NOAM GREENBERG AND ANTONIO MONTALBÁN

Abstract. ATR_0 is the natural subsystem of second order arithmetic in which one can develop a decent theory of ordinals ([Sim99]). We investigate classes of structures which are in a sense the "well-founded part" of a larger, simpler class, for example, superatomic Boolean algebras (within the class of all Boolean algebras). The other classes we study are: well-founded trees, reduced Abelian p-groups, and countable, compact topological spaces. Using computable reductions between these classes, we show that Arithmetic Transfinite Recursion is the natural system for working with them: natural statements (such as comparability of structures in the class) are equivalent to ATR_0. The reductions themselves are also objects of interest.

1. Introduction

Classification of mathematical objects is often achieved by finding *invariants* for a class of objects - a method of representing the equivalence classes of some notion of sameness (such as isomorphism, elementary equivalence, bi-embeddability) by simple objects (such as natural numbers or ordinals). A related logical issue is the question of complexity: if the invariants exist, how complicated must they be; when does complexity of the class make the existence of invariants impossible; and how much information is implied by the statement that invariants of certain type exist. To mention a far from exhaustive list of examples: in descriptive set theory, Hjorth and Kechris ([HK95]) investigated the complexity of the existence of Ulm-type classification (and of the invariants themselves) in terms of the Borel and projective hierarchy; see also Camerlo and Gao ([CG01]) and Gao ([Gao04]). In computability theory, complexity of index-sets of isomorphism relations on structures have been studied, among others, by Goncharov and Knight ([GK02]) and Calvert ([Cal04]); index-sets for elementary equivalence are considered as well (Selivanov [Sel91]); and in reverse mathematics, the proof-theoretic strength of the statement of existence of invariants was studied by Shore ([Sho]).

For some classes, closely connected to invariants is the notion of rank. For example, the Cantor-Bendixon rank of a countable compact metric space is obtained by an iterated process of weeding out isolated points. This rank, then, together with the number of points left at the last step, constitutes an invariant for the homeomorphism relation. Similar processes of iterating some derivative can be used to classify well-founded trees, superatomic Boolean algebras, and reduced Abelian

We would like to thank our advisor Richard A. Shore for introducing us to questions that are discussed in this paper, and for many useful conversations. Both authors were partially supported by NSF Grant DMS-0100035. This paper will be part of the second author's doctoral thesis ([Mon05a]).

p-groups. In this paper we investigate the proof-theoretic strength of various statements directly relating to the existence of invariants and ranks on these classes. We do it from the viewpoint of Reverse Mathematics; we refer the reader to [Sim99] for more information about the program of Reverse Mathematics. We assume that the reader is familiar with at least the introductory chapter of [Sim99].

It turns out that in some sense, the structures under consideration effectively code the ordinals which are their ranks. Thus, the study of these structures is closely related to two issues: the translation processes between these classes, which reduce statements about one class to another (and in particular, to ordinals); and the strength of related questions for the class of ordinals. The corresponding subsystem of second-order arithmetic is ATR_0, the system which allows us to iterate arithmetic comprehension along ordinals. To quote Simpson ([Sim99, Page 176]):

> ... ATR_0 is the weakest set of axioms which permits the development of a decent theory of countable ordinals.

Our general aim is to demonstrate that a similar statement can be made for well-founded trees, superatomic Boolean algebras, etc.; general statements about these classes will be shown to be equivalent to ATR_0, thereby implying the necessary use of ordinal ranks in the investigation of these classes. Our work continues investigations of ordinals (Friedman and Hirst [FH90], see [Hir] for a survey), of reduced Abelian p-groups (see Simpson [Sim99] and Friedman, Simpson and Smith [FSS83]), of countable compact metric spaces (see Friedman [Fria] and Frideman and Hirst [FH91]), and of well-founded directed graphs (Hirst [Hir00]).

As we mentioned, key tools for establishing our results are reductions between various classes of objects. These reductions are an interesting object of study in their own right. Indeed we have two points of view: classical - we investigate when there are continuous (or even computable) reductions of one class to another; and proof-theoretic - we ask in what system can one show that these reductions indeed preserve notions such as isomorphism and embeddability.

1.1. **Reverse mathematics.** In this paper we only use common subsystems of second-order arithmetic. The base theory we use will usually be RCA_0 - the system that consists of the semi-ring axioms, Δ^0_1 comprehension and Σ^0_1 induction. We often use the stronger system ACA_0 which adds comprehension for arithmetic formulas. We note that over RCA_0, ACA_0 is equivalent to the existence of the range of any one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$.

The focus, though, is Friedman's even stronger system ATR_0 which enables us to iterate arithmetical comprehension along any well-ordering. As mentioned above, this is the system which is both sufficient and necessary for a theory of ordinals in second-order arithmetic. For example, comparability of well-orderings is equivalent to ATR_0 over RCA_0.

For recursion-theoretic intuition, we mention that ATR_0 is equivalent to the statement that for every $X \subset \mathbb{N}$ and every ordinal α, $X^{(\alpha)}$, the α^{th} iterate of the Turing jump of X, exists.

1.2. **The classes.** We discuss the various classes of objects with which we deal only briefly in this introduction, as greater detail will be given at the beginning of each section. The common feature of these classes is that they form the "well-founded part" of a larger class which is simply (arithmetically) definable (whereas the classes themselves are usually Π^1_1). [All structures are naturally coded as subsets of $\mathbb{N}$ and so the classes can be considered as sets of reals.] The fact that the larger class has both well-founded and ill-founded elements will usually imply large complexity: the isomorphism relation will be Σ^1_1-complete, and natural statements about the class will require Π^1_1-comprehension. On the other hand, when we focus our attention on the well-founded part, the hyperarithmetic hierarchy (and ATR_0) suffice.

When we discuss each class in detail, we specify a notion of isomorphism $\cong$ and a notion of embedding $\preccurlyeq$; we also define the notion of rank and describe which structures are ranked.

- The class of ordinals, that is, well-orderings of natural numbers (which we denote by $\mathcal{O}n$), is of course a sub-class of the class of linear orderings. For embedding we use weak embeddings (one-to-one, order preserving maps).
- We let $\mathcal{WFT}$ denote the class of well-founded trees, a sub-class of the class of trees of height $\leqslant \omega$. As the tree structure we take not only the partial ordering but also the predecessor relation; we thus may assume that all trees are trees of finite sequences of natural numbers (with the extension relation). The notion of embedding only requires preservation of strict order, so an embedding isn't necessarily one-to-one.
- $\mathcal{SABA}$ denotes the class of superatomic Boolean algebras (a sub-class of the class of Boolean algebras). As far as we know, this class has not been discussed in the setting of reverse mathematics, and so we give a detailed treatment of various definitions and their proof-theoretic content.
- Fixing a prime number p, we let $\mathcal{R}$-p-$\mathcal{G}$ denote the class of reduced Abelian p-groups, a sub-class of the class of all p-groups.
- On the analytic side, we let $\mathcal{CCS}$ denote the class of compact, very countable topological spaces. *Very countable* means Hausdorff, countable and second countable. It turns out that each countable, compact Hausdorff space is second countable, but in the setting of second-order arithmetic, we can only treat very countable spaces as reals (so this last statement is not expressible in this setting). In fact, we show that the compact spaces are all metrizable, and so the class coincides with countable, compact metric spaces. However, all properties we discuss are purely topological and so we pick the topological presentation. To be strict, the class of compact spaces does not consist of all "well-founded" very countable spaces; the latter class (the class of *scattered* spaces) is larger, but ill behaved, so we restrict ourselves to the compact case. As isomorphisms we take homeomorphisms, and as embeddings we take one-to-one, continuous and open maps.

1.3. **The statements.** We now discuss the various statements we analyze. Let $\mathcal{X}$ be a class of structures as above, equipped with a notion of isomorphism $\cong$, a notion of embeddability $\preccurlyeq$, and a subclass of ranked structures.

1.3.1. *Rank.* For each of the classes we study, there is a notion of derivative which is analogous to the Cantor-Bendixon operation of removing isolated points (such as removing leaves from trees or eliminating atoms in Boolean algebras by means of a quotient). Iterating the derivative yields a rank and an invariant, which has the expected properties (for example, it characterizes the isomorphism relation and is well-behaved with regards to the embeddability relation). When dealing with the class $\mathcal{X}$, we define this rank formally and thus the class of structures in $\mathcal{X}$ which are ranked. We thus define the following statement:

$\mathsf{RK}(\mathcal{X})$: Every structure in $\mathcal{X}$ is ranked.

We remark that we often show directly that $\mathsf{RK}(\mathcal{X})$ implies other statements $\varphi(\mathcal{X})$ (without appealing to ATR_0).

1.3.2. *Implications of invariants.* Suppose that an invariant for isomorphism for the class $\mathcal{X}$ exists. Now as this is a third-order statement, we follow Shore ([Sho]) and discuss a statement which is immediately implied by this existence. Suppose that a sequence $\{A_n\}_{n\in\mathbb{N}}$ of structures in $\mathcal{X}$ is given; if each A_n is uniformly assigned a simple object which characterized its isomorphism type, then we could uniformly decide which pairs (A_n, A_m) are isomorphic. We thus define:

$\exists$-$\mathsf{ISO}(\mathcal{X})$: If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, then the set $\{(n,m) : A_n \cong A_m\}$ exists.

Suppose that the invariant is even stronger; that the simpler objects assigned are quasi-ordered and that the invariant preserves the notion of embeddability. Then, as above, we could decide the embeddability relation. Thus we define:

$\exists$-$\mathsf{EMB}(\mathcal{X})$: If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, then the set $\{(n,m) : A_n \preccurlyeq A_m\}$ exists.

1.3.3. *Natural statements.* We define simple statements which are elementary in the analysis of the class $\mathcal{X}$. Relating to Simpson's words, we consider these statements (when true) necessary for the study of $\mathcal{X}$.

$\mathsf{COMP}(\mathcal{X})$: For every A and B in $\mathcal{X}$, either $A \preccurlyeq B$ or $B \preccurlyeq A$.

$\mathsf{EQU{=}ISO}(\mathcal{X})$: For every A and B in $\mathcal{X}$, if $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \cong B$.

1.3.4. *The structure of the embeddability relation.* It turns out that for the classes that we study, the embeddability relation is well-founded; moreover, it forms a *well-quasi ordering*: whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{X}$, there are some $n < m$ such that $A_n \preccurlyeq A_m$. This fact can be added to our list.

[Another familiar definition for the notion of well-quasi-orderings is a quasi-ordering which has no infinite descending sequences and no infinite antichains. However, this equivalence uses Ramsey's theorem for pairs, and so cannot be carried out in our base theory RCA_0. In fact, the equivalence uses the existence of the embeddability relation (to which Ramsey's theorem is applied) and so by our results the standard proof uses ATR_0. (See [CMS04] for a comparison of the different definitions of well-quasi-orderings from the viewpoint of reverse mathematics.)]

$\mathsf{WQO}(\mathcal{X})$: The class $\mathcal{X}$, quasi-ordered by $\preccurlyeq$, forms a well-quasi ordering.

1.4. **Reductions.** As we mentioned, reductions between classes of structures provide means of proving equivalences to ATR_0 be means of reducing statements from class to class. However, these reductions are interesting in their own right. It turns out many of the classes in question are as equivalent as they can be.

We consider two kinds of reductions. One should perhaps be called "effective Wadge" reducibility. The classes we consider are complicated in the sense that membership is often Π^1_1-complete. Recall that each class $\mathcal{X}$ we consider is the well-founded part of a simpler class $\mathcal{Y}$ which is arithmetic. "Effective Wadge" reducibility is the analogue of many-one reducibility in the context of sets of reals.

Definition 1.1. For a pair of classes $\mathcal{X}_1, \mathcal{X}_2$ which are subclasses of simpler classes $\mathcal{Y}_1$ and $\mathcal{Y}_2$, we say that $\mathcal{X}_1$ *is EW-reducible to* $\mathcal{X}_2$ *within* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ (and write $\mathcal{X}_1 \leqslant_{EW} \mathcal{X}_2$) if there is some computable functional $\Phi\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $\Phi^{-1}\mathcal{X}_2 = \mathcal{X}_1$.

The idea is that the question of membership in $\mathcal{X}_1$ is effectively reduced to an oracle which gives us membership for $\mathcal{X}_2$.

Another notion of reducibility is closer to the notion of Borel reducibility for Borel equivalence relations, which is extensively investigated by descriptive set-theorists (see, for instance, [HK01]). Here we consider not the elements of classes $\mathcal{X}_1$ and $\mathcal{X}_2$ but rather the collection of isomorphism types of these classes, and we look for an embedding of one class into the other which is induced by a computable transformation. The structures we work with may have domain which is a proper subset of ω. If we would like to factor out the influence of the complexity of the domain, we arrive at the following definition made by Calvert, Cummins, Knight and S. Miller ([CCKM]), which is in fact stronger than a mere embedding induced by a computable function:

Definition 1.2. A *computable transformation* of a class of structures $\mathcal{X}_1$ to another class $\mathcal{X}_2$ is a function $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ for which there is some recursively enumerable collection Φ such that for all $A \in \mathcal{X}_1$, for every finite collection of statements b in the language of $\mathcal{X}_2$, $b \subset D(f(A))$ (the atomic diagram of $f(A)$) iff there is some finite collection $a \subset D(A)$ such that $(a, b) \in \Phi$. A computable transformation f is an *embedding* if f preserves $\cong$ and $\ncong$. We write $\mathcal{X}_1 \leqslant_c \mathcal{X}_2$ if there is a computable embedding of $\mathcal{X}_1$ into $\mathcal{X}_2$.

Another way to think of computable transformations is as functionals which from any enumeration of $D(A)$ produce, uniformly, an enumeration of $D(f(A))$.

Computable embeddings (unlike Turing embeddings) preserve the substructure relation, hence preserve embeddability.

In some cases, we cannot have computable reductions: for example, from well-founded trees to ordinals - simply because $\mathsf{EQU}{=}\mathsf{ISO}(\mathcal{WFT})$ fails. If instead of isomorphism classes we consider equimorphism (bi-embeddability) classes, we get a slightly different notion of reduction. This reduction is not just a one-to-one map of $\mathcal{X}_1$-equimorphism types into $\mathcal{X}_2$-equimorphism types; it preserves the partial ordering on these equivalence classes induced by embeddability.

Definition 1.3. For classes of structures $\mathcal{X}_1$ and $\mathcal{X}_2$, $\mathcal{X}_1$ is *equicomputably reducible* to $\mathcal{X}_2$ (we write $\mathcal{X}_1 \leqslant_{ec} \mathcal{X}_2$) if there is a computable transformation $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ which preserves both $\preccurlyeq$ and $\not\preccurlyeq$.

We introduce notation which indicates that two reductions are induced by the same function. For example, $\mathcal{X}_1 \leqslant_{EW,c} \mathcal{X}_2$ if there is some computable transformation f which is both an EW- and a c-reduction of $\mathcal{X}_1$ to $\mathcal{X}_2$.

1.5. **Results.** We first consider the proof-theoretic strength of the various statements we discussed earlier.

Theorem 1.4.

	RK	∃-ISO	∃-EMB	COMP	EQU=ISO	WQO
$\mathcal{O}n$	N/A	✓	✓	✓	✓	✓
$\mathcal{WFT}$	✓	✓	✓	✓	F	✓
$\mathcal{SABA}$	✓	✓	✓	✓	✓	✓
$\mathcal{R}$-p-$\mathcal{G}$	✓	✓	✓	F	F	✓
$\mathcal{CCS}$	✓	✓	✓	✓	✓	✓

For a statement φ and class $\mathcal{X}$, a ✓ indicates that $\varphi(\mathcal{X})$ is equivalent to ATR_0 *over* RCA_0*. A square labelled by "F" indicates that $\varphi(\mathcal{X})$ is false. A square labelled by* "N/A" *indicates that $\varphi(\mathcal{X})$ is meaningless.*

Of course, not all of these results are new. Friedman and Hirst ([FH90]) showed that both $\mathsf{COMP}(\mathcal{O}n)$ and $\mathsf{EQU{=}ISO}(\mathcal{O}n)$ are equivalent to ATR_0 over RCA_0. Shore ([Sho93]) showed that $\mathsf{WQO}(\mathcal{O}n)$ is equivalent to ATR_0 over RCA_0. Hirst ([Hir00]) showed that ATR_0 implies $\mathsf{RK}(\mathcal{WFT})$ (actually he proved that every well-founded directed graph is ranked, which implies the result for trees.) Friedman, Simpson and Smith showed that ATR_0 is equivalent to $\mathsf{RK}(\mathcal{R}\text{-p-}\mathcal{G})$ over RCA_0 (see [Sim99, Theorem V.7.3]). In [Frib], Friedman shows that over ACA_0, $\mathsf{WQO}(\mathcal{R}\text{-p-}\mathcal{G})$ is equivalent to ATR_0, and leaves open the question of whether the equivalence can be proved over RCA_0. Shore and Solomon (unpublished) proved that $\exists\text{-}\mathsf{ISO}(\mathcal{R}\text{-p-}\mathcal{G})$ is equivalent to ATR_0 over RCA_0. That $\mathsf{COMP}(\mathcal{CCS})$ is equivalent to ATR_0 (over ACA_0) can be deduced from results in either [FH91] or [Fria].

Next, we turn to reducibilities. The classes we deal with are all highly equivalent. In second order arithmetic we often find that ATR_0 shows the existence of reductions between the classes; in fact, usually what we really use is the fact that structures are ranked. As this comes for free when the original class is the class of ordinals we can usually show reductions from ordinals to other classes in weaker systems. However, we do not have reversals to ATR_0 from the statements asserting the existence of computable reductions starting from other classes. In particular it is interesting to know if for a class $\mathcal{X}$, the existence of a computable reduction from $\mathcal{X}$ to the ordinals is as strong as ATR_0, as we can think of that statement as another way to say that invariants for $\mathcal{X}$ exist.

Theorem 1.5.

(1) *Let $\mathcal{X} \in \{\mathcal{WFT}, \mathcal{SABA}, \mathcal{R}\text{-p-}\mathcal{G}, \mathcal{CCS}\}$. Then in* RCA_0 *we can show that $\mathcal{O}n \leqslant_{EW,c} \mathcal{X}$. In* ACA_0 *we can show that $\mathcal{O}n \leqslant_{EW,c,ec} \mathcal{X}$.*
(2) ATR_0 *implies the following: $\mathcal{WFT} \leqslant_{EW,ec} \mathcal{O}n$, $\mathcal{SABA} \leqslant_{EW,c,ec} \mathcal{O}n$ and $\mathcal{R}\text{-p-}\mathcal{G} \leqslant_{EW} \mathcal{O}n$.*

We remark that EW-equivalence of all of our classes follows immediately from the fact that $\mathcal{O}n$ is EW-reducible to every other class. For $\mathcal{O}n$ is Π^1_1-complete, and these reductions show that each of our classes is Π^1_1-complete, hence all EW-equivalent. The extra information here is that these reductions can be made by computable transformations (rather than merely Turing reductions), and furthermore these transformations often preserve isomorphism, non-isomorphism, etc.

Note that we don't have reductions from $\mathcal{CCS}$ to other classes. Turing reductions can be found, but computable transformations have not been found yet.

We also note that $\mathcal{R}$-p-$\mathcal{G} \not\leqslant_{ec} \mathcal{O}n$. This is because $\preccurlyeq$ is not a total relation on $\mathcal{R}$-p-$\mathcal{G}$.

Proofs of the various parts of the theorems appear in the relevant sections.

1.6. **More Results.** The last section of this paper is not about Reverse Mathematics as are the previous ones. Rather, it is about a property shared by all the classes of structures we study.

Clifford Spector proved the following well known classical theorem in Computable Mathematics.

Theorem 1.6. [Spe55] *Every hyperarithmetic well ordering is isomorphic to a recursive one.*

This result was later extended in [Mon05b] as follows.

Theorem 1.7. *Every hyperarithmetic linear ordering is equimorphic to a recursive one.*

Note that Theorem 1.7 extends Spector's theorem because if a linear ordering is equimorphic to an ordinal, it is actually isomorphic to it.

As for the connection to ATR_0, this result can be extended to classes of structures studied in this paper. For example, Ash and Knight mention the following:

Theorem 1.8. [AK00] *Every hyperarithmetic superatomic Boolean algebra is isomorphic to a recursive one.*

The two following theorems are straightforward, the third less so. We give proofs for all in section 7.

Theorem 1.9.

(1) *Every hyperarithmetic tree is equimorphic with a recursive one.*
(2) *Every hyperarithmetic Boolean algebra is equimorphic with a recursive one.*

Theorem 1.10. *Every hyperarithmetic compact metric space is isomorphic to a computable one.*

Theorem 1.11. *Every hyperarithmetic Abelian p-group is equimorphic with a recursive one.*

We refer the reader to [AK00, Chapter 5] or to [Sac90] for background on hyperarithmetic theory.

2. Ordinals

A survey of the theory of ordinals in reverse mathematics can be found in [Hir]. We follow his notation and definitions. As our notion of embedding we take $\leqslant_w$, an order-preserving injection.

We first show below (proposition 2.1) that the statement $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0; we then mention some facts about the Kleene-Brouwer ordering of a tree - these will be useful also in later sections). Using this results we show (2.6) that $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0. Finally (2.7) we reduce the base to RCA_0 for both statements.

Equivalences of other statements about ordinals to ATR_0 are not new; references were made in the introduction.

2.1. Equivalence over ACA_0.

Proposition 2.1 (ACA_0). *$\exists$-$\mathsf{ISO}(\mathcal{O}n)$ is equivalent to* ATR_0.

Proof. First assume ATR_0 and let $\{\alpha_n : n \in \mathbb{N}\}$ be a sequence of ordinals. For all $i < j$, there is a unique comparison map between α_i and α_j. This shows that $\{(i,j) : \alpha_i \equiv \alpha_j\}$ is Δ^1_1-definable. By Δ^1_1-comprehension, which holds in ATR_0 ([Sim99, Lemma VIII.4.1]), this set exists. Thus $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ holds.

Suppose now that $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ holds. We will prove $\mathsf{COMP}(\mathcal{O}n)$, which implies ATR_0. Let α and β be ordinals. Let

$$F = \{(x,y) \in \alpha \times \beta : \alpha \restriction x \cong \beta \restriction y\}$$

(where $\alpha \restriction x$ is the induced ordering from α on the collection of α-predecessors of x). This set exists by $\exists$-$\mathsf{ISO}(\mathcal{O}n)$. We claim that F itself is a comparison map between α and β.

Recall that no ordinal can be isomorphic to any of its proper initial segments. It follows that F is a one-to-one function on its domain. Further, we observe that if $(x,y),(x',y') \in F$ then $x < x'$ (in α) if and only if $y < y'$ (in β). For if not, say $x < x'$ and $y > y'$, we compose the isomorphisms $\alpha \restriction x' \to \beta \restriction y'$ and $\beta \restriction y \to \alpha \restriction x$ to get an isomorphism between $\alpha \restriction x'$ and an initial segment of $\alpha \restriction x$. Also not hard to prove is that $\operatorname{dom} F$ and $\operatorname{range} F$ are initial segments of α and β (this is where we use ACA_0).

F is an isomorphism between $\operatorname{dom} F$ and $\operatorname{range} F$. If they are both proper initial segments of α and β respectively, let $\alpha \restriction x = \operatorname{dom} F$, $\beta \restriction y = \operatorname{range} F$. Then F witnesses that $(x,y) \in F$ for a contradiction. □

We could prove that $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0 over ACA_0 using a similar argument. Instead we give a different proof.

Remark 2.2. In the following, we use effective (Δ^0_1)-transfinite recursion in RCA_0. The proof that it works is the classical one (using the recursion (fixed-point) theorem). Also, we may perform the recursion along any well-founded relation (not necessarily linear).

Remark 2.3. Recall that in RCA_0, for every ordinal α one can construct the linear ordering ω^α. In fact, for any linear ordering L, one can construct ω^L in an analogous fashion. However, one needs ACA_0 to show that if α is an ordinal then so is ω^α. See Hirst [Hir94].

Recall the following: a *tree* is a downwards closed subset of $\mathbb{N}^{<\mathbb{N}}$; a tree is well-founded if it does not have an infinite path (all common definitions coincide in RCA_0). For linear orderings X and Y, $T(X,Y)$ denotes the *tree of double descent* for X which consists of the descending sequences in the partial ordering $X \times Y$. For a tree T, $\mathrm{KB}(T)$ denotes the Kleene-Brouwer ordering on T ([Sim99, Section V.1]);

$X * Y = \mathrm{KB}(T(X,Y))$. In RCA_0 we know that if either X or Y are well-founded then so is $T(X,Y)$. In ACA_0 we know that T is well-founded iff $\mathrm{KB}(T)$ is.

Lemma 2.4 (RCA_0)**.** *Let α be an ordinal and L be a linear ordering. Then there is an embedding of $\alpha * L$ into $\omega^\alpha + 1$.*

Proof. Let $T = T(\alpha, L)$; we know that T is well-founded. By effective transfinite recursion on T we construct, for every $\sigma \in T$ with last element (β, l), a recursive function $i_\sigma \colon T_\sigma \to \omega^\beta + 1$ (if $\sigma = \langle\rangle$ then $\beta = \alpha$), where $T_\sigma = \{\tau \in T : \sigma \subseteq \tau\}$. Given $i_{\sigma^\frown x}$ for every $x < \omega$ (such that $\sigma^\frown x \in T$), we construct i_σ by pasting these $i_{\sigma^\frown x}$s linearly and placing σ at the end. In detail: Let $S = \{x \in \mathbb{N} : \sigma^\frown x \in T\}$. For $x \in S$, let $\beta_x = (x)_0$. We have $i_\sigma \colon T_{\sigma^\frown x} \to \omega^{\beta_x}$. For $x \in S$ let

$$\gamma_x = \sum_{y <_{\mathbb{N}} x, y \in S} (\omega^{\beta_y} + 1)$$

which is smaller than ω^β; for $\tau \in T_{\sigma^\frown x}$ let $i_\sigma(\tau) = \gamma_x + i_{\sigma^\frown x}(\tau)$. Finally let $i_\sigma(\sigma) = \omega^\beta$.

Now by Π^0_1-transfinite induction on T, which holds in RCA_0 ([Hir]), we can show that for all $\sigma \in T$, for all $\tau_0, \tau_1 \in T_\sigma$, $i_\sigma(\tau_0) < i_\sigma(\tau_1)$ iff $\tau_0 <_{\mathrm{KB}} \tau_1$. □

Remark 2.5 (ACA_0). For the next proof, we need the fact that if α is an ordinal and L is a non-well-founded linear ordering, then α embeds into $\alpha * L$ ([Sim99, Lemma V.6.5]). The embedding is obtained by considering $T(\alpha)$, the tree of (single) descent of elements of α. We first embed α into $\mathrm{KB}(T(\alpha))$ by taking $\beta < \alpha$ to the KB-least $\sigma \in T(\alpha)$ whose last element is β. Next we embed $T(\alpha)$ into $T(\alpha, L)$ by fixing a descending sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ of L and taking $\langle \beta_1, \dots, \beta_n \rangle$ to $\langle (\beta_1, x_1), \dots, (\beta_n, x_n) \rangle$. To see that this embedding induces an embedding of $\mathrm{KB}(T(\alpha))$ into $\mathrm{KB}(T(\alpha, L))$ we note that if $a <_{\mathbb{N}} b$ then for all x, $(a, x) <_{\mathbb{N}} (b, x)$.

Proposition 2.6 (ACA_0)**.** *$\exists$-$\mathsf{EMB}(\mathcal{O}n)$ is equivalent to ATR_0.*

Proof. We show ATR_0 by showing the equivalent principle of Σ^1_1-separation ([Sim99, Theorem V.5.1]). Suppose that φ, ψ are Σ^1_1 formulas which define disjoint classes of natural numbers. From φ and ψ we can manufacture sequences $\langle X_n \rangle$ and $\langle Y_n \rangle$ of linear orderings such that for all n, X_n is a well-ordering iff $\neg\varphi(n)$ and Y_n is a well-ordering iff $\neg\psi(n)$. Let

$$\alpha_n = (\omega^{X_n} + 2) * Y_n$$

and

$$\beta_n = X_n * (\omega^{Y_n} + 2).$$

Note that at least one of X_n and Y_n are well-founded (and X_n is well-founded implies ω^{X_n} well-founded) thus both α_n and β_n are indeed ordinals.

Suppose that X_n is well-founded and that Y_n is not. We claim that β_n embeds into α_n but α_n does not embed into β_n. By Lemma 2.4, $\beta_n \preccurlyeq \omega^{X_n} + 1$ and by Remark 2.5, $\omega^{X_n} + 2 \preccurlyeq \alpha_n$. So $\beta_n \preccurlyeq \alpha_n$, but we cannot have $\alpha_n \preccurlyeq \beta_n$ or we would have $\omega^{X_n} + 2 \preccurlyeq \omega^{X_n} + 1$ which is impossible.

We can thus let $A = \{n : \beta_n \preccurlyeq \alpha_n\}$. By $\exists$-$\mathsf{EMB}(\mathcal{O}n)$, A exists. If $\psi(n)$ holds then X_n is an ordinal and Y_n is not, and so $n \in A$. If $\varphi(n)$ holds then by a similar argument we get $\beta_n \not\preccurlyeq \alpha_n$ so $n \notin A$, as required. □

2.2. Proofs of arithmetic comprehension.

Proposition 2.7 (RCA_0)**.** *Both* $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ *and* $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ *imply* ACA_0*.*

Proof. Let φ be a Σ^0_1 formula. For each n, construct an ordinal α_n by letting $\alpha_n \cong 3$ if $\neg\varphi(n)$ and $\alpha_n \cong 17$ if $\varphi(n)$.

Now

$$\{n : \varphi(n)\} = \{n : \alpha_n \cong 17\} = \{n : \alpha_n \not\preccurlyeq 5\}.$$

$\exists$-$\mathsf{ISO}(\mathcal{O}n)$ implies the second set exists; $\exists$-$\mathsf{EMB}(\mathcal{O}n)$ implies that the third set exists. □

3. Well-founded trees

We denote the class of well-founded trees by $\mathcal{WFT}$. If T, S are trees then $T \preccurlyeq S$ if there is some $f\colon T \to S$ which preserves strict inclusion. Note that f does not need to preserve non-inclusion, in fact f may be not injective.

The layout of this section is fairly straightforward. The standard rank of a well-founded tree is defined in the language of second-order arithmetic; we mention that Hirst showed that ATR_0 implies that every well-founded tree is ranked. We then show how to get the other (true) statements from $\mathsf{RK}(\mathcal{WFT})$, except for $\exists$-$\mathsf{ISO}(\mathcal{WFT})$, which follows directly from ATR_0.

We then define the reduction $L \mapsto T(L)$ which maps ordinals to well-founded trees, and use this reduction to get reversals. To get the reduction from trees to ordinals, we need the notion of a *fat tree* which we discuss in subsection 3.2.2. In the last subsection we derive ACA_0 from the statements for which the previous reversals required this comprehension.

Notation. Let T be a tree and $\sigma \in T$. Then $T[\sigma] = \{\tau \in T : \tau \not\supseteq \sigma\}$, $T - \sigma = \{\tau : \sigma^\frown\tau \in T\}$ and $\sigma^\frown T = \{\sigma^\frown\tau : \tau \in T\}$.

3.1. Ranked Trees.

Definition 3.1. Let T be a tree. A node $\tau \in T$ is an *immediate successor* of a node σ if $\sigma \subset \tau$ and $|\tau| = |\sigma| + 1$. A function $\mathrm{rk}\colon T \to \alpha$ for some ordinal α is *a rank function* for T if for every $\sigma \in T$, $\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau) + 1 : \tau \text{ is an immediate successor of } \sigma \text{ on } T\}$, and further $\alpha = \mathrm{rk}(\langle\rangle) + 1$. We say that a tree T is *ranked* if a rank function of T exists.

Lemma 3.2 (RCA_0)**.** *Let $f : T \to \alpha$ be a rank function on a tree T. Then* $\mathrm{range}\, f = \alpha$*.*

Proof. Let T be a well-founded tree and $\mathrm{rk}\colon T \to \alpha$ a rank function on it. Suppose, toward a contradiction, that there is a $\gamma < \alpha$ not in the range of rk. We prove by Π^0_1-transfinite induction that every β such that $\gamma < \beta < \alpha$ is not in the range of rk. This will contradict that $\alpha = \mathrm{rk}(\langle\rangle) + 1$.

Suppose that every β' between γ and β is not in the range of rk. Then for no node σ can we have $\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau) + 1 : \tau \text{ is an immediate successor of } \sigma \text{ on } T\}$. □

Let T be a tree and rk be a rank function on T. Let $\sigma \in T$. By Π^0_1-induction on $|\tau|$ we can show in RCA_0 that if $\sigma \subsetneq \tau$ then $\mathrm{rk}(\tau) < \mathrm{rk}(\sigma)$. It follows that

$$\mathrm{rk}(\sigma) = \sup\{\mathrm{rk}(\tau) + 1 : \tau \in T, \sigma \subsetneq \tau\}.$$

Another immediate corollary is:

Lemma 3.3 (RCA_0)**.** *Every ranked tree is well-founded.*

(As an infinite path through T would give rise to a descending sequence in T's rank.)

The following two propositions are proved in [Hir00] for well-founded directed graphs, a class which essentially contains the class of trees.

Proposition 3.4 (RCA_0)**.** *Let T be a tree and let $f_1 \colon T \to \alpha_1$ and $f_2 \colon T \to \alpha_2$ be rank functions. Then there is a bijection $g \colon \alpha_1 \to \alpha_2$ such that $f_2 = g \circ f_1$.*

Thus ranks are unique up to isomorphism; if T is ranked by a function $f \colon T \to \alpha$ then we let $\mathrm{rk}(T) = \alpha - 1 = f(\langle\rangle)$. (Note though that most set theory texts let $\mathrm{rk}(T) = \alpha$).

Proposition 3.5 (ATR_0)**.** *Every well founded tree is ranked.*

3.1.1. *Implications of rank.*

Lemma 3.6 (RCA_0)**.** *Suppose that S and T are ranked trees and that $\mathrm{rk}(S) \preccurlyeq \mathrm{rk}(T)$. Then $S \preccurlyeq T$.*

Proof. Let $g \colon \mathrm{rk}(S) \to \mathrm{rk}(T)$ be an embedding of ordinals. For each $\sigma \in S$, we define $f(\sigma) \in T$ by induction on $|\sigma|$. Along the construction we make sure at every step that for every $\sigma \in S$, $g(\mathrm{rk}_S(\sigma)) \leqslant \mathrm{rk}_T(f(\sigma))$. Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(\sigma)$ and we want to define $f(\tau)$ where τ is an immediate successor of σ on S. Since $g(\mathrm{rk}_S(\tau)) < g(\mathrm{rk}_S(\sigma)) \leqslant \mathrm{rk}_T(f(\sigma))$, there exists $\pi \supsetneq f(\sigma)$ with $\mathrm{rk}_T(\pi) \geqslant g(\mathrm{rk}_S(\tau))$. Let $f(\tau)$ be the $<_{\mathbb{N}}$-least such π. □

Lemma 3.7 (ACA_0)**.** *Let S and T be ranked trees and assume that $S \preccurlyeq T$. Then $\mathrm{rk}(S) \preccurlyeq \mathrm{rk}(T)$.*

Proof. Suppose first that there is an embedding $f \colon S \to T$; we want to construct an embedding $g \colon \mathrm{rk}(S) \to \mathrm{rk}(T)$. Given $\alpha < \mathrm{rk}(S)$, let $g(\alpha) = \min(\mathrm{rk}_T(f(\sigma)) : \sigma \in S \ \&\ \mathrm{rk}_S(\sigma) = \alpha)$. We claim that g is an embedding of $\mathrm{rk}(S)$ into $\mathrm{rk}(T)$. Consider $\alpha_0 < \alpha_1 < \mathrm{rk}(S)$. Let $\sigma \in S$ be such that $\mathrm{rk}_S(\sigma) = \alpha_1$ and $g(\alpha_1) = \mathrm{rk}_T(f(\sigma))$. Let $\tau \supsetneq \sigma$ be such that $\mathrm{rk}_S(\tau) = \alpha_0$. Such τ exists by Lemma 3.2 applied to S_σ. Since $f(\tau) \supsetneq f(\sigma)$, $g(\alpha_1) = \mathrm{rk}_T(f(\sigma)) > \mathrm{rk}_T(f(\tau)) \geqslant g(\alpha_0)$. □

Corollary 3.8 (ACA_0)**.** *If every well-founded tree is ranked then $\exists$-$\mathsf{EMB}(\mathcal{WFT})$ holds.*

Proof. Let $\langle T_n \rangle$ be a sequence of well-founded trees. Let $T = \bigoplus T_n$; this is the tree obtained by placing a common root below all of the T_ns: $T = \{\langle\rangle\} \cup \bigcup_n \langle n \rangle^\frown T_n$. The tree T is well-founded and so has a rank function rk_T.

For $n, m \in \mathbb{N}$, $\mathrm{rk}(T_n) \preccurlyeq \mathrm{rk}(T_m)$ iff $\mathrm{rk}_T(\langle n \rangle) \leqslant \mathrm{rk}_T(\langle m \rangle)$. This is because $\mathrm{rk}_T \restriction T_n$ is a rank function for T_n (of course we mean $\langle n \rangle^\frown T_n$); and because for $\beta, \gamma < \mathrm{rk}(T)$, $\beta \preccurlyeq \gamma$ iff $\beta \leqslant \gamma$. It follows that $T_n \preccurlyeq T_m$ iff $\mathrm{rk}_T(\langle m \rangle) \leqslant \mathrm{rk}_T(\langle n \rangle)$ so the set $\{(n, m) : T_n \preccurlyeq T_m\}$ exists. □

Corollary 3.9 (RCA_0)**.** *If every well-founded tree is ranked then $\mathsf{COMP}(\mathcal{WFT})$ holds.*

Proof. Let T, S be well-founded trees; let rk^* be a rank function on $T \oplus S$ (as before this is $\{\langle\rangle\} \cup 0^\frown T \cup 1^\frown S$). Let $\alpha = \mathrm{rk}^*(\langle 0 \rangle)$ and $\beta = \mathrm{rk}^*(\langle 1 \rangle)$. Now, since $\alpha, \beta < \mathrm{rk}(T \oplus S)$, either $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$; suppose the former. Then $\mathrm{rk}(T) \preccurlyeq \mathrm{rk}(S)$. It follows that $T \preccurlyeq S$. □

Corollary 3.10 (RCA_0)**.** *If every well-founded tree is ranked then* $\mathsf{WQO}(\mathcal{WFT})$ *holds.*

Proof. Let $\langle T_n \rangle$ be a sequence of well-founded trees. Let $T = \bigoplus T_n$ and let rk_T be a rank function on T. Now $\langle \mathrm{rk}_T(\langle n \rangle) \rangle_{n \in \mathbb{N}}$ cannot be strictly decreasing. It follows that for some $n \leqslant m$ we have $\mathrm{rk}_T \langle n \rangle \leqslant \mathrm{rk}_T \langle m \rangle$ so $\mathrm{rk}(T_n) \preccurlyeq \mathrm{rk}(T_m)$ so $T_n \preccurlyeq T_m$. □

We have no direct argument to get $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ from $\mathsf{RK}(\mathcal{WFT})$. Rather, we give an argument from ATR_0.

Proposition 3.11 (ATR_0)**.** $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ *holds.*

Proof. We prove that given two recursive trees T and S, both of rank α, we can decide whether $T \cong S$ recursively uniformly in $0^{(3\alpha+3)}$, which exists by ATR_0. We do it by effective transfinite induction. For each i let $T_i = T - \langle i \rangle, S_i = S - \langle i \rangle$. Observe that $T \cong S$ if and only if for every i, the number of trees T_j such that $T_j \cong T_i$ is equal to the number of trees S_j such that $S_j \cong T_i$ (this number is possibly infinite). We can check whether $T_i \cong T_j$ and whether $T_i \cong S_j$ recursively uniformly in $0^{(3\alpha)}$, so we can check whether the sentence above holds recursively in $0^{(3\alpha+3)}$. □

3.1.2. *A reversal.* We will later get all reversals by translating ordinals into trees. However, we also have one direct reversal akin to the proof for ordinals (proposition 2.6); it is simpler. For any tree T, temporarily let $1 + T = \{\langle\rangle\} \cup \langle 0 \rangle^\frown T$. As for ordinals, if T is well-founded then we cannot have $1 + T \preccurlyeq T$; for iterating the embedding on $\langle\rangle$ would yield a path in T.

Proposition 3.12 (ACA_0)**.** $\exists$-$\mathsf{EMB}(\mathcal{WFT})$ *implies* ATR_0.

Proof. We show Σ^1_1-separation. Suppose that φ, ψ are Σ^1_1 formulas which define disjoint classes. From φ and ψ we can manufacture sequences $\langle T_n \rangle$ and $\langle S_n \rangle$ of trees such that for all n, T_n is a well-founded iff $\neg\varphi(n)$ and S_n is well-founded iff $\neg\psi(n)$.

Consider $A_n = T_n \times (1 + S_n)$ and $B_n = (1 + T_n) \times S_n$. Both A_n and B_n are well-founded for all n. Suppose that T_n is well founded and that S_n is not. We always have $T_n \preccurlyeq 1 + T_n$; since S_n is not well-founded we have $1 + S_n \preccurlyeq S_n$ (map everything onto an infinite path). Thus $A_n \preccurlyeq B_n$.

On the other hand, again since S_n is not well-founded, there is an embedding of $1 + T_n$ into B_n (again use an infinite path for the second coordinate). By omitting the second coordinate, we have $A_n \preccurlyeq T_n$. It follows that we cannot have $B_n \preccurlyeq A_n$, or we would have $1 + T_n \preccurlyeq T_n$. We can thus again let the separator be $\{n : B_n \not\preccurlyeq A_n\}$. □

3.2. **Reductions.**

3.2.1. *From Ordinals to Trees.*

Definition 3.13 (RCA_0)**.** Given a linear ordering L, let $T(L)$ be the tree of L-decreasing sequences of elements of L.

It is easy to show in RCA_0 that for a linear ordering L, $T(L)$ is well-founded iff L is, so we get an EW-reduction. In RCA_0 we can show that $L \mapsto T(L)$ is a computable transformation.

Lemma 3.14 (RCA_0)**.** *For every α, $T(\alpha)$ is ranked and has rank α.*

Proof. For every nonzero $\sigma \in T(\alpha)$ let $\mathrm{rk}(\sigma)$ be the last element of σ, and let $\mathrm{rk}(\langle\rangle) = \alpha$. rk is indeed a rank function because for all $\sigma \in T$ of rank β, the set of ranks of immediate successors of σ is exactly all $\gamma < \beta$. □

The next corollary follows from 3.4; the one after it follows from 3.6 and 3.7.

Corollary 3.15 (RCA_0)**.** *For all ordinals α and β, $\alpha \cong \beta$ iff $T(\alpha) \cong T(\beta)$.*

Corollary 3.16 (ACA_0)**.** *Let α and β be ordinals. Then $\alpha \preccurlyeq \beta$ iff $T(\alpha) \preccurlyeq T(\beta)$.*

Proposition 3.17 (RCA_0)**.** $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ *implies* ATR_0.

Proof. Corollary 3.15 shows that $\exists$-$\mathsf{ISO}(\mathcal{WFT})$ implies $\exists$-$\mathsf{ISO}(\mathcal{O}n)$. □

Proposition 3.18 (RCA_0)**.** $\mathsf{RK}(\mathcal{WFT})$ *implies* ATR_0.

Proof. We show that $\exists$-$\mathsf{ISO}(\mathcal{O}n)$ holds. Let $\langle\alpha_n\rangle$ be a sequence of ordinals; consider $T = \bigoplus T(\alpha_n)$. Let rk be a rank function on T. Then $\alpha_n \cong \alpha_m$ iff $T(\alpha_n) \cong T(\alpha_m)$ iff $\mathrm{rk}(\langle n\rangle) = \mathrm{rk}(\langle m\rangle)$. □

Proposition 3.19 (ACA_0)**.** $\exists$-$\mathsf{EMB}(\mathcal{WFT})$, $\mathsf{COMP}(\mathcal{WFT})$ *and* $\mathsf{WQO}(\mathcal{WFT})$ *imply* ATR_0.

Proof. It follows from the previous lemma that each of these three statements imply the corresponding ones for ordinals, and hence ATR_0. □

3.2.2. *From trees to ordinals.* We describe fat trees.

Definition 3.20. Given a tree T, let T^∞ be the tree consisting of sequences of the form $\langle(\sigma_0, n_0), \dots, (\sigma_k, n_k)\rangle$ where $\langle\rangle \neq \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k \in T$ and $n_i \in \mathbb{N}$.

Of course, in RCA_0, T is well-founded iff T^∞ is.

Let T be a tree; for $p \in T^\infty$ ending with the pair (τ, n) we let $i(p) = \tau$. We also let $i(\langle\rangle) = \langle\rangle$.

Lemma 3.21 (RCA_0)**.** *Let T be a tree. Then $(T^\infty)^\infty \cong T^\infty$.*

Proof. We define a map f from T^∞ to $(T^\infty)^\infty$ by induction. At every step we make sure that $i(p) = i(i(f(p)))$. Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(q) = \bar{\sigma} \in (T^\infty)^\infty$ and we want to define $f(q^\frown\langle\sigma, n\rangle)$ for some $q^\frown\langle\sigma, n\rangle \in T^\infty$. Let $B_{q,\sigma} = \{\langle\sigma, m\rangle \in T \times \mathbb{N} : q^\frown\langle\sigma, m\rangle \in T^\infty\}$. Let $A_{\bar{\sigma},\sigma} = \{\langle p, m\rangle \in T^\infty \times \mathbb{N} : i(p) = \sigma \;\&\; \bar{\sigma}^\frown\langle p, m\rangle \in (T^\infty)^\infty\}$. Both $\mathcal{B}_{q,\sigma}$ and $A_{\bar{\sigma},\sigma}$ are infinite because $\sigma \supset i(p) = i(i(\bar{\sigma}))$, so we can find a bijection $g_{q,\sigma}$ between them. Let $f(q^\frown\langle\sigma, n\rangle) = \bar{\sigma}^\frown g_{q,\sigma}(\langle\sigma, n\rangle)$. □

Definition 3.22. A tree T is called *fat* if $T^\infty \cong T$.

Of course, a tree is fat iff it is isomorphic to T^∞ for some tree T. Note that for all T and $p \in T^\infty$ we have $(T \restriction i(p))^\infty \cong T^\infty \restriction p$. It follows that if T is fat and $\sigma \in T$ then $T \restriction \sigma$ is fat.

Also, if T is fat, then for all $\sigma \in T$ and for all successors τ of σ in T, there are infinitely many τ' which are immediate successors of σ and such that $T \restriction \tau \cong T \restriction \tau'$. (Work in T^∞; consider all τ' such that $i(\tau') = i(\tau)$.)

Lemma 3.23 (RCA_0)**.** *Let T be a tree. T is ranked iff T^∞ is ranked and in that case,* $\mathrm{rk}(T) = \mathrm{rk}(T^\infty)$.

Proof. Suppose that T is ranked. For all $p \in T^\infty$, let $\mathrm{rk}^\infty(p) = \mathrm{rk}_T(i(p))$. It is immediate that rk^∞ is a rank function on T^∞ and that $\mathrm{rk}(T) = \mathrm{rk}(T^\infty)$.

Suppose that T^∞ is ranked by a rank function rk_{T^∞}. We note that for all $p, q \in T^\infty$, if $i(p) = i(q)$ then $\mathrm{rk}_{T^\infty}(p) = \mathrm{rk}_{T^\infty}(q)$. This is because $T^\infty \upharpoonright p \cong T^\infty \upharpoonright q$ as they are both isomorphic to $(T \upharpoonright i(p))^\infty$. It follows that we can define rk_T on T by letting $\mathrm{rk}_T(\sigma) = \mathrm{rk}_{T^\infty}(p)$ where p is any such that $i(p) = \sigma$. It is clear that rk_T is order-inversing. If $i(p) = \sigma$ then the collection of $i(q)$s of immediate successors q of p is exactly the collection of all extensions of σ in T. It follows that rk_T is indeed a rank function on T. □

Together with lemmas 3.6 and 3.7, we get:

Corollary 3.24 (ACA_0). *Let S and T be ranked trees. Then $S \preccurlyeq T$ iff $S^\infty \preccurlyeq T^\infty$.*

In fact, EQU=ISO holds for fat trees:

Lemma 3.25 (RCA_0). *Let T be a ranked fat tree. Then for every $\sigma \in T$ and every $\gamma < \mathrm{rk}_T(\sigma)$ there are infinitely many immediate successors τ of σ such that $\mathrm{rk}(\tau) = \gamma$.*

(So a ranked fat tree is saturated within its rank.)

Proof. Work with T^∞. Let $p \in T^\infty$ have rank β; we know that $\mathrm{rk}_T(i(p)) = \mathrm{rk}_{T^\infty}(p)$. Let $\gamma < \beta$. As the rank function on $T \upharpoonright i(p)$ is onto $\beta + 1$, there is some extension τ of σ such that $\mathrm{rk}_T(\tau) = \gamma$. Then for every $n \in \mathbb{N}$, the immediate extension q of p which is determined by adding (τ, n) as last pair has rank γ. □

Corollary 3.26 (RCA_0). *Suppose that T and S are ranked fat trees. If $\mathrm{rk}(T) \cong \mathrm{rk}(S)$ then $T \cong S$.*

Proof. We define $f : T \to S$ by induction on the levels of T. Suppose that $f(\sigma)$ is defined. For each $\gamma < \mathrm{rk}(\sigma)$, consider the set $A_{\sigma,\gamma}$ which consists of those immediate extensions of σ of rank γ; similarly $B_{\sigma,\gamma}$ consists of those immediate extension of $f(\sigma)$ of rank γ. For all $\gamma < \mathrm{rk}(\sigma)$, $A_{\sigma,\gamma}$ and $B_{\sigma,\gamma}$ are infinite; we can thus build bijections between them and thus extend f. □

Next we go from fat trees to ordinals. We need the following because $T \to \mathrm{rk}(T)$ is far from computable.

Lemma 3.27 (RCA_0). *Let T be a ranked fat tree. Then $\mathrm{KB}(T)$ is isomorphic to $\omega^{\mathrm{rk}(T)} + 1$.*

Proof. We can mimic the construction in the proof of Lemma 2.4: by effective transfinite recursion we construct maps from $T \upharpoonright \sigma$ to $\omega^{\mathrm{rk}(\sigma)} + 1$ which preserve $<_{\mathrm{KB}}$. We then use Π^0_1-transfinite induction on T to show that each such map is onto its range. In fact, we can directly compute the final embedding: For any $\sigma \in T$, let P_σ be the collection of those $\tau \in T$ which lie lexicographically to the left of σ but such that $\tau \upharpoonright |\tau| - 1 \subset \sigma$. This is in fact finite. Order P_σ by $<_{\mathrm{KB}}$ (which is the same as the lexicographic ordering on P_σ) as $\langle \tau_0, \dots, \tau_k \rangle$. We let $f(\sigma) = \omega^{\mathrm{rk}(\tau_0)} + 1 + \omega^{\mathrm{rk}(\tau_1)} + 1 + \cdots + \omega^{\mathrm{rk}(\tau_k)} + 1 + \omega^{\mathrm{rk}(\sigma)}$. □

3.3. **Proofs of arithmetic comprehension.**

Proposition 3.28 (RCA_0)**.** $\exists$-$\mathsf{EMB}(\mathcal{WFT})$ *implies* ACA_0.

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. We construct a sequence of trees $\langle T_n \rangle_{n\in\mathbb{N}}$. We have $\langle\rangle \in T_n$ for all n; further, we put $\langle x\rangle$ in T_n if $f(x) = n$. Let $T = \{\langle\rangle\}$. Let $A = \{n \in \mathbb{N} : T_n \preccurlyeq T\}$. We observe that $\mathbb{N} \smallsetminus A = \operatorname{range} f$. ☐

Proposition 3.29 (RCA_0)**.** $\mathsf{COMP}(\mathcal{WFT})$ *implies* ACA_0.

Proof. This is similar to ideas of Friedman's [Frib] for p-groups. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. We construct two trees by defining their leaves. Let the leaves of T be $\{\langle 0\rangle^\frown\langle n\rangle^n : n \in \mathbb{N}\}$; let the leaves of S be $\{\langle nn\rangle^\frown\langle m\rangle^m : n = f(m)\} \cup \{\langle nn\rangle : n \notin \operatorname{range} f\}$.

T does not embed in S: say $g\colon T \to S$ is an embedding. Let $\sigma = g(\langle 0\rangle)$. There is some n such that $\langle n\rangle \subset \sigma$. If $n \notin \operatorname{range} f$ then $g(\langle 022\rangle)$ cannot be defined. If $n = f(m)$ then $g(\langle 0\rangle^\frown\langle m+2\rangle^{m+2})$ cannot be defined.

Thus let $g\colon S \to T$ be an embedding. If $f(m) = n$ then $g(\langle nn\rangle)$ must extend $\langle 0k\rangle$ for some k, and in fact we must have $k > m$ because in this case $g(\langle nn\rangle^\frown\langle m\rangle^m)$ must be defined. Thus $n \in \operatorname{range} f$ iff for the unique k such that $\langle 0k\rangle \subset g(\langle nn\rangle)$, $n \in \operatorname{range} f \upharpoonright k$. ☐

3.3.1. *WQO.* We will first prove that $\mathsf{WQO}(\mathcal{WFT})$ implies ACA_0 using RCA_2, and then show that over RCA_0, $\mathsf{WQO}(\mathcal{WFT})$ implies RCA_2 (recall that RCA_2 is RCA_0 together with Σ^0_2-induction.) Our proofs of $\mathsf{WQO}(\mathcal{X}) \to \mathsf{ACA}_0$ in this and later sections are motivated by Shore's technique of proving $\mathsf{WQO}(\mathcal{O}n) \to \mathsf{ACA}_0$ [Sho93, ??].

Proposition 3.30 (RCA_2)**.** $\mathsf{WQO}(\mathcal{WFT})$ *implies* ACA_0.

Proof. Let $\langle k_s\rangle$ be an effective enumeration of $0'$. $s \in \mathbb{N}$ is a *true stage* of this enumeration if for all $t > s$, $k_t > k_s$. $s \in \mathbb{N}$ *appears to be a true stage at stage* $t > s$ if for all $r \in (s,t]$ we have $k_r > k_s$.

Let T consist of all sequences $\sigma^\frown\langle t\rangle$ where $t \in \mathbb{N}$ and σ is an increasing enumeration of the stages which appear to be true at t. T is indeed a tree because if $t_1 < t_2$ and t_1 appears to be true at t_2, then for all $s < t_1$, s appears to be true at t_1 iff it appears to be true at t_2. We let $\max\sigma$ denote the last element of a sequence $\sigma \in T$.

Assume for contradiction that $0'$ does not exist. Then T is well-founded: If f is an infinite path in T then every $s \in \operatorname{range} f$ is a true stage, since it appears to be true at unboundedly many later stages.

For $n \in \mathbb{N}$, let $T_n = \{\tau - \tau \upharpoonright n : \tau \in T\}$ (i.e. the elements of T_n are the final segments of sequences in T, the first n elements removed.) Each T_n is a well-founded tree because T is.

For $\sigma \in T_n$, we say that σ is *true* if $\max\sigma$ is a true stage.

By assumption, there are some $n < m$ and an embedding $g\colon T_n \to T_m$. We claim that the image, under g, of a true sequence, is also true. This is because if $\tau \in T_m$ is not true, then $T_n[\tau]$ is finite. On the other hand, for any number r, there is a true string $\sigma \in T_n$ of length $> r$ (this requires Σ^0_2-induction). And of course, the true strings on T_n are linearly ordered. Thus if σ is true then $g(\sigma)$ can never be off a true string, for in that case g would be "stuck".

The second point is that if $\sigma \in T_n$ is true, then $\max g(\sigma) > \max\sigma$; this is because $|g(\sigma)| \geqslant |\sigma|$ and the fact that in T_m we "chopped off" more of the beginning of each string. This shows that given any true stage, we can manufacture a bigger true stage; iterating, we compute $0'$. □

To derive $I\Sigma_2$, we use ideas of Shore ([Sho93, Theorem 3.1]). Let $\psi(x) = \exists u \forall v \phi(x,u,v)$ be a Σ^0_2 formula, and fix $n \in \mathbb{N}$. We let $Z = \{x < n : \psi(x)\}$ and for $p \in \mathbb{N}$, we let $Z_p = \{x < n : \exists u \leqslant p \, \forall v \phi(x,u,v)\}$. Obviously if $p < q$ then $Z_p \subset Z_q \subset Z$. Each Z_p exists (by bounded Σ^0_1-comprehension, which holds in RCA_0. See [Sim99, Definition II.3.8 and Theorem II.3.9]), so if for some p we have $Z_p = Z$ then Z exists. This is enough to get induction on ψ up to n.

Lemma 3.31 (RCA_0)**.** *There is a sequence $\langle \alpha_p \rangle_{p \in \mathbb{N}}$ of ordinals such that for all p, if $Z_p \neq Z$ then for all $q > p$, $\alpha_q + 1 \preccurlyeq \alpha_p$.*

Proof. In Shore's construction, each α_i (M_i for Shore), is defined as a sum $\sum_{j<n} N_{i,j}$, where $N_{i,n-j}$ is isomorphic to ω^{3j+1} if $\neg\psi(n-j)$, and $\omega^{3j} \cdot u_j - i + \omega^{3j-1}$ if $\psi(n-j)$ and u_j is the least witness, i.e, the least number u such that $\forall v \phi(n-j,u,v)$. (If there is a witness, a least one exists by Σ^0_1-induction.) Of course, if $u_j < i$ we let $u_j - i = 0$. Now, if $Z_p \subsetneq Z$, there exists $0 < j \leqslant n$ such that $\psi(n-j)$ but $\neg \exists u \leqslant p \forall v \phi(n-j,u,v)$, or in other words, $u_j > p$. Then, $\sum_{k=n-j,\dots,n-1} N_{q,k} + 1 \preccurlyeq \omega^{3j} \cdot u_j - q + \omega^{3j-1} \cdot 2 \preccurlyeq \omega^{3j} \cdot u_j - p \preccurlyeq N_{p,n-j} \preccurlyeq \sum_{k=n-j,\dots,n-1} N_{p,k}$. Then, since for all $k < n-j$, $N_{q,k} \preccurlyeq N_{p,k}$, we have that $\alpha_q + 1 \preccurlyeq \alpha_p$. □

Proposition 3.32 (RCA_0)**.** *$\mathsf{WQO}(\mathcal{WFT})$ implies Σ^0_2 induction.*

Proof. Let ψ, Z and Z_p be as above, and let $\langle \alpha_p \rangle$ be the sequence given by Lemma 3.31

By $\mathsf{WQO}(\mathcal{WFT})$, there exist $p < q$ such that $T(\alpha_p) \preccurlyeq T(\alpha_q)$. In RCA_0 we cannot deduce that $\alpha_p \preccurlyeq \alpha_q$, but we can deduce that $\alpha_q + 1 \not\preccurlyeq \alpha_p$. This is because otherwise, we would have $1 + T(\alpha_q) \preccurlyeq T(\alpha_q + 1) \preccurlyeq T(\alpha_q)$, contradicting the well-foundedness of $T(\alpha_q)$. Thus $Z_p = Z$ and we're done. □

4. Superatomic Boolean algebras

Superatomic Boolean algebras have not, as far as we can tell, been studied in the context of reverse mathematics. This is why we first discuss various possible definitions for this class and see how they relate (for some of the equivalences we seem to require ACA_0). We then follow the plan that was executed in the last section.

4.1. Definitions.

4.1.1. *Boolean Algebras.* In this subsection, unless we mention otherwise, we work in RCA_0.

Definition 4.1. A *Boolean algebra* is a set A endowed with two binary operations $\wedge$ and $\vee$, a unary operation $\neg$ and a distinguished element 0_A, which satisfies the familiar axioms of Boolean algebras. (For basic properties, see [Kop89].)

The partial ordering on A given by $x \vee y = y$ (equivalently $x \wedge y = x$) also exists; $\vee$ and $\wedge$ are indeed the least upper bound and greatest lower bound, and so a Boolean algebra is indeed a complemented distributive lattice. We use $\triangle$ to denote the symmetric difference operation: $x \triangle y = (x-y) \cup (y-x)$, where $x - y = x \cap \neg y$.

The notions of an ideal of a Boolean algebra, a homomorphism of Boolean algebras, products (and infinite sums) of Boolean algebras and subalgebras can be copied from the algebra textbooks and carried out in RCA_0. We can also formalize the notion of the quotient algebra:

Definition 4.2. Let I be an ideal of a Boolean algebra A. For $a, b \in A$, let $a =_I b$ if $a \triangle b \in I$. Let B be the collection of $a \in A$ which are the $<_{\mathbb{N}}$-least elements of their $=_I$-equivalence class. B exists as it is Δ^0_0. For $a, b \in B$, let $a \vee_B b = c$ if $a \vee_A b =_I c$, and similarly for $\wedge, \neg, 0$. Again these operations exist, and the resulting structure is a Boolean algebra; this is the *quotient algebra* A/I.

A particular example is the free Boolean algebra. Let V be a set. We let $\mathtt{Prop}(V)$ be the collection of all propositional formulas with variables in V (this can be effectively coded). We let $\Leftrightarrow$ denote logical equivalence on propositional formulas (it exists as it is computable by using truth tables). We let $\mathbb{B}(V) = \mathtt{Prop}(V)/\Leftrightarrow$; this is a Boolean algebra which we call the *free Boolean algebra over* V.

In particular, we fix some infinite set of variables V^* and let $\mathtt{Prop} = \mathtt{Prop}(V^*)$. For $\varphi(\bar{x}) \in \mathtt{Prop}$, a Boolean algebra A and $\bar{a} \in A$, $\varphi^A(\bar{a})$ is well-defined (by induction on φ). If A is a Boolean algebra and $X \subset A$ then we let

$$\langle X \rangle_A = \{\varphi^A(\bar{a}) \,:\, \varphi \in \mathtt{Prop}, \bar{a} \in X\}.$$

In ACA_0 we can show that $\langle X \rangle_A$ (which we call the *subalgebra of A generated by X*) indeed exists, and we can also show that $\langle X \rangle_A$ is the inclusion-wise smallest subalgebra of A containing X. In fact the existence of subalgebras generated by sets is equivalent to ACA_0. However, we note that if X is finite, then in RCA_0 we can show that $\langle X \rangle_A$ exists, as there are only $2^{|X|}$ many propositional formulas with variables in X.

4.1.2. *Superatomicity in* RCA_0. We turn to describe superatomic algebras. We need some definitions.

Definition 4.3. Let A be a Boolean algebra. An element $x \in A$ is an *atom* if $x > 0$ but there is no $y \in A$, $0 < y < x$. A Boolean algebra is *atomless* if it has no atoms.

For example, if V is infinite then $\mathbb{B}(V)$ is atomless.

If A is a Boolean algebra then we let $A^+ = A \setminus \{0_A\}$.

Definition 4.4. An embedding of the full binary tree into a Boolean algebra A is a map $f \colon 2^{<\mathbb{N}} \to A^+$ such that for all $\sigma, \tau \in 2^{<\mathbb{N}}$, $\sigma \subset \tau$ implies $f(\tau) \leqslant f(\sigma)$ and $\sigma \perp \tau$ implies $f(\sigma) \wedge f(\tau) = 0$.

Lemma 4.5 (RCA_0)**.** *If A is atomless then there is an embedding of the full binary tree into A.*

Proof. For $\sigma \in 2^{<\mathbb{N}}$, we define $f(\sigma)$ by induction on $|\sigma|$. Let $s(\langle\rangle) = 1_A$. Say that $f(\sigma)$ is defined. It is not an atom; so we can let $f(\sigma^\frown 0)$ be the $<_{\mathbb{N}}$-least $y \in A$ such that $0 < y < f(\sigma)$, and let $f(\sigma^\frown 1) = f(\sigma) - f(\sigma^\frown 0)$. □

Definition 4.6. Let A be a Boolean algebra. A set $X \subset A$ is *free* if for all $\bar{a} \in X$ and $\varphi(\bar{x}) \in \mathtt{Prop}$, if $\varphi^A(\bar{a}) = 0$ then φ is logically false.

For example, for any set V, V is free in $\mathbb{B}(V)$.

Lemma 4.7 (RCA_0)**.** *Let A be a Boolean algebra. Then there is an embedding of the full binary tree into A iff there is some infinite free set $X \subset A$.*

Proof. In one direction, suppose that $X \subset A$ is infinite and free; let $g\colon \mathbb{N} \to X$ be a one-to-one enumeration of X. For any $a \in A$, let $a^0 = a$ and $a^1 = \neg a$. We define $f\colon 2^{<\mathbb{N}} \to A^+$ by letting $f(\sigma) = \bigwedge_{n<|\sigma|} g(n)^{\sigma(n)}$. That f preserves extension and incompatibility is immediate; the point is that for all σ, $f(\sigma) > 0$. This follows from freeness; $f(\sigma) \neq 0$ because $\bigwedge_{n<\sigma} x_n^{\sigma(n)}$ is not logically false.

In the other direction, let $f\colon 2^{<\mathbb{N}} \to A^+$ be an embedding of the full binary tree into A. Let $g(n) = \bigvee_{\sigma\in 2^n} f(\sigma^\frown 0)$ (so $g(0) = f(0)$). We first show that g has "free range": for all distinct $n_0, \dots, n_k \in \mathbb{N}$ and for all $\varphi(x_0, \dots, x_k) \in \mathtt{Prop}$, if $\varphi^A(f(n_0), \dots, f(n_k)) = 0_A$ then φ is logically false. Let $n > 1$ and let $\varphi(x_0, \dots, x_{n-1})$ be a propositional sentence. For $\sigma \in 2^n$, let $\varphi_\sigma = \bigwedge_{k<n} x_k^{\sigma(k)}$. We have $\varphi_\sigma^A(g(0), \dots, g(n-1)) = f(\sigma)$. For some $F \subset 2^n$, φ is equivalent to $\bigvee_{\sigma\in F} \varphi_\sigma$. If φ is not logically false then $F \neq 0$; take some $\sigma \in F$. We have $\varphi^A(g(0), \dots, g(n-1)) \geqslant \varphi_\sigma^A(g(0), \dots, g(n-1)) > 0$.

Now g is one-to-one; it follows that its range contains an infinite set X; X must be free. $\square$

It is clear that if A has an atomless subalgebra then there is an embedding of the full binary tree into A. In fact these notions are equivalent.

Lemma 4.8 (RCA_0)**.** *Suppose that a Boolean algebra A contains an infinite free set. Then A has an atomless subalgebra.*

Proof. Let $X \subset A$ be an infinite free set. We construct $B \subset A$ as an increasing union of finite subalgebras $\langle B_n \rangle$ which we construct by induction. Suppose that we have $B_n = \langle x_0, \dots, x_{n-1} \rangle_A$ where $x_i \in X$. Let m_n be the $<_\mathbb{N}$-least natural number which is $<_\mathbb{N}$-greater than all $y \in B_n$. Let x_n be the $<_\mathbb{N}$-least element of X such that $\langle x_0, \dots, x_n \rangle_A \cap \{0, \dots, m_n - 1\} = B_n$; one exists by freeness: if $y, z \in X$, and y, z and the x_i's are distinct then $\langle B_n \cup \{y\} \rangle_A \cap \langle B_n \cup \{z\} \rangle_A = B_n$, so the possibilities below m_n soon exhaust themselves.

Now $B = \cup B_n$ is isomorphic to the free Boolean algebra on infinitely many elements, and so is atomless. $\square$

We thus make the following definition:

Definition 4.9. A Boolean algebra is *superatomic* if it has no atomless subalgebra.

Let $\mathcal{SABA}$ denote the class of superatomic Boolean algebras.

4.1.3. *Superatomicity in* ACA_0*.*

Lemma 4.10 (RCA_0)**.** *A superatomic Boolean algebra has no atomless quotient.*

Proof. Suppose that a Boolean algebra A has an atomless quotient $B = A/I$. Let $X \subset B$ be infinite and free. Recall that we designed our subalgebras such that as sets, $B \subset A$. So we have $X \subset A$ and X is free in A. $\square$

We work toward the converse. Recall that a set $X \subset A$ is *dense* if for all nonzero $a \in A$ there is some nonzero $b \in X$ such that $b \leqslant a$. If I is an ideal of A, B is a subalgebra of A and $I \cap B = \{0\}$ then the quotient map $\pi\colon A \to A/I$ is one-to-one on B, and we identify B with its image π"B (which exists under ACA_0).

Lemma 4.11 (ACA_0)**.** *If B is a subalgebra of a Boolean algebra A then there is an ideal I of A such that $I \cap B = \{0\}$ and B is dense in A/I.*

Proof. Let B be a subalgebra of A. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be an enumeration of the elements of A. We define a sequence of ideals I_n of A such that for all n, $I_n \cap B = \{0\}$. We let $I_0 = \{0_A\}$. At stage n, given I_n, we ask if a_n bounds some element of B^+ modulo I_n. If so, we let $I_{n+1} = I_n$; Otherwise, we let I_{n+1} be the ideal generated by I_n together with a_n.

Now the sequence $\langle I_n \rangle_{n\in\mathbb{N}}$ exists because each I_n is finitely generated and we can keep track of the finite sets of generators. Thus we can let $I = \cup_n I_n$. By the construction, $I \cap B = \{0\}$. Also, B is dense in A/I, because each element at its turn is either discovered to bound some element of B^+ modulo an ideal contained in I, or is thrown into I. ☐

Corollary 4.12 (ACA_0)**.** *A Boolean algebra is superatomic iff it has no atomless quotient.*

Proof. Suppose that A is not superatomic; it has some atomless subalgebra B. Let I be given by Lemma 4.11. Then A/I is atomless, because B is a dense, atomless subalgebra of A/I. ☐

Question 4.13. Does the statement of corollary 4.12 imply ACA_0?

Remark 4.14. Lemma 4.11 is equivalent to ACA_0 over RCA_0.

4.2. **Ranked Boolean algebras.** In this section we define rank functions on superatomic Boolean algebras; this follows Simpson [Sim99, Section V.7], where reduced Abelian p-groups and their Ulm resolutions are discussed.

Let B be a Boolean algebra. The *Cantor-Bendixon derivative* of B is the quotient B/I, where I is the ideal generated by the atoms of B. I is called the *Cantor-Bendixon ideal* of B.

Let α be an ordinal. A *partial resolution* of B along α is a sequence of ideals $\langle I_\beta \rangle_{\beta<\alpha}$ such that $I_0 = \{0_B\}$, if $\beta+1 < \alpha$ then $I_{\beta+1}$ is the (pullback to B) of the CB-ideal of B/I_β and for limit $\beta < \alpha$, $I_\beta = \cup_{\gamma<\beta} I_\gamma$. A *resolution* of B is a partial resolution of B along some ordinal $\alpha+1$ such that $I_\alpha = B$ and for all $\beta < \alpha$ $I_\beta \neq B$.

Suppose that $\langle I_\beta \rangle_{\beta\leqslant\alpha}$ is a resolution of a Boolean algebra B. We define associated rank and degree functions. For $x \in B$, $\mathrm{rk}(x)$ is the unique $\beta < \alpha$ such that $x \in I_{\beta+1} \setminus I_\beta$; and $\deg(x) = n$ if in $B/I_{\mathrm{rk}(x)}$, x is the join of n many atoms. We let $\mathrm{inv}(x) = (\mathrm{rk}(x), \deg(x))$; we let $\mathrm{rk}(B) = \mathrm{rk}(1_B)$, $\deg(B) = \deg(1_B)$ and $\mathrm{inv}(B) = \mathrm{inv}(1_B)$.

Definition 4.15. A Boolean algebra is *ranked* if it has some resolution such that the associated invariant function exists.

Lemma 4.16 (RCA_0)**.** *Let B be a superatomic Boolean algebra. Let α, α' be ordinals and suppose that $\langle I_\beta \rangle_{\beta\leqslant\alpha}$ and $\langle I'_\beta \rangle_{\beta\leqslant\alpha'}$ are two resolutions of B. Further assume that the associated invariant functions* inv *and* inv′ *exist. Then $\alpha \cong \alpha'$, and the isomorphism commutes with* rk, rk′ *(i.e. if $f\colon \alpha \to \alpha'$ is the isomorphism then $f \circ \mathrm{rk} = \mathrm{rk}'$).*

Proof. We remark that for all $\beta < \alpha, \beta' < \alpha'$, if $I_\beta \subset I'_{\beta'}$ then $I_{\beta+1} \subset I'_{\beta'+1}$. For take some atom x of B/I_β. For all $z \leqslant x$, either $z \in I_\beta$ or $z =_{I_\beta} x$. It follows that for all $z \leqslant x$, either $z \in I'_{\beta'}$ or $z =_{I'_{\beta'}} x$. It follows that $x \in I'_{\beta'}$ or x is an atom of

$B/I'_{\beta'}$. Thus every finite join of atoms of B/I_β is in $I'_{\beta'}$ or is a finite join of atoms of $B/I'_{\beta'}$.

Of course, rk and rk$'$ are symmetric here. Thus, if $I_\beta = I'_{\beta'}$ then $I_{\beta+1} = I'_{\beta'+1}$.

Using Π^0_1-transfinite induction on $\beta < \alpha$ we show that for all $x \in B$, if $\mathrm{rk}(x) = \beta$ then $I_\beta = I'_{\mathrm{rk}'(x)}$.

Suppose that the claim is verified up to β; let $x \in B$ be such that $\mathrm{rk}(x) = \beta$. Let $\beta' = \mathrm{rk}'(x)$. We need to see that $I_\beta = I'_{\beta'}$.

Let $y \in I_\beta$; let $\gamma = \mathrm{rk}(y)$. Now $\gamma < \beta$ so $x \notin I_{\gamma+1}$. Let $\gamma' = \mathrm{rk}'(y)$. By induction, $I_\gamma = I'_{\gamma'}$ and so $I_{\gamma+1} = I'_{\gamma'+1}$ and so $x \notin I'_{\gamma'+1}$; as $x \in I_{\beta'+1}$ it follows that $\gamma' < \beta'$. Thus $I_\beta \subset I'_{\beta'}$.

Next, we note that for no $\gamma' < \beta'$ can we have $I_\beta \subset I'_{\gamma'}$; for then we would have $I_{\beta+1} \subset I'_{\gamma'+1} \subset I'_{\beta'}$, contrary to $\mathrm{rk}'(x) = \beta'$.

Let $\gamma' < \beta'$. Then $I_\beta \not\subset I'_{\gamma'}$; so there is some $y \in I_\beta$ such that $\mathrm{rk}'(y) \geqslant \gamma'$. Let $\delta = \mathrm{rk}(y)$; $\delta < \beta$ so by induction, $I_{\delta+1} = I_{\mathrm{rk}'(y)+1}$. Thus $I_{\gamma'+1} \subset I_\beta$. As $I'_{\beta'} = \cup_{\gamma'<\beta'} I'_{\gamma'+1}$, we have $I'_{\beta'} \subset I_\beta$ as required.

Now we can define a function $f\colon \alpha \to \alpha'$ by letting $f(\beta) = \beta'$ if for some (all) x such that $\mathrm{rk}(x) = \beta$ we have $\mathrm{rk}'(x) = \beta'$. The function f exists as it is Δ^0_1-definable. We show that f is an isomorphism. Of course, dom $f = \alpha$ as the sequence of ideals $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ is strictly increasing. By the same argument, range $f = \alpha'$. f is order preserving: say $\gamma < \beta < \alpha$. Then $I_\beta = I'_{f(\beta)}$ and $I_\gamma = I'_{f(\gamma)}$. Also, $I_\gamma \subsetneq I_\beta$. It follows that $I'_{f(\gamma)} \subsetneq I'_{f(\beta)}$ and so $f(\gamma) < f(\beta)$. □

Corollary 4.17 (RCA_0)**.** *Suppose that A, B are ranked Boolean algebras and that $f\colon A \to B$ is an isomorphism. Let $\mathrm{rk}_A\colon A \to \alpha$ and $\mathrm{rk}_B\colon B \to \alpha'$ be the rank functions. Then $\alpha \cong \alpha'$ and the isomorphism g commutes with $\mathrm{rk}_A, \mathrm{rk}_B, f$ (i.e. $\mathrm{rk}_B \circ f = g \circ \mathrm{rk}_A$).*

Lemma 4.18 (RCA_0)**.** *A ranked Boolean algebra is superatomic.*

Proof. Let B be a ranked Boolean algebra of rank α. Suppose toward a contradiction that B is not superatomic. Then, there is an embedding f of the full binary tree into B. By recursion we construct a decreasing sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}} \subset 2^{<\omega}$ such that $\langle \mathrm{inv}(f(\sigma_n)) \rangle_{n\in\mathbb{N}}$ is a descending sequence in $\alpha\times\omega$, getting a contradiction. Let $\sigma_0 = \langle\rangle$. Given σ_n, either $\mathrm{inv}(f(\sigma_n^\frown 0)) < \mathrm{inv}(f(\sigma_n))$ or $\mathrm{inv}(f(\sigma_n^\frown 1)) < \mathrm{inv}(f(\sigma_n))$; for if $\mathrm{inv}(f(\sigma_n)) = \mathrm{inv}(f(\sigma_n^\frown 0)) = \mathrm{inv}(f(\sigma_n^\frown 1)) = (\beta, n)$, then necessarily $f(\sigma_n) = f(\sigma_n^\frown 0) = f(\sigma_n^\frown 1)$ in B/I_β, contradicting $f(\sigma_n^\frown 0) \wedge f(\sigma_n^\frown 1) = 0_B$. Let σ_{n+1} one of $\sigma_n^\frown 0$ or $\sigma_n^\frown 1$ which has smaller invariant. □

Lemma 4.19 (ATR_0)**.** *Every superatomic Boolean algebra is ranked.*

Proof. Let B be a Boolean algebra, and assume that there is no ordinal α such that a full iteration of the derivative of B along $\alpha+1$ exists.

Let $\varphi(L, \langle I_a \rangle_{a\in L})$ say that L is a linear ordering, that for all $a \in L$, I_a is a proper ideal of B, and that if $a <_L b$ then the (pullback to B of) the CB-ideal of B/I_a is contained in I_b.

For every ordinal α, by arithmetic transfinite recursion we can construct an iteration $\langle I_\beta \rangle_{\beta<\alpha}$ of the derivative of B along α. Then $\varphi(\alpha, \langle I_\beta \rangle_{\beta<\alpha})$ holds.

Since the collection of ordinals is not Σ^1_1-definable (this is provable in ACA_0; see [Sim99, V.1.9]), there is some linear ordering L which is not well-founded and such that there is a sequence $\langle I_a \rangle_{a\in L}$ such that $\varphi(L, \langle I_a \rangle_{a\in L})$ holds.

Let $a_0 >_L a_1 >_L a_2 >_L \dots$ be an infinite descending sequence in L. Let $I = \cap_{n\in\mathbb{N}} I_{a_n}$. Then B/I is atomless (showing that B is not superatomic). For if $y \in B \setminus I$ then for some n, $y \notin I_{a_n}$. If in B/I, y is an atom, then y is an atom in $B/I_{a_{n+1}}$, in which case we would have $y \in I_{a_n}$.

Thus if B is superatomic then there is a full iteration $\langle I_\beta \rangle_{\beta\leqslant\alpha}$ along some ordinal $\alpha+1$. For every $\beta < \alpha$, The collection of atoms of B/I_α exists. Given any $b \in B$, we can find the unique $\beta < \alpha$ such that $b \in I_{\beta+1} \setminus I_\beta$; and then find the finite set F of atoms of B/I_β such that $b = \vee F$. Then the invariant of b in B is $(\beta, |F|)$. □

4.2.1. *Implications of rank.* The following is a converse to corollary 4.17.

Lemma 4.20 (RCA_0). *Suppose that A, B are ranked Boolean algebras which have the same CB invariant. Then $A \cong B$.*

Proof. This is a back-and-forth construction; we define $f\colon A \to B$. Let $f(0_A) = 0_B$ and $f(1_A) = 1_B$.

Let $a_0 \in A$. We look for $b_0 \in B$ such that $\text{inv}_A(a_0) = \text{inv}_B(b_0)$ and $\text{inv}_A(\neg a_0) = \text{inv}_B(\neg b_0)$. Why does such exist? We first note that $\text{inv}_A(a_0) \leqslant_{\text{lex}} \text{inv}(A)$ and that for all pairs $(\gamma, n) \leqslant_{\text{lex}} \text{inv}(B)$ there is some $c \in B$ such that $\text{inv}_B(c) = (\gamma, n)$. Next we note that for all $(\gamma, n) <_{\text{lex}} \text{inv}(A)$ there is a unique $(\beta, m) \leqslant_{\text{lex}} \text{inv}(A)$ such that for all $a \in A$ such that $\text{inv}_A(a) = (\gamma, n)$ we have $\text{inv}_A(\neg a) = (\beta, m)$. By replacing a_0 by its complement, if necessary, we can assume that $\text{inv}_A(a_0) <_{\text{lex}} \text{inv}(A)$. Thus we can pick any $b_0 \in B$ such that $\text{inv}_B(b_0) = \text{inv}_A(a_0)$. We let $f(a_0) = b_0$ and $f(\neg a_0) = \neg b_0$.

We now repeat the process **back**ward, in the other direction, in the Boolean algebras $B(\leqslant b_0)$ and $B(\leqslant \neg b_0)$; we pick new elements, find their equivalents in $A(\leqslant a_0)$ and $A(\leqslant \neg a_0)$ as above, and extend f to be defined on the subalgebra of A generated by all the elements picked so far. We then repeat inside the four new smaller algebras we got, and so **forth**. □

Lemma 4.21 (RCA_0). *A (finite or infinite) direct sum of superatomic Boolean algebras is superatomic.*

Proof. Let A and B be Boolean algebras, and suppose that $f\colon 2^{<\mathbb{N}} \to A \times B$ is an embedding of the full binary tree into $A \times B$; write $f(\sigma) = (a_\sigma, b_\sigma)$. There are two possibilities. Suppose that there is some σ such that $a_\sigma = 0_A$. Then we can define $g\colon 2^{<\mathbb{N}} \to B$ by letting $g(\tau) = b_{\sigma^\frown\tau}$. For all τ, $g(\tau) > 0$, as $f(\sigma^\frown\tau) > 0$. If $\tau_1 \perp \tau_2$ then $\sigma^\frown\tau_1 \perp \sigma^\frown\tau_2$ so $f(\sigma^\frown\tau_0) \perp f(\sigma^\frown\tau_1)$ so $g(\tau_0) \perp g(\tau_1)$. Thus B is not superatomic. Similarly, if for some σ we have $b_\sigma = 0$ then A is not superatomic.

Otherwise, we can let $g(\sigma) = a_\sigma$. By assumption, $g(\sigma) > 0$ for all σ; and is $\tau_0 \perp \tau_1$ then $f(\tau_0) \perp f(\tau_1)$ which implies $g(\tau_0) \perp g(\tau_1)$. Then A (and B) are not superatomic.

Let $\langle B_n \rangle_{n\in\mathbb{N}}$ be a sequence of Boolean algebras, and let $B = \bigoplus_{n\in\mathbb{N}} B_n$. Recall that the elements of B are those sequences of $\prod_n B_n$ which have either almost all elements 1 or almost all elements 0. Suppose that $f\colon 2^{<\mathbb{N}} \to B$ is an embedding of the full binary tree into B. Then for either $f(0)$ or $f(1)$, almost all elements are 0, which essentially means that either $f(0)$ or $f(1)$ (and all extensions) lie in some finite product. Thus some finite product of the B_ns is not superatomic, so some B_n is not superatomic. $\square$

Corollary 4.22 (RCA_0)**.** *Suppose that every superatomic Boolean algebra is ranked. Then* $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ *holds.*

Proof. Let $\langle B_n \rangle$ be a sequence of superatomic Boolean algebras. Let $B = \bigoplus_n B_n$. Then B is superatomic; let inv_B be the invariant function for B. We claim that $\mathrm{inv}_B \upharpoonright B_n$ is the invariant function for B_n (where we identify $b \in B_n$ with the sequence containing b and otherwise only 0s): by Π^0_1-transfinite induction on $\beta < \mathrm{rk}(B)$ we can see that $I_\beta(B_n) = I_\beta(B) \cap B_n$. For the successor step note that B_n is an initial segment of B; at limit stages take unions.

By corollary 4.17 and Lemma 4.20, $B_n \cong B_m$ iff $\mathrm{inv}_B(B_n) = \mathrm{inv}_B(B_m)$. This is equality of elements of $\mathrm{inv}(B)$ rather than merely isomorphism of ordinals; we use the fact that if $\beta, \gamma < \alpha$ then $\beta \cong \gamma$ iff $\beta = \gamma$. Thus $\{(n,m) : B_n \cong B_m\}$ exists. $\square$

If we care for only one direction, we have the following.

Lemma 4.23 (RCA_0)**.** *Suppose that A, B are ranked Boolean algebras and that* $\mathrm{inv}(A) \leqslant_{\mathrm{lex}} \mathrm{inv}(B)$*. Then there is an embedding of A into B.*

The proof is similar to the proof of Lemma 4.20, without going back; f, instead of preserving the invariant, simply does not decrease it (lexicographically).

Corollary 4.24 (RCA_0)**.** *Assume that every superatomic Boolean algebra is ranked. Then* $\mathsf{COMP}(\mathcal{SABA})$ *holds.*

Proof. Let A, B be superatomic Boolean algebras; get an invariant inv on $A \times B$. We know that either $\mathrm{inv}(1_A) \leqslant_{\mathrm{lex}} \mathrm{inv}(1_B)$ or vice-versa; thus $A \preccurlyeq B$ or $B \preccurlyeq A$. $\square$

Corollary 4.25 (RCA_0)**.** *Assume that every superatomic Boolean algebra is ranked. Then* $\mathsf{WQO}(\mathcal{SABA})$ *holds.*

Proof. Let $\langle B_n \rangle$ be a sequence of superatomic Boolean algebras. Let $B = \bigoplus_n B_n$ and let inv be an invariant on B. Since $\langle \mathrm{inv}(1_{B_n}) \rangle$ cannot be a strictly $<_{\mathrm{lex}}$-decreasing sequence, we must have some $n < m$ such that $\mathrm{inv}(1_{B_n}) \leqslant_{\mathrm{lex}} \mathrm{inv}(1_{B_m})$. It follows that $B_n \preccurlyeq B_m$. $\square$

The analog of corollary 4.17 (i.e. the converse of Lemma 4.23) seems to require ACA_0.

Lemma 4.26 (ACA_0)**.** *Let A, B be ranked Boolean algebras such that A embeds into B. Then* $\mathrm{rk}(A) \preccurlyeq \mathrm{rk}(B)$*.*

Proof. Let $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ be the CB resolution for A and let $\langle J_\beta \rangle_{\beta \leqslant \alpha'}$ be the CB resolution for B. Let $f \colon A \to B$ be an embedding. Define $g(\beta) = \beta'$ if β' is the least ordinal below α' such that there is some $x \in A$ such that $\mathrm{rk}(x) = \beta$ and $\mathrm{rk}(f(x)) = \beta'$.

Now we prove that g is order preserving. Let $\beta < \gamma < \alpha$. Take $x \in A$ such that $\mathrm{rk}(x) = \gamma$ and $g(\gamma) = \mathrm{rk}(f(x))$. $x \notin I_{\beta+1}$ so in A/I_β there are infinitely many atoms below x. By induction we can pick an infinite collection $X \subset A$ such that for all $y \in X$, $y \leqslant x$ and in A/I_β, y is an atom (so $\mathrm{rk}(y) = \beta$); and further, for distinct $y, y' \in X$ we have $y \wedge y' = 0$. For all $y \in X$, $f(y) \notin J_{g(\beta)}$, $f(y) \leqslant f(x)$, and the $f(y)$'s are pairwise disjoint. Also, f is one-to-one so $f``X$ is infinite. It follows that $f(x) \notin J_{g(\beta)+1}$, and hence that $g(\beta) < g(\gamma)$. $\square$

Remark 4.27 (ACA_0). Suppose that A, B are ranked Boolean algebras and that $\mathrm{rk}(A) = \mathrm{rk}(B)$. Then $A \preccurlyeq B$ iff $\deg(A) \leqslant \deg(B)$. For if $\deg(A) \leqslant \deg(B)$ then $\mathrm{inv}(A) \leqslant_{\mathrm{lex}} \mathrm{inv}(B)$ so $A \preccurlyeq B$ (Lemma 4.23). Suppose that $f\colon A \to B$ is an embedding; let $\alpha = \mathrm{rk}(A)$ and $n = \deg(A)$. Let $X \subset A$ be a set of size n of pairwise disjoint elements of rank α. By Lemma 4.26, for each $x \in X$, $\mathrm{rk}_B(f(x)) = \mathrm{rk}(B(\leqslant f(x))) \geqslant \mathrm{rk}(A(\leqslant x)) = \mathrm{rk}_A(x) = \alpha$, so in B there are n pairwise disjoint elements of rank α; it follows that $\deg(B) \geqslant n$.

Corollary 4.28 (ACA_0)**.** *Assume that every superatomic Boolean algebra is ranked. Then* $\exists$-$\mathsf{EMB}(\mathcal{SABA})$ *holds.*

Proof. The proof is similar to that of corollary 4.22. We are given a sequence $\langle B_n \rangle$ of superatomic Boolean algebras and get a rank on $B = \bigoplus_n B_n$. Now if $\mathrm{inv}(B_n) \leqslant_{\mathrm{lex}} \mathrm{inv}(B_m)$ then B_n embeds into B_m. On the other hand, if B_n embeds into B_m then $\mathrm{rk}(B_n) \preccurlyeq \mathrm{rk}(B_m)$ and if they are equal then $\deg(B_n) \leqslant \deg(B_m)$. However, as all of these ordinals are initial segments of $\mathrm{rk}(B)$, $\preccurlyeq$ and $\leqslant$ coincide, so if B_n embeds into B_m then $\mathrm{inv}(B_n) \leqslant_{\mathrm{lex}} \mathrm{inv}(B_m)$. The conclusion follows. ☐

Corollary 4.29 (ACA_0)**.** *Assume that every superatomic Boolean algebra is ranked. Then* $\mathsf{EQU}{=}\mathsf{ISO}(\mathcal{SABA})$ *holds.*

Proof. Let A, B be superatomic Boolean algebras such that $A \preccurlyeq B$ and $B \preccurlyeq A$. Again let inv be an invariant on $A \times B$. By Lemma 4.26 and Remark 4.27, we have that $\mathrm{inv}(1_A) = \mathrm{inv}(1_B)$. It follows that $A \cong B$. ☐

4.3. **Reductions.**

4.3.1. *Ordinals to superatomic Boolean algebras.* Let L be a linear ordering. We let $\mathrm{Int}(L)$ be the Boolean algebra consisting of finite unions of half open intervals of L of the form $[a, b)$ (where we allow $b = \infty$). [In RCA_0, the elements of this Boolean algebra are coded by the finite sequences of the pairs of endpoints of these intervals; all Boolean operations exist.]

For any linear ordering ordinal L, let $\mathbb{B}(L) = \mathrm{Int}(\omega^L)$.

Lemma 4.30 (RCA_0)**.** *For all* α*,* $\mathbb{B}(\alpha)$ *is ranked, and its invariant is* $(\alpha, 1)$*.*

Proof. The important thing to recall is that the operations of ordinal addition and subtraction ($\alpha - \beta = \gamma$ if $\beta + \gamma = \alpha$) below ω^α exist; also, taking the logarithm of base ω exists. That is, given $\gamma = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k$ where $\beta_1 > \beta_2 > \cdots > \beta_k$ are elements of α and $n_i \in \omega$, we know that β_1 is the greatest element β of α such that $\omega^\beta \leqslant \gamma$; we in fact let $\mathrm{inv}(\gamma) = (\beta_1, n_1)$ and for every interval $[\beta, \gamma)$ (where $\beta < \gamma \leqslant \omega^\alpha$) we let $\mathrm{inv}([\beta, \gamma)) = \mathrm{inv}(\gamma - \beta)$. Note that for the special case $\gamma = \infty (= \omega^\alpha)$ we always have $\omega^\alpha - \beta = \omega^\alpha$ and $\mathrm{inv}(\omega^\alpha) = (\alpha, 1)$. When faced by a finite disjoint union of intervals, we take the natural "sum" of invariants, namely the invariant is (β, n) where β is the maximal rank of the intervals and n is the sum of the degrees of those intervals which have rank β.

The fact that inv is indeed the correct invariant for $\mathbb{B}(\alpha)$ follows from three facts (all of which are properties of ordinal addition up to ω^α): one, that finite unions do not increase rank; second, that the inclusion relation respects the lexicographic ordering of invariants (so in particular, I_γ, the collection of all elements of $\mathbb{B}(\alpha)$ which have rank below γ, forms an ideal); and third, that an interval of length ω^γ cannot include two disjoint subintervals of length ω^γ; this implies that the intervals of invariant $(\gamma, 1)$ (which modulo I_γ have length ω^γ) are the atoms of $\mathbb{B}(\alpha)/I_\gamma$. ☐

Lemma 4.31 (RCA_0). *For all linear orderings L, L is well-founded iff $\mathbb{B}(L)$ is superatomic.*

Proof. The direction from left to right follows from the previous lemma and from Lemma 4.18. For the other direction suppose that L is not well-founded and that $\{a_n\}_{n\in\mathbb{N}}$ is a descending sequence in L. We define an embedding f of the full binary tree into $\mathbb{B}(L)$. Given $\sigma \in 2^{<\mathbb{N}}$, we let $x_\sigma = \sum_{i:\sigma(i)=1} \omega^{a_{i+1}}$ and we let

$$f(\sigma) = [x_\sigma, x_\sigma + \omega^{a_{|\sigma|}}).$$

Is not hard to check that f is as wanted. □

We get (aided by 4.17 and 4.20):

Corollary 4.32 (RCA_0). *Let α, β be ordinals. Then $\alpha \cong \beta$ iff $\mathbb{B}(\alpha) \cong \mathbb{B}(\beta)$.*

The following is immediate;

Lemma 4.33 (RCA_0). *Let α, β be ordinals and suppose that $\alpha \preccurlyeq \beta$. Then $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$.*

By 4.26, we get

Corollary 4.34 (ACA_0). *Let α, β be ordinals. Then $\alpha \preccurlyeq \beta$ iff $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$.*

As a corollary we have:

Corollary 4.35 (RCA_0). *The statement $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ implies $\exists$-$\mathsf{ISO}(\mathcal{O}n)$, and $\mathsf{EQU{=}ISO}(\mathcal{SABA})$ implies $\mathsf{EQU{=}ISO}(\mathcal{O}n)$. Therefore, both $\exists$-$\mathsf{ISO}(\mathcal{SABA})$ and $\mathsf{EQU{=}ISO}(\mathcal{SABA})$ are equivalent to ATR_0.*

Corollary 4.36 (ACA_0). *The statement $\exists$-$\mathsf{EMB}(\mathcal{SABA})$ implies $\exists$-$\mathsf{EMB}(\mathcal{O}n)$, the statement $\mathsf{COMP}(\mathcal{SABA})$ implies $\mathsf{COMP}(\mathcal{O}n)$ and the statement $\mathsf{WQO}(\mathcal{SABA})$ implies $\mathsf{WQO}(\mathcal{O}n)$. Therefore, $\exists$-$\mathsf{EMB}(\mathcal{SABA})$, $\mathsf{COMP}(\mathcal{SABA})$ and $\mathsf{WQO}(\mathcal{SABA})$ are equivalent to ATR_0 over ACA_0.*

4.3.2. *Well-founded trees to superatomic Boolean algebras.* We do not need the following for the proof of Theorem 1.5, but we thought to include a natural operation which takes us directly from trees to Boolean algebras.

Let T be a tree. We let $\mathbb{B}(T)$ be the *tree algebra* of T, as described in [Kop89, Section 16]. There, it is defined as the subalgebra of $\mathcal{P}(T)$ generated by the cones $\{\sigma \in T : \tau \subseteq \sigma\}$ (for all $\tau \in T$). Of course, this definition only makes sense in ACA_0, but in fact we can get the tree algebra in RCA_0. We let $\mathbb{B}_0(T)$ be a Boolean algebra generated freely over a basis $\{b_\sigma : \sigma \in T\}$; We let I be the ideal generated by the equations $b_\sigma \leqslant b_\tau$ if $\tau \subset \sigma$ and $b_\sigma \wedge b_\tau = 0$ if $\sigma \perp \tau$. The ideal I exists: by reducing to disjunctive normal form, it is sufficient to decide whether an element of the form $a = b_{\sigma_1} \wedge \cdots \wedge b_{\sigma_n} \wedge -b_{\tau_1} \wedge \cdots \wedge -b_{\tau_m}$ is in I. If the σ_i are not linearly ordered then $a \in I$. Otherwise, we let σ be the greatest σ_i; modulo I, $a = b_\sigma \wedge -b_{\tau_1} \wedge \cdots \wedge -b_{\tau_m}$. If for some i we have $\sigma \supset \tau_i$ then $a \in I$; otherwise $a \notin I$. Finally we let $\mathbb{B}(T) = \mathbb{B}_0(T)/I$.

Remark 4.37. The argument showing that I exists and some more work yield a normal form for elements of $\mathbb{B}(T)$; all the elements can be written as *disjoint* sums of elements of the form $b_\sigma - \sum_{\tau\in S} b_\tau$, where S is a finite antichain in T and for all $\tau \in S$, $\sigma \subset \tau$. Again, see [Kop89].

Recall that for a tree T and $\tau \in T$, $T - \tau = \{\sigma : \tau^\frown\sigma \in T\}$.

Fact. Let T be a tree. For all $\tau \in T$, $\mathbb{B}(T - \tau) \cong \mathbb{B}(T)(\leqslant b_\tau)$.

In fact, we can break up the tree algebra into pieces. Let $\tau \in T$ let $S(\tau)$ be the collection of immediate successors of τ on T. If $S(\tau)$ is infinite, then

$$\mathbb{B}(T - \tau) \cong \oplus \{\mathbb{B}(T - \sigma) \,:\, \sigma \in S(\tau)\}.$$

However, if $S(\tau)$ is finite (for example, if τ is a leaf), then $b_\tau - \sum_{\sigma \in S(\tau)}$ is an atom of $\mathbb{B}(T - \tau)$ and so we have

$$\mathbb{B}(T - \tau) \cong \{0, 1\} \oplus \{\mathbb{B}(T - \sigma) \,:\, \sigma \in S(\tau)\}$$

(So if τ is a leaf then b_τ is an atom of $\mathbb{B}(T)$ and $\mathbb{B}(T - \tau) \cong \{0, 1\}$.

If T is well-founded then we can build $\mathbb{B}(T)$ from the leaves up.

Lemma 4.38 (ACA_0)**.** *If T is well-founded then $\mathbb{B}(T)$ is superatomic.*

Proof. Suppose that f is an embedding of the full binary tree into $\mathbb{B}(T)$. By induction, we find some $\sigma_n \in T$ and some embedding f_n of the full binary tree into $\mathbb{B}(T \upharpoonright \sigma_n)$. Given $f_n(\langle\rangle) = b_{\sigma_n} = 1_{\mathbb{B}(T - \sigma_n)}$, for some $i < 2$, there is some finite $S \subset S(\sigma_n)$ such that $f_n(i) \leqslant \sum_{\tau \in S} b_\tau$. We can thus view $\rho \mapsto f_n(i^\frown \rho)$ as an embedding of the full binary tree into $\oplus\{\mathbb{B}(T - \tau) \,:\, \tau \in S\}$. By asking finitely many questions, as in the proof of Lemma 4.21, we find some coordinate from which we can pick a σ_{n+1} and let f_{n+1} be the adequate restriction. □

It is clear that if $T \cong T'$ then $\mathbb{B}(T) \cong \mathbb{B}(T')$.

4.3.3. *Superatomic Boolean algebras to well-founded trees.*

Definition 4.39. A *uniformly splitting tree* is a tree of the form $\{\sigma \in \omega^{<\alpha} : \forall i < |\sigma| (\sigma(i) < m_i)\}$ where $\alpha \in \mathbb{N} \cup \{\mathbb{N}\}$ and for all i, $m_i \in \mathbb{N}$ and $m_i \geqslant 2$. A *finite tree embedding* into a Boolean algebra B is a partial function $g \colon T \to B^+$ where T is a finite uniformly splitting tree and g preserves $\leqslant$, $\perp$ and 1 (i.e. $g(\langle\rangle) = 1_B$).

Fix a sequence of constants $\langle q_n \rangle_{n \geqslant 2}$ which grows sufficiently quickly so that for all $l, k < n$, $q_l + q_k \leqslant q_n$ (for example, let $q_k = 2^{2^k}$). Also, fix an infinite recursive set $\mathcal{C} = \{c_n \,:\, n \geqslant 2\}$ (say $c_n = \langle 0, n \rangle$) and fix a coding of all finite sequences and functions such that the collection of code numbers is disjoint from $\mathcal{C}$. For $m \in \mathbb{N}$, $n \geqslant 2$ we let $\varrho(n; m) = \langle c_n \rangle^m$ and let $\varrho_n = \varrho(n; q_n - 1)$.

Given a Boolean algebra B we will code all finite tree embeddings into B using a tree. Let $T \subset m_1 \times \cdots \times m_k$ be a finite uniformly splitting tree and let $g \colon T \to B$ be a finite tree embedding. For $l \leqslant k$ let $T_l = T \cap \omega^l$. We let the *code* for g, be the string

$$\varpi(g) = \varrho_{m_1}{}^\frown \langle g \upharpoonright T_1 \rangle^\frown \varrho_{m_2}{}^\frown \langle g \upharpoonright T_2 \rangle^\frown \cdots^\frown \varrho_{m_k}{}^\frown \langle g \upharpoonright T_k \rangle.$$

Note that we also allow g to be defined on $T_0 = \{\langle\rangle\}$; there is a unique such g (we must have $g(\langle\rangle) = 1_B$) and in this case $\varpi(g) = \langle\rangle$. The function $g \mapsto \varpi(g)$ is one-to-one and its range is computable from B, so it exists (in RCA_0).

The tree $T(B)$ consists of all of the strings $\varpi(g)^\frown \varrho(n; m)$ where $n \geqslant 2$ and $m < q_n$, and g is a finite tree embedding into B. In RCA_0 we can show that $T(B)$ exists. Also, $T(B)$ is closed under initial segments so it is indeed a tree.

Lemma 4.40 (RCA_0)**.** *B is superatomic iff $T(B)$ is well-founded.*

Proof. Suppose that B is not superatomic. Then there is an embedding of the full binary tree into B, and by shifting elements around we may assume that the top of the embedding is 1_B. This embedding would yield a path in $T(B)$.

On the other hand, from a path in $T(B)$ we can recover an embedding of an infinite uniformly splitting tree into B. The reason is that on an infinite path, codes for functions must occur infinitely often: if f is an infinite path and $f(n)$ is a code for a function then if $c_m = f(n+1)$ we know that $f(n+q_m)$ must be a code for a function. Restricting the embedding we get to $2^{<\omega}$ we get an embedding of the full binary tree into B. □

For the purpose of the following computation, we let $q_1 = 0$. Also recall that $I_0 = \{0\}$ so that $\mathrm{rk}(0_B) = -1$ but for every finite join of atoms x, $\mathrm{rk}(x) = 0$.

Lemma 4.41 (RCA_0)**.** *Suppose that B is a ranked Boolean algebra and $\mathrm{inv}(B) = (\alpha, n)$. Then $T(B)$ is ranked and has rank $\omega(2\alpha+1)+q_n$.*

(Recall that if γ is limit and $\alpha = \gamma + k$ then $2\alpha = \gamma + 2k$.)

Proof. For a finite tree embedding g into B we let $\mathrm{inv}(g) = \min\{\mathrm{inv}(x) : x \in \mathrm{range}\, g\}$, where the ordering is of course the lexicographic one. For each such g, if $\mathrm{inv}(g) = (\beta, k)$ then we let $f(\varpi(g)) = \omega(2\beta+1)+q_k$.

Now we claim that f extends to a rank function on $T(B)$. We first discuss how we should define f on nodes of the form $\varpi(g)^\frown\varrho_n$ for $n \geqslant 2$ and finite tree embeddings g into B. Let g be such and let $(\beta,k) = \mathrm{inv}(g)$; let $x \in \mathrm{range}\, g$ such that $\mathrm{inv}(x) = (\beta, k)$. Let $n \geqslant 2$. If $n > k$ then we cannot split x into n many disjoint elements of rank β; on the other hand, we can split x into n many disjoint elements, each of which have invariant below $(\beta, 0)$ but as large as we like below $(\beta, 0)$. It follows that the immediate extensions of $\varpi(g)^\frown\varrho_n$ in $T(B)$ are of the form $\varpi(g')$ where $\mathrm{inv}(g') < (\beta, 0)$ (so if $\beta = 0$ there are no such g' and $\varpi(g)^\frown\varrho_n$ is terminal), and all invariants below $(\beta,0)$ occur; then the supremum of f on the immediate extensions of $\varpi(g)^\frown\varrho_n$ is $\sup\{\omega(2\gamma+1)+l : \gamma < \beta, l \in \mathbb{N}\} = \omega(2\beta)$. We thus let $f(\varpi(g)^\frown\varrho_n) = \omega(2\beta)$ for such n.

If, however, $n \leqslant k$, then x can be split into n disjoint elements of rank β. In fact, there is then an immediate successor $\varpi(g')$ of $\varpi(g)^\frown\varrho_n$ in $T(B)$ of maximal invariant (β, l) (where l is the largest possible size of a smallest set in a partition of k into n nonempty sets). We can thus let $f(\varpi(g)^\frown\varrho_n) = \omega(2\beta+1)+q_l+1$. We note that $l = \lfloor k/n \rfloor$.

Having defined f on the nodes $\varpi(g)^\frown\varrho_n$ we can extend it to nodes $\varpi(g)^\frown\varrho(n;m)$ because from $\varpi(g)^\frown\varrho(n;1)$ to $\varpi(g)^\frown\varrho_n$ there is no splitting on $T(B)$. We thus have $f(\varpi(g)^\frown\varrho(n;1)) = \omega(2\beta)+q_n-2$ if $n > k$, $\omega(2\beta+1)+q_l+q_n-1$ if $n \leqslant k$ where $\lfloor k/n <\rfloor$. Since we chose the q_ks to rise quickly we have $q_l + q_n \leqslant q_k$ for $n < k$. Thus if $k \geqslant 2$ then we indeed have that $\varpi(g)^\frown\varrho(k;1)$ has maximal rank among the immediate successors of $\varpi(g)$, and indeed we assigned it rank $f(\varpi(g))-1$, so f is continuous at $\varpi(g)$. If $k = 1$ then for all n, $f(\varpi(g)^\frown\varrho(n;1)) = \omega(2\beta)+q_n-2$ and $q_n \to \infty$, so again f is continuous at $\varpi(g)$ as required. □

Corollary 4.42 (ACA_0)**.** *Suppose that A, B are ranked Boolean algebras. Then $A \cong B$ iff $T(A)^\infty \cong T(B)^\infty$ and $A \preccurlyeq B$ iff $T(A)^\infty \preccurlyeq T(B)^\infty$.*

Note that this yields another proof that ATR_0 implies the various statements for superatomic Boolean algebras; we deduce them from $\mathsf{RK}(\mathcal{SABA})$ and the corresponding statements for fat trees.

4.4. **Proofs of arithmetic comprehension.**

Proposition 4.43 (RCA_0)**.** $\exists$-$\mathsf{EMB}(\mathcal{SABA})$ *implies* ACA_0*.*

Proof. This is immediate (same as Proposition 3.28). □

Proposition 4.44 (RCA_0)**.** $\mathsf{COMP}(\mathcal{SABA})$ *implies* ACA_0*.*

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. For each $n \in \mathbb{N}$ let B_n be a finite boolean algebra such that if n is not in the range of f, it has only one atom, and if $n = f(m)$, it has m many atoms. Let $B = \bigoplus_{n\in\mathbb{N}} B_n$ and let B' be a ranked boolean algebra of invariant $(\omega, 2)$. First we observe that $B' \not\preccurlyeq B$: B' has two elements with meet $0_{B'}$ and infinitely many atoms below, but B does not. So, $\mathsf{COMP}(\mathcal{SABA})$ implies $B \preccurlyeq B'$. Let g be such an embedding. For at most two (actually one) values of n we may have $\mathrm{rk}(g(1_{B_n})) = 1$. For every other n we have $\mathrm{inv}(g(1_{B_n})) = (0, k_n)$. As in the proof for trees, we observe that n is in the range of f iff it is in the range of $f \restriction k_n$. □

4.4.1. *WQO.* As in subsection 3.3.1, we go via RCA_2. As mentioned in that subsection, we make use of some ideas of Shore's proof that $\mathsf{WQO}(\mathcal{O}n)$ implies ACA_0 [Sho93, Theorems 2.17 and 3.1].

Proposition 4.45 (RCA_2)**.** $\mathsf{WQO}(\mathcal{SABA})$ *implies* ACA_0*.*

Proof. Again fix an enumeration of $0'$. Let $\beta^n_t = \omega + 1$ if t is the k^{th} true stage of the enumeration for some $k > n$, and finite otherwise. [Set $1 <^* 2 <^* 3 <^* \cdots <^* 0$. At stage $s > t$, determine that $s \in \beta^n_t$ if at s, t appears to be a true stage and there are n other stages before t which also appear to be true at s. If t is not true then eventually this will be found out. If t is true then already at t we know all the true stages $< t$, so if there are fewer than n true stages $< t$, then no $s > t$ is in β^n_t.]

Let $\alpha_n = \sum_{t\in\mathbb{N}} \beta^n_t$ and let $B_n = \mathrm{Int}(\alpha_n)$. Each B_n is superatomic (to see this quickly, note that $B_n = \oplus_t \mathrm{Int}(\beta^n_t)$, and see Lemma 4.21). Suppose that $n < m$ and that $f\colon B_n \to B_m$ is an embedding.

By Σ^0_2-induction, there are more than n many true stages. Suppose then that t is the $(n+k)^{th}$ true stage for some $k > 0$. Thus the interval $I = \cup_{s\leqslant t}\beta^n_s$ has exactly k limit points, which in B_n means that it is the join of k many elements below each of which there are infinitely many atoms. This has to be true of $f(I)$ in B_m. Suppose that $\sup f(I) \in \beta^m_r$. Then r bounds a true stage larger than t (for example, the $(m+k)^{th}$ true stage). The stage u between t and r at which the smallest number ever to be enumerated between t and r was actually enumerated, is a true stage $> t$. This process can be iterated to get infinitely many true stages and thus $0'$. □

Now we observe that in light of Lemma 3.31 and the proof of proposition 3.32, in order to show

Proposition 4.46 (RCA_0)**.** $\mathsf{WQO}(\mathcal{SABA})$ *implies* $I\Sigma_2$*.*

it is enough to show two things in RCA_0: first, that if α, β are ordinals and $\alpha \preccurlyeq \beta$ then $\mathbb{B}(\alpha) \preccurlyeq \mathbb{B}(\beta)$ (this is Lemma 4.33), and that if α is an ordinal, then $\mathbb{B}(\alpha + 1)$

does not embed into $\mathbb{B}(\alpha)$. Armed with these facts, the proof follows exactly as it did for trees. The second statement can be shown using:

Fact (RCA_0). Suppose that A and B are ranked Boolean algebras with resolutions $\langle I_\beta\rangle_{\beta<\alpha}$ and $\langle J_\beta\rangle_{\beta<\alpha}$ along the *same ordinal* α (we allow for a final segment of $I_\beta = A$ or $J_\beta = B$). Suppose that $f\colon A \to B$ is an embedding. Then for all $\beta < \alpha$ and $x \in A$, if $\mathrm{rk}(x) \geqslant \beta$ then $\mathrm{rk}(f(x)) \geqslant \beta$. This is shown by Π^0_1-transfinite induction on α.

5. Reduced p-groups

Fix a prime number p. A *p-group* is a group in which every element has order a power of p.

Convention. From now on, all groups are Abelian.

A group G is *divisible* if for every $a \in G$ and every $n \in \mathbb{N}$, there exists $b \in G$ such that $nb = a$.

Definition 5.1. A group is *reduced* if it has no divisible subgroup.

Fact (ACA_0). A p-group G is reduced iff there is no sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of elements of G such that for all n, $pg_{n+1} = g_n$. This is because the subgroup generated by the g_ns is the direct limit of $\mathbb{Z}_{p^n}$, and in each $\mathbb{Z}_{p^n}$ one can divide by numbers not divisible by p.

Again, reduced groups are the "well-founded part" of the collection of groups, and it takes Π^1_1-comprehension to weed out this part. A classic result is the following theorem of Friedman, Simpson and Smith:

Theorem 5.2 ([FSS83])**.** *The statement "every group is the direct product of a reduced group and a divisible groups" is equivalent to* Π^1_1-CA_0 *over* RCA_0.

As our notion of embedding $\preccurlyeq$ we take the usual notion of group embedding (one-to-one homomorphism).

5.1. Ranked p-groups.

We define rank functions for reduced p-groups. Let α be an ordinal and G a p-group. A *partial Ulm resolution* of G along α is a sequence of subgroups $\langle G_\beta\rangle_{\beta<\alpha}$ such that $G_0 = G$, if $\beta+1 < \alpha$ then $G_{\beta+1} = \{pg : g \in G_\beta\}$, and for limit $\lambda < \alpha$, $G_\lambda = \bigcap_{\beta<\lambda} G_\beta$ (we sometimes write $p^\beta G$ for G_β). An *Ulm resolution* of G is a partial Ulm resolution of G along some ordinal $\alpha+1$ such that $G_\alpha = \{0\}$ and for all $\beta < \alpha$, $G_\beta \neq \{0\}$. We call such an α the *length* of G.

Notation (RCA_0). If G is a p-group then $G[p]$, its *socle*, is the subgroup consisting of elements of G of order p.

Suppose that $\langle G_\beta\rangle_{\beta\leqslant\alpha}$ is a resolution of a p-group G; we define an associated Ulm sequence. For each $\beta < \alpha$, $G_\beta[p]/G_{\beta+1}[p]$ is a vector space over $\mathbb{Z}_p$; we let $U_G(\alpha)$ be its dimension. The sequence $\langle U_G(\beta) : \beta < \alpha\rangle$ is called the *Ulm sequence* of G, and it characterizes G up to isomorphism. We also define an associated rank function: for $x \in G$, let $\mathrm{rk}_G(x)$ be the unique $\beta < \alpha$ such that $x \in G_\beta \smallsetminus G_{\beta+1}$. (In ACA_0, if a resolution exists then so does the sequence and the rank function; but not in RCA_0.)

Definition 5.3. A p-group G is *weakly ranked* if it has an Ulm resolution and the associated rank function exists. It is *ranked* if further, the associated Ulm sequence exists.

RCA_0 is enough to prove that reduced p-groups with the same Ulm sequence are isomorphic [Sim99, Theorem V.7.1]. Simpson [Sim99, Lemma V.7.2] uses ACA_0 to prove that Ulm sequences are unique up to isomorphisms of ordinals.

Lemma 5.4 (RCA_0)**.** *Let G be a reduced p-group. Let α and α' be ordinals and suppose that $\langle G_\beta\rangle_{\beta<\alpha}$ and $\langle G'_\beta\rangle_{\beta<\alpha'}$ are two resolutions of G. Further, assume that the associated rank function* rk *and* rk$'$ *exist. Then $\alpha \cong \alpha'$, and the isomorphism commutes with* rk*,* rk$'$*.*

Proof. This is similar (but not identical) to the proof of Lemma 4.16. By Π^0_1-transfinite induction on $\beta < \alpha$ we show that for all x of rank β, $G_{\mathrm{rk}(x)} = G'_{\mathrm{rk}(x')}$ (in particular it follows that if $\mathrm{rk}(x) = \mathrm{rk}(y) = \beta$ then $\mathrm{rk}'(x) = \mathrm{rk}'(y)$). Suppose that the claim is verified up to β. There are two options.

First, suppose that β is a limit ordinal. Then $G_\beta = \cap_{\gamma<\beta} G_\gamma$. Let x have rank β and let $\beta' = \mathrm{rk}'(x)$. So far, we have a map $f\colon \beta \to \alpha'$ which is defined by taking $\gamma < \beta$ to the unique γ' such that for some (all) $y \in G$ of rk γ, $\mathrm{rk}'(y) = \gamma'$ (so $G_\gamma = G'_{\gamma'}$). For all $\gamma < \beta$ we thus have $x \in G'_{f(\gamma)}$ so range $f \subset \beta'$. Of course, f is order-preserving, and since β is limit, range f has no last element. Suppose that f is not cofinal in β': that there is some $\delta' < \beta'$ such that for all $\gamma' \in$ range f, $\gamma' \leqslant \delta'$. But then we have $G_\beta = \cap_{\gamma'\in \mathrm{range}\, f} G'_{\gamma'} \supset G'_{\delta'}$; on the other hand, $\beta' > \delta'$ implies that there is some $y \in G'_{\delta'}$ such that $py = x$. But there is no such y in G_β. Thus $\beta' = \sup \mathrm{range}\, f$ is limit and $G'_{\beta'} = G_\beta$ as required.

Next, suppose that $\beta = \gamma + 1$. Take $x \in G$ of rank β; let $\beta' = \mathrm{rk}'(x)$. Since $x \in G_{\gamma+1}$, $p|x$ in G_γ so we can find $y \in G_\gamma$ such that $py = x$. We cannot have $y \in G_\beta$ (or $x \in G_{\beta+1}$); so $\mathrm{rk}(y) = \gamma$. Let $\gamma' = \mathrm{rk}'(y)$. By induction, $G_\gamma = G'_{\gamma'}$, so $G_\beta = G'_{\gamma'+1}$. But $\beta' = \mathrm{rk}'(x) = \gamma' + 1$. Why is that? Otherwise, we have $\mathrm{rk}'(x) > \gamma'+1$ so $x \in G'_{\gamma'+2}$; thus there is some $y \in G'_{\gamma'+1}$ such that $py = x$; and so there is some $z \in G'_{\gamma'}$ such that $pz = y$. But then $z \in G_\gamma$ and so $x \in G_{\gamma+2}$ which is false. Thus $G_\beta = G'_{\beta'}$ as required.

When we are done with the induction we define $f\colon \alpha \to \alpha'$ as we did in limit stages and get the desired isomorphism. □

Lemma 5.5 (RCA_0)**.** *Every ranked p-group is reduced.*

Proof. Suppose G is a p-group which is not reduced. Let H be a divisible subgroup of G and let $x_0 \in H$. By primitive recursion construct a sequence $\langle x_i : i \in \mathbb{N}\rangle$ such that for each i, $px_{i+1} = x_i$. Note now that for each i, $\mathrm{rk}(x_{i+1}) < \mathrm{rk}(x_i)$, because if $x_{i+1} \in G_\beta$, then $x_i \in G_{\beta+1}$. So, we have a contradiction because the length of G is well-founded. □

The statement that every reduced p-group is weakly ranked is equivalent to ATR_0 over RCA_0 [Sim99, Theorem V.7.3]. It follows that the statement that every reduced p-group is ranked is also equivalent to ATR_0 over RCA_0, because as mentioned earlier, ACA_0 is enough to compute the Ulm sequence.

Given a group G, a set $A \subseteq G \setminus \{0\}$ is *independent* if for any $a_1, \dots, a_k \in A$, if $n_1a_1 + \dots + n_ka_k = 0$, then $n_1a_1 = \dots = n_ka_k = 0$. Let G be a p-group. Then if

$a_1, \dots, a_k \in G$ are independent and the order of a_i is p^{n_i} then $p^{n_1-1}a_1, \dots, p^{n_k-1}a_k$ is an independent subset of $G[p]$, which is a vector space. Hence all maximal independent sets in G have the same cardinality which we denote by $r(G)$, the (for our purposes unfortunately named) *rank* of G; $r(G) = r(G[p])$.

Lemma 5.6 (ACA_0)**.** *Suppose that G, G' are ranked p-groups (with resolutions $\langle G_\beta \rangle_{\beta \leqslant \alpha}$, $\langle G'_{\beta'} \rangle_{\beta' \leqslant \alpha'}$). Suppose that $f\colon G \to G'$ is an embedding. Then there is some embedding $h\colon \alpha \to \alpha'$ such that for all $\beta < \alpha$, $r(G_\beta) \leqslant r(G'_{h(\beta)})$.*

(This extends the discussion about embeddings in [Frib].)

Proof. For $\beta < \alpha$, we let $h(\beta)$ be the maximal $\beta' < \alpha$ such that $f``G_\beta \subset G'_{\beta'}$. h is strictly order-preserving because for all $\beta < \alpha$, $G_{\beta+1} = pG_\beta \subset G'_{h(\beta)+1}$. Now $h \restriction G_\beta[p]$ is a vector space embedding of $G_\beta[p]$ into $G'_{h(\beta')}[p]$ so the rank cannot decrease. □

Remark 5.7. Suppose that $\langle G_\beta \rangle_{\beta \leqslant \alpha}$ is the Ulm resolution of a p-group G. Then for $\beta < \gamma \leqslant \alpha$ we have $r(G_\beta) - r(G_\gamma) = \sum_{\delta \in [\beta,\gamma)} U_G(\delta)$. Thus, if $\alpha = \epsilon + n$ where ϵ is limit, then for all $k < n$, $r(G_{\epsilon+k}) = U_G(\epsilon + k) + \cdots + U_G(\epsilon + n - 1)$. If $\beta < \epsilon$ then there are infinitely many $\delta \in [\beta, \epsilon)$ such that $U_G(\delta) > 0$ and so $r(G_\beta) = \omega$. (See Barwise and Eklof [BE71].)

We, in fact, have a converse for Lemma 5.6.

Lemma 5.8 (ACA_0)**.** *Suppose that G, G' are ranked p-groups (with resolutions $\langle G_\beta \rangle_{\beta \leqslant \alpha}$, $\langle G'_{\beta'} \rangle_{\beta' \leqslant \alpha'}$). Suppose that there is an embedding $h\colon \alpha \to \alpha'$ such that for all $\beta < \alpha$, $r(G_\beta) \leqslant r(G'_{h(\beta)})$. Then $G \preccurlyeq G'$.*

Proof. See [BE71, Corollary 5.4] and [Frib]. □

As before, we need to see how ranks correspond to direct sums.

Lemma 5.9 (RCA_0)**.** *Suppose that $\langle H_n \rangle$ is a sequence of reduced p-groups. Then $G = \bigoplus_n H_n$ is reduced. If $\langle G_\beta \rangle_{\beta \leqslant \alpha}$ is the Ulm resolution for G, then for all $n \in \mathbb{N}$, $\langle G_\beta \cap H_n \rangle_{\beta \leqslant \alpha}$ is an Ulm resolution of H_n (we may need to trim the end of the sequence though).* □

Proposition 5.10 (ACA_0)**.** $\mathsf{RK}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$ *implies* $\exists\text{-}\mathsf{EMB}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$.

Proof. This is similar to what we did before. Let $\langle H_n \rangle$ be a sequence of reduced p-groups. Let $G = \bigoplus_n H_n$ and get a resolution of G; we get the induced resolutions of the H_ns uniformly. Again comparability of the ordinals in question is equivalent to their position in the length of G; we can check ranks of tail-ends to see if the condition in lemmas 5.6, 5.8 holds. □

Similarly,

Proposition 5.11 (ACA_0)**.** $\mathsf{RK}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$ *implies* $\exists\text{-}\mathsf{ISO}(\mathcal{R}\text{-}p\text{-}\mathcal{G})$.

Proof. Note that we need ACA_0 to check not only equality of lengths but also of the Ulm function along the length. □

5.2. **Reductions.**

5.2.1. *Ordinals to groups.*

Definition 5.12. Given a tree T we let $\mathbb{G}(T)$ be the (Abelian) group generated freely by the elements of T, modulo the relations $\langle\rangle = 0$ and $p\tau = \sigma$ whenever τ is an immediate successor of σ on T. Given an ordinal α we let $\mathbb{G}(\alpha) = \mathbb{G}(T(\alpha))$.

Despite the presentation as generators / relations, the group $\mathbb{G}(T)$ is computable from T. For more information on the reduction $\mathbb{G}(T)$, as for example how to compute its Ulm Sequence, see [Bar95].

The following is proved in [Frib]:

Lemma 5.13 (RCA_0)**.** *For any tree T, T is well-founded iff $\mathbb{G}(T)$ is reduced.*

Also,

Lemma 5.14 (RCA_0)**.** *For any ordinal α, $\mathbb{G}(\alpha)$ is a weakly ranked p-group of length α.*

As usual, in ACA_0 we can also prove that $\mathbb{G}(\alpha)$ is ranked.

Proof, following [Frib]*.* Given $\beta < \alpha$, let G_β be the set of elements of $\mathbb{G}(\alpha)$ of the form $n_0\sigma_0 + \cdots + n_k\sigma_k$ such that for all $i \leq k$, the last element of σ_i is at least β. We claim that $\langle G_\beta : \beta < \alpha\rangle$ is an Ulm resolution for $\mathbb{G}(\alpha)$. Clearly for λ limit $G_\lambda = \cap_{\beta<\lambda} G_\beta$. Now consider $x = n_0\sigma_0 + \cdots + n_k\sigma_k$. If $x \in G_{\beta+1}$, then $y = n_0\sigma_0{}^\frown\beta + \cdots + n_k\sigma_k{}^\frown\beta \in G_\beta$ and $py = x$. Conversely, if $x \notin G_{\beta+1}$, then for some σ_i, the last element of σ_i is at most β. So, there is no $y \in G_\beta$ such that $py = x$. Thus $\langle G_\beta\rangle_{\beta\leqslant\alpha}$ is indeed an Ulm resolution of $G(\alpha)$. It is also easy to find the rank of any element. □

Lemma 5.15 (RCA_0)**.** *Given ordinals α and β we have that $\alpha \cong \beta \leftrightarrow \mathbb{G}(\alpha) \cong \mathbb{G}(\beta)$.*

Proof. Clearly if $\alpha \cong \beta$ then $\mathbb{G}(\alpha) \cong \mathbb{G}(\beta)$. Suppose now that $\mathbb{G}(\alpha) \cong \mathbb{G}(\beta)$. By the previous lemma $\mathbb{G}(\alpha)$ and $\mathbb{G}(\beta)$ are both weakly ranked and have lengths α and β. By Lemma 5.4, we have that $\alpha \cong \beta$. □

Corollary 5.16 (RCA_0)**.** *$\exists$-$\mathsf{ISO}(\mathcal{R}$-p-$\mathcal{G})$ implies ATR_0. Therefore, $\mathsf{RK}(\mathcal{R}$-p-$\mathcal{G})$ and $\exists$-$\mathsf{ISO}(\mathcal{R}$-p-$\mathcal{G})$ are equivalent to ATR_0.*

Lemma 5.17 (ACA_0)**.** *Given ordinals α and β we have that $\alpha \preccurlyeq \beta \leftrightarrow \mathbb{G}(\alpha) \preccurlyeq \mathbb{G}(\beta)$.*

Proof. If $\mathbb{G}(\alpha) \preccurlyeq \mathbb{G}(\beta)$ then by Lemma 5.8, $\alpha \preccurlyeq \beta$. If $\alpha \preccurlyeq \beta$ then we can directly construct an embedding of $\mathbb{G}(\alpha)$ into $\mathbb{G}(\beta)$. □

Corollary 5.18 (ACA_0)**.** *$\exists$-$\mathsf{EMB}(\mathcal{R}$-p-$\mathcal{G})$ implies ATR_0.*

5.2.2. *Groups to trees.* Given a p-group G, we let $T(G)$ (essentially) consist of the elements of G: we declare that 0_G corresponds to $\langle\rangle$, and $x \in G$ of order p^n is identified with the sequence $\langle p^{n-1}x, \ldots, px, x\rangle$. It is immediate that G is reduced iff $T(G)$ is well-founded; if G is ranked then so is $T(G)$ (the rank of every nonzero x on $T(G)$ is its rank in G, the rank of the tree is the length of G). As we noticed, however, because of the intricate structure of the equimorphism classes of reduced p-groups, this operation can preserve neither non-isomorphism nor non-embedding.

5.3. Proofs of arithmetic comprehension.

Proposition 5.19 (RCA_0)**.** *$\exists$-$\mathsf{EMB}(\mathcal{R}$-p-$\mathcal{G})$ implies ACA_0, and hence it is equivalent to ATR_0.*

Proof. Let φ be a Σ^0_1 formula. For each n, construct a p-group G_n by letting $G_n = \mathbb{Z}_p$ if $\neg\varphi(n)$ and $G_n = \mathbb{Z}_{p^2}$ if $\varphi(n)$. Then $\{n : \varphi(n)\} = \{n : \mathbb{Z}_{p^2} \preccurlyeq G_n\}$. □

We next show that $\mathsf{WQO}(\mathcal{R}$-p-$\mathcal{G})$ implies ACA_0. As before we go through Σ^0_2-induction. Let T and the sequence $\langle T_n\rangle$ be as in subsection 3.3.1, and let $G_n = \mathbb{G}(T_n)$. Again assuming that $0'$ does not exist, by the results of Friedman quoted earlier (Lemma 5.13), each G_n is reduced.

Suppose $n < m$ and that $g\colon G_n \to G_m$ is an embedding. Suppose that $\sigma \in T_n$ is a true string. Considered as an element of G_n, we write $g(\sigma)$ in normal form as $\sum_{i<k} m_i\sigma_i$, where $\sigma_i \in T_m$ and $m_i \in \mathbb{Z}_p$. We claim that every σ_i is true. Suppose that some σ_j is not true; let r be the height of the (finite) tree $T_m[\sigma_j]$. By Σ^0_2-induction, there is some true $\tau \supset \sigma$ on T_n which is sufficiently long so that $p^s\tau = \sigma$, where $s > r - |\sigma_j|$. Then $p^s g(\tau) = g(\sigma)$. Writing $g(\tau)$ in normal form as $\sum_{i<k'} n_i\tau_i$ and multiplying by p^s, we get

$$g(\sigma) = \sum_{i<k} m_i\,\sigma_i = \sum_{i<k'} n_i\,\tau_i',$$

where τ_i' is τ_i with the last s bits chopped off. Thus the set of the τ_i's equals the set of the σ_is, which shows that some τ_i is an extension of σ_j of length $> r$. This is impossible.

By the same kind of calculation, we see that if $\sigma_0 \subsetneq \sigma_1 \in T_n$ are true, then each τ appearing in the normal form of $g(\sigma_0)$ is properly extended by some τ' which appears in $g(\sigma_1)$. This shows that if $\sigma \in T_n$ is true then via g we can obtain some true $\tau \in T_m$ of length at least $|\sigma|$. This allows us to iterate and get $0'$.

Next, we see that Σ^0_2-induction follows from $\mathsf{WQO}(\mathcal{R}$-p-$\mathcal{G})$. This follows the proof of Lemma 3.32. As for Boolean algebras, all we really need is that in RCA_0:

(1) If α, β are ordinals and $\alpha \preccurlyeq \beta$ then $G(\alpha) \preccurlyeq G(\beta)$.
(2) For any ordinal α, $G(\alpha + 1)$ does not embed into $G(\alpha)$.

As for Boolean algebras (subsection 4.4.1), the first follows from a direct construction, and the second follows from an analogue of fact 4.4.1 (with the same proof). Thus:

Proposition 5.20 (RCA_0)**.** *$\mathsf{WQO}(\mathcal{R}$-p-$\mathcal{G})$ implies ACA_0.*

6. Scattered and compact spaces

In this section we introduce the class of very countable topological spaces. Unfortunately, the well founded part (the collection of *scattered* spaces) of our class is not very-well behaved. We thus leave open the analysis of the class of scattered spaces and concentrate on compact spaces, which turn out to be metrizable. This allows us to refer to a rich body of research on metric spaces in reverse mathematics.

Remark 6.1. We do not give a reduction from topological spaces to other classes. In fact, we do not know whether such computable embeddings exist. There are Turing reductions from compact spaces to well-founded trees and ordinals which preserve embedding, non-embedding, isomorphism and non-isomorphism; however, they make use of a particular listing of the points of the space or of its basic open sets.

The most natural reduction from compact spaces is to the class of superatomic Boolean algebras - Stone duality. All facts about Stone duality can, in fact, be proved in ATR_0 (including the fact that the corresponding Boolean algebra is countable); however, this is not a continuous operation, so we do not consider it in this paper.

6.1. **Definitions.**

Definition 6.2. A *very countable topological space* is a set $X \subset \mathbb{N}$, equipped with a (countable) collection of subsets $\mathcal{O}_X$ which are a basis for a topology on X (i.e., for every finite subset F of $\mathcal{O}_X$ and every $x \in \bigcap F$, there exists $U \in \mathcal{O}_X$ such that $x \in U \subseteq \bigcap F$).

We assume all usual topological notions (see, for example, [Mun00]). So, for instance, $V \subseteq X$ is an *open set* (or an *$\mathcal{O}_X$-open set*) if $\forall x \in V \exists U \in \mathcal{O}_X (x \in V \subseteq U)$. Two topologies $\mathcal{O}_X$ and $\mathcal{O}'_X$ on a same set X are *equivalent* if the $\mathcal{O}_X$-opens sets are exactly the $\mathcal{O}'_X$-open sets. Note that, up to equivalence, we can assume that $\mathcal{O}_X$ is always closed under finite intersections, since closing up under finite intersections is a computable operation.

As *isomorphism* we use homeomorphisms. A one-to-one map $f\colon X \to Y$ is *bi-continuous* if it is continuous, and its inverse is continuous as well (for formalization in RCA_0, we do not assume that a map necessarily has a range; the notion of continuity still makes sense, because the range is definable.) An *embedding* of a space X into a space Y is a one-to-one, bi-continuous function. An embedding $f\colon X \to Y$ is a homeomorphism of X onto its perhaps non-existent range. We note that the standard definition of the subspace topology makes sense in this setting.

All spaces we deal with are very countable, and so we drop this prefix. Also, unless otherwise stated, all spaces are *Hausdorff.* (That is, for every $x, y \in X$, there exists disjoint $U, V \in \mathcal{O}_X$ such that $x \in U$, $y \in V$.) In Hausdorff spaces we can use familiar notions such as converging sequences and limit points. So, when we say *space* we mean very countable Hausdorff topological space.

We note that very countable topological spaces are just countable, second countable spaces. The reader familiar with these concepts should note that when a space is countable, the notions of first countable (N1) and second countable (N2) coincide. We deal with very countable topological spaces because they are the ones that can be easily encoded in Second Order Arithmetic as a set of natural numbers (rather than a class of reals).

Example (RCA_0). Let L be a linear ordering. There is a natural topology on L, the *order topology*, which makes L a very countable space: $\mathcal{O}_L$ is the set of open intervals of L, defined by endpoints in $A \cup \{-\infty, \infty\}$.

We observe that if B is a sub-ordering of A, then the order topology on B is equivalent to the subspace topology.

A "well-founded" topological space is called *scattered*.

Definition 6.3. Let X be a space. We say that $y \in Y$ is *isolated* if $\{y\} \in \mathcal{O}_Y$. A set $Y \subset X$ is *dense in itself* if as a subspace, Y has no isolated points. A space is *scattered* if it contains no subset which is dense in itself.

6.2. **Metrizable spaces.** A *metric space* consists of a set $M \subset \mathbb{N}$ and a sequence $\langle r(a,b)\rangle_{a,b\in M}$ of real numbers (real numbers in the sense of [Sim99, Chapter II], quickly converging Cauchy sequences of rationals,) satisfying the classical properties of a metric.

Recall that a metric on a set M, induces a topology on it: $\mathcal{O}_M = \{B_{x,r} : x \in M, r \in \mathbb{Q}\}$. (Of course, we can then close $\mathcal{O}_M$ under finite intersections.) A topological space is *metrizable* if there is a metric on X consistent with its topology. That is, a metric such that the topology it induces is equivalent to the topology on X.

Example (RCA_0). If α is an ordinal, then there is a canonical embedding of α into the interval $(0,1)$ - see [FH91]. This embedding is bi-continuous with respect to the order topology on α, which shows that α, as a topological space, is metrizable.

A topological space X is *normal* if it is Hausdorff, and for every disjoint closed sets $C, D \subset X$, there exists disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. X is *regular* if the condition above holds when C is a singleton.

Lemma 6.4. *A space X is regular iff for all $x \in X$ and all open neighborhoods U of x, there is some neighborhood V of X such that $\bar{V} \subset U$.*

The proof in [Mun00, Theorem 31.1] goes through in ACA_0.

We note that being an open subset of X is an arithmetic property. Similarly, the relation $x \in \bar{A}$ is arithmetic, as the closure of $A \subset X$ is the collection of points which do not have basic open neighborhoods disjoint from A. As a corollary of the previous lemma, we notice that regularity is arithmetically definable, as a space X is regular iff for all $x \in X$ and all *basic* open neighborhoods U of x there is some *basic* neighborhood V of x such that $\bar{V} \subset U$. Further, we note that if X is regular, then the function taking some $x \in X$ and a basic neighborhood U of x to the least (according to some fixed enumeration) basic neighborhood V of x such that $\bar{V} \subset U$ is arithmetically definable. It is not "topological" as it depends on the enumeration of $\mathcal{O}_X$ - so does not respect homeomorphism. Of course, we view the structure X as equipped with some enumeration of $\mathcal{O}_X$, so the function is indeed definable from X.

Similarly, the function taking some closed $A \subset X$ and $x \in X \setminus A$ to the least basic neighborhood of x disjoint from B is arithmetically definable.

Lemma 6.5 (ACA_0). *Every regular space is normal.*

Proof. Essentially, the proof is the standard one, given for example in [Mun00, Theorem 32.1]. All we need is to note that when, for each $x \in A$, we choose a basic open neighborhood of x whose closure is disjoint from B, we can do that arithmetically, and so the resulting cover exists. □

The proof in fact yields more: if X is regular, then there is an arithmetically definable function which takes disjoint, closed $A, B \subset X$ to a pair of open subsets of X which separate A and B. Or, equivalently, given some closed A and open $B \supset A$ we can get in an arithmetical way some open $C \supset A$ such that $\bar{C} \subset B$.

We need to refine even further. Let X be a normal space. Fix some enumeration $\langle U_n\rangle_{n\in\mathbb{N}}$ of $\mathcal{O}_X$. Suppose that $A \subset X$ is open. An *open presentation* of A is a set $N \subset \mathbb{N}$ such that $A = \cup_{n\in N} U_n$. A *closed presentation* of a closed set $B \subset X$ is an open presentation of $X \setminus B$. Among all open presentations of some open $A \subset X$

there is one maximal; it is of course $\{n : U_n \subset A\}$. Given some open set $A \subset X$, ACA_0 ensures that some presentation of A, in fact its maximal one, exists.

ACA_0 ensures the existence of sets such as $\{(x,n) : x \in \bar{U}_n\}$, $\{(n,m) : \bar{U}_n \subset U_m\}$ and functions such as the one taking n to the maximal closed presentation of $\bar{U}_n$ and (n,m) to the maximal closed presentation of $U_n \cap U_m$. When we fix these sets and functions as *oracle*, we can, by the proof of Lemma 6.5, *effectively* construct, given some closed presentations of disjoint $A, B \subset X$, open sets U, V separating A and B; further, we can effectively construct open presentations of U and V. This allows us to iterate the process of finding separators, which shows that the proof of Urysohn's Lemma goes through in ACA_0:

Lemma 6.6 (ACA_0)**.** *(Urysohn's Lemma* [Mun00, Theorem 33.1]*) If X is a normal space and A, B are disjoint closed subsets of X, then there is some continuous $f : X \to [0,1]$ such that $A \subset f^{-1}\{0\}$ and $B \subset f^{-1}\{1\}$.*

Remark 6.7. We may assume that $\operatorname{range} f \subset \mathbb{Q}$. For given f, we can use the standard "forth" argument to get an order-preserving, hence continuous, $g : \operatorname{range} f \to \mathbb{Q}$; ACA_0 ensures that $<\restriction \operatorname{range} f$ exists.

We also remark that f is obtained effectively given the discussed oracle, uniformly in (closed presentations of) A and B. This uniformity allows us to see that if X is regular then there is a countable collection of functions $f : X \to [0,1]$ such that for any $x \in X$ and any neighborhood U of x, some f in the collection is positive at x and vanishes outside U. The rest of the proof of the Urysohn metrization theorem goes through in ACA_0:

Theorem 6.8 (ACA_0)**.** *(Urysohn metrization theorem* [Mun00, Theorem 34.1]*) Let X be a space. The following are equivalent.*

(1) *X is regular.*
(2) *X is normal.*
(3) *X is metrizable.*

Moreover, given a sequence of regular topological spaces, there is a sequence of metrics for them.

Example (ACA_0). The order topology of every linear ordering is regular and hence metrizable.

6.3. **Compact spaces.**

Definition 6.9. A space X is *compact* if every open covering of X which consists of basic open sets, contains a finite sub-covering. That is, if for every sequence of basic open sets $\langle U_n \rangle_{n \in \mathbb{N}}$ such that $X \subseteq \bigcup_{n \in \mathbb{N}} U_n$, there exists a finite $F \subset \mathbb{N}$ such that $X \subseteq \bigcup_{n \in F} U_n$.

We note that when working in ZFC, the definition of compactness requires consideration of uncountable open coverings. But if a space is countable, then every open covering has a countable sub-covering. Further, if a space is very countable, then from an arbitrary countable open covering, we can find a refinement which is both countable and consists of basic open sets. So, when dealing with very countable spaces, the definition of compactness given above is equivalent to the usual one.

Definition 6.10. A space X is *sequentially compact* if every infinite sequence of elements of X has a converging subsequence.

The trick of having every infinite Σ^0_1 class containing an infinite set yields the following.

Lemma 6.11 (RCA_0). *A space X is sequentially compact iff every infinite subset of X has a limit point in X.*

The standard proofs of equivalence, formalized, yield the following:

Lemma 6.12 (RCA_0). *Every sequentially compact space is compact.*

Lemma 6.13 (ACA_0). *Every compact space is sequentially compact.*

We observe that when working in ZFC with countable, compact, Hausdorff spaces, the condition of very countability comes for free.

Observation. (ZFC) Every countable, compact, Hausdroff topological space is second countable, and hence very countable.

Proof. Suppose that X is not second countable. Then, it is not first countable either. That means that there exists an $x \in X$ which has no countable basis of open neighborhoods, i.e., such that for every decreasing sequence $\langle U_n \rangle_{n\in\mathbb{N}}$ of open neighborhoods of x, there exists an open neighborhood V of x such that for no n, $U_n \subseteq V$.

Let $\langle x_n \rangle_{n\in\omega}$ be an enumeration of $X \smallsetminus \{x\}$. We will construct a subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ with no converging subsequence, contradicting the compactness of X. First, using the fact that X is Hausdorff construct two sequences of open sets $\langle U_n \rangle_{n\in\omega}$ and $\langle V_n \rangle_{n\in\omega}$ such that for every n, U_n and V_n are disjoint neighborhoods of x and x_n respectively, $U_{n+1} \subsetneq U_n$. Now there is an open set W containing x such that for no n, $U_n \subseteq W$. Define $\langle x_{n_k} \rangle_{k\in\omega}$ as follows: Let $x_{n_0} \in U_0 \setminus W$; given x_{n_k}, let $m > n_k$ be such that $x_{n_k} \notin U_m$ and let $x_{n_{k+1}} \in U_m \smallsetminus W$. Now, x is not a limit of any subsequence of $\langle x_{n_k} \rangle_{k\in\omega}$ because W is a neighborhood of x which contains no point in that sequence. Also, any point x_n is not a limit of any subsequence of $\langle x_{n_k} \rangle_{k\in\omega}$ because V_n is a neighborhood of x_n which contains no point x_{n_k} for $n_k > n$. □

Observation. The above proof can be carried through in a subsystem of second order arithmetic with sufficiently much choice, provided that it is meaningful: that is, when the given space is a definable class.

We now see that compact spaces are nice and well-founded. For the first, the standard proof will do.

Lemma 6.14 (ACA_0). *Every compact space is normal.*

Lemma 6.15 (ACA_0). *Every compact space is scattered.*

Proof. Suppose that X is a topological space and suppose that $Y \subset X$ is dense in itself. We will construct a sequence $\langle y_n \rangle_{n\in\omega} \subseteq Y$ with no convergent subsequence. Let $\langle x_n \rangle_{n\in\omega}$ be an enumeration of X. Let y_0 be such that there exits disjoint basic open sets U_0 and V_0 around y_0 and x_0 respectively. Suppose we have already defined y_n and U_n, and $y_n \in U_n$. Let $y_{n+1} \in U_n$ be any point different from x_{n+1}, which exists because Y is dense in itself. Let $U_{n+1} \subseteq U_n$ and V_{n+1} be basic open neighborhoods of y_{n+1} and x_{n+1} respectively. Now, since for every n, $x_n \in V_n$ and $\forall m \geq n \ (y_m \notin V_n)$, $\langle y_n \rangle$ has no convergent subsequence. □

We shall need to following basic facts:

(1) If X is compact and $Y \subset X$ is closed, then Y is compact (in the subspace topology).
(2) If $Y \subset X$ is compact and X is Hausdorff then Y is closed in X.

Standard proof go through in ACA_0, so we get (remembering that all spaces are Hausdorff):

Fact (ACA_0). If X is compact and $f\colon X \to Y$ is one-to-one and continuous, then f is closed (so f is an embedding).

6.3.1. *Spaces of well-orderings.* In the following, by *complete* spaces we of course mean perhaps uncountable, definable spaces.

Lemma 6.16 (ATR_0)**.** [FH91] *Every countable, closed, totally bounded subset of a complete separable metric space is homeomorphic to the canonical metric space of some well-ordered set.*

Lemma 6.17 (ACA_0)**.** *Let X be a topological space. The following are equivalent.*

(1) *X is compact.*
(2) *X is homeomorphic to a countable, closed, totally bounded subset of a complete, separable metric space.*

Proof. If X is compact, then we can consider it as a metric space. We let $\bar{X}$ be the completion of X (see [Sim99, Section II.5]). This is a complete, separable metric space. Since X is compact, it is a closed in $\bar{X}$ and is also totally bounded.

For the other direction see [Mun00, Theorem 45.1]. □

Thus every compact space is homeomorphic to the order topology of some ordinal.

Lemma 6.18 (RCA_0)**.** *Let α be an ordinal. Then $\alpha + 1$ is a compact space.*

This can be derived in ACA_0 from theorems 2.2 and 2.3 of [FH91].

Proof. Let $\langle U_n \rangle$ be a sequence of basic open sets which cover $\alpha + 1$. The point is that from n we can get the pair (a_n, b_n) defining U_n. Thus, if $\langle U_n \rangle$ does not have a finite sub-cover, we can inductively choose a descending sequence $\langle c_k \rangle$ in α as follows: together with $\langle c_k \rangle$, we find a sequence $\langle U_{n_k} \rangle$ such that for each k, $(c_k, \infty) \subset \cup_{l \leqslant k} U_{n_l}$. Given c_k, we let $U_{n_{k+1}}$ be some open set on the list which contains c_k; and we let $c_{k+1} = a_{n_{k+1}}$. □

We can now define the reduction from ordinals to compact spaces:

Definition 6.19. Let α be an ordinal and $n \in \mathbb{N}$. The space $\mathbb{C}(\alpha, n)$ is the topological space given by the order topology on $\omega^\alpha \cdot n + 1$. We let $\mathbb{C}(\alpha) = \mathbb{C}(\alpha, 1)$.

6.4. **Ranked spaces.** The *Cantor-Bendixon derivative* X' of a space X is the collection of limit points of X (i.e., X with its isolated points removed.) Let α be an ordinal; a *partial Cantor-Bendixon resolution* of X along α is a sequence of subspaces $\langle X_\beta \rangle_{\beta < \alpha}$ such that $X_0 = X$, if $\beta + 1 < \alpha$ then $X_{\beta+1} = X'_\beta$, and for limit $\lambda < \alpha$, $X_\lambda = \bigcap_{\beta < \lambda} X_\beta$. A *Cantor-Bendixon resolution* of X is a partial Cantor-Bendixon resolution of X along some ordinal $\alpha + 1$ such that $X_\alpha = \emptyset$ but for $\beta < \alpha$, $X_\beta \neq \emptyset$. If there is a resolution of X along $\alpha + 1$ then we let $\mathrm{rk}(X) = \alpha$ (which is also called the *length* of X). If α is a successor ordinal, then the *degree* of

X is the number of points in $X_{\alpha-1}$; if α is a limit ordinal we let $\deg(X) = 0$. We define $\mathrm{inv}(X) = (\mathrm{rk}(X), \deg(X))$. We also define an associated rank function; for $x \in X$, we let $\mathrm{rk}(x)$ be the unique $\beta < \alpha$ such that $x \in X_\beta \smallsetminus X_{\beta+1}$.

Definition 6.20. A space X is *ranked* if it has a Cantor-Bendixon resolution and the associated rank function exists.

As usual, in RCA_0 we can show that any two rankings of a ranked space X are isomorphic. See, for example, 4.16.

Lemma 6.21 (ATR_0)**.** *Every scattered space is ranked.*

Proof. Friedman [Fria] proved this result for countable metric spaces. The same proof works for topological spaces. □

Lemma 6.22 (RCA_0)**.** *Every ranked space is scattered.*

Proof. Suppose that $Y \subset X$ is dense in itself. Let $\langle X_\beta \rangle_{\beta<\alpha}$ be any partial resolution of X. Then by Π^0_1-transfinite induction on $\beta < \alpha$ we can show that $Y \subset X_\beta$ for every $\beta < \alpha$. Thus $\langle X_\beta \rangle$ cannot be a full resolution of X. □

Lemma 6.23 (RCA_0)**.** *Let α be and ordinal and $n \in \mathbb{N}$. Then $\mathbb{C}(\alpha, n)$ is scattered, indeed it is ranked, and its invariant is $(\alpha + 1, n)$.*

Of course, in RCA_0, $\mathbb{C}(\alpha, n)$ is compact.

Proof. Let $X = \mathbb{C}(\alpha, n)$. Given $x = \sum_{i<k} \omega^{\beta_i} \cdot n_i \in X$, we let $\mathrm{rk}(x) = \min\{\beta_i : i < k\}$. For $\beta \leqslant \alpha+1$, let $X_\beta = \{x \in X : \mathrm{rk}(x) \geqslant \beta\}$. We claim that $\langle X_\beta : \beta \leqslant \alpha+1\rangle$ is a Cantor-Bendixon resolution of X and rk is a rank function for X. It is clear that when γ is a limit ordinal, $X_\gamma = \bigcap_{\delta<\gamma} X_\delta$. We then have to prove that for every $\gamma \leqslant \alpha$, $X_{\gamma+1} = X'_\gamma$. This follows from the fact that $X_\gamma = \{\omega^\gamma \cdot \delta : \delta \leqslant \omega^{\alpha'} \cdot n\}$, and $X_{\gamma+1} = \{\omega^\gamma \cdot \delta : \delta \leqslant \omega^{\alpha'} \cdot n$ and δ is a limit ordinal$\}$, where α' is such that $\gamma + \alpha' = \alpha$ (note that we do not assume that we can regard α' as an initial segment of α).

It follows that $\mathrm{rk}(X) = \alpha+1$ and since $X_\alpha = \{\omega^\alpha \cdot i : 0 < i \leq n\}$, $\deg(X) = n$. □

The importance of the Cantor-Bendixon invariant is that it classifies compact spaces up to isomorphism, and is also compatible with the embedding relation. For the case of countable metric spaces, Friedman essentially proved the following lemma.

Lemma 6.24 (ACA_0)**.** *([Fria]) Let X and Y be ranked countable metric spaces with invariants $\langle \alpha, n\rangle$ and $\langle \beta, m\rangle$ respectively. Then, there is an one-to-one, continuous function $f\colon X \to Y$ if and only if $\langle \alpha, n\rangle \leqslant_{\mathrm{lex}} \langle \beta, m\rangle$.*

Recalling fact 6.3, we get:

Corollary 6.25 (ACA_0)**.** *Let X and Y be ranked compact spaces. Then $X \preccurlyeq Y$ iff* $\mathrm{inv}(X) \leqslant_{\mathrm{lex}} \mathrm{inv}(Y)$.

As in earlier sections, we will want to get uniform rankings of a sequence of spaces.

Lemma 6.26 (ACA_0)**.** *Assume that every compact space is ranked. Let $\langle X_n\rangle$ be a sequence of compact spaces. Then there is an ordinal α and a sequence of functions $f_n\colon X_n \to \alpha$ such that each f_n is a rank function for X_n.*

Proof. Let Y be the disjoint union of the X_ns and let X be a simple version of the 1-point compactification of Y: The basic open subsets of X are the basic open subsets of Y, together with the sets $X \setminus X_n$ for $n \in \mathbb{N}$. It is straightforward to check that X is compact and that each X_n is an open subset of X. Thus, a ranking of X gives uniform rankings of all the X_ns (see Friedman, [Fria], to see that if $A \subset B$ is open in B and $\langle B_\gamma \rangle$ is a resolution of B, then $\langle A \cap B_\gamma \rangle$ is a resolution of A.) □

Corollary 6.27 (ACA_0)**.** $\mathsf{RK}(\mathcal{CCS})$ *implies* $\exists$-$\mathsf{EMB}(\mathcal{CCS})$, $\mathsf{COMP}(\mathcal{CCS})$ *and* $\mathsf{WQO}(\mathcal{CCS})$.

Corollary 6.28 (ATR_0)**.** *Every compact space X is homeomorphic to $\mathbb{C}(\alpha, n)$, where $(\alpha+1, n) = \mathrm{inv}(X)$.*

Proof. By lemmas 6.16 and 6.17, X is homeomorphic to the canonical metric space of some ordinal β. (Note that β has to be a successor ordinal, because otherwise X would not be compact.) Using the Cantor normal form, write β as:

$$\beta = \omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha_0 > \alpha_1 > \ldots > \alpha_k$. We can write β as

$$(\omega^{\alpha_0} \cdot n_0 + 1) + (\omega^{\alpha_1} \cdot n_1 + 1) + \ldots + (\omega^{\alpha_{k'}} \cdot n'_{k'} + 1).$$

It is not hard to prove that β is homeomorphic to

$$(\omega^{\alpha_{k'}} \cdot n'_{k'} + 1) + \ldots + (\omega^{\alpha_1} \cdot n_1 + 1) + (\omega^{\alpha_0} \cdot n_0 + 1),$$

which, as an ordinal, is isomorphic to $\omega^{\alpha_0} \cdot n_0 + 1$. Finally, $(\alpha+1, n) = \mathrm{inv}(X) = \mathrm{inv}(\omega^{\alpha_0} \cdot n_0 + 1) = (\alpha_0 + 1, n_0)$, so $\alpha_0 = \alpha$ and $n_0 = n$. □

Corollary 6.29 (ATR_0)**.** *Let X an Y be compact spaces with invariants $\langle \alpha, n \rangle$ and $\langle \beta, m \rangle$ respectively. Then, X and Y are homeomorphic if and only if $\langle \alpha, n \rangle = \langle \beta, m \rangle$.*

Corollary 6.30. ATR_0 *implies the statements* $\exists$-$\mathsf{ISO}(\mathcal{CCS})$ *and* EQU=$\mathsf{ISO}(\mathcal{CCS})$.

6.5. **Reversals.** To get reversals, we need to apply some of the aforementioned results in weaker systems. The next lemma follows from Lemma 6.23.

Lemma 6.31 (RCA_0)**.** *Let α and β be ordinals. Then $\alpha \cong \beta$ iff $\mathbb{C}(\alpha)$ and $\mathbb{C}(\beta)$ are homeomorphic.*

Corollary 6.32 (RCA_0)**.** $\exists$-$\mathsf{ISO}(\mathcal{CCS})$ *is equivalent to* ATR_0.

The next lemma follows from Lemma 6.23 and corollary 6.25.

Lemma 6.33 (ACA_0)**.** *Let α and β be ordinals. Then, $\mathbb{C}(\alpha)$ embeds in $\mathbb{C}(\beta)$ if and only if $\alpha \preccurlyeq \beta$.*

Corollary 6.34 (ACA_0)**.** *Each of* $\exists$-$\mathsf{EMB}(\mathcal{CCS})$, $\mathsf{COMP}(\mathcal{CCS})$, EQU=$\mathsf{ISO}(\mathcal{CCS})$, *and* $\mathsf{WQO}(\mathcal{CCS})$, *is equivalent to* ATR_0.

To get the EW-reduction, we first observe the following. Suppose that L is an ill-founded linear ordering; let $\langle a_n \rangle$ be an infinite descending sequence in L. We can construct a copy of $1 + \mathbb{Q}$ inside ω^L by considering all elements of the form $\omega^{a_1} n_1 + \omega^{a_2} n_2 + \ldots \omega^{a_k} n_k$ for $k, n_l \in \mathbb{N}$. Thus ω^L is not scattered. As a conclusion, we see that for all linear orderings L, if L is a well-ordering then $\mathbb{C}(L)$ is compact, and if L is ill-founded, then $\mathbb{C}(L)$ is not scattered, and so not compact. All this can be done in RCA_0.

6.6. **Proofs of arithmetic comprehension.**

Proposition 6.35 (RCA_0)**.** $\exists$-$\mathsf{EMB}(\mathcal{CCS})$ *implies* ACA_0.

Proof. See Proposition 3.28 or 5.19. □

Proposition 6.36 (RCA_0)**.** $\mathsf{RK}(\mathcal{CCS})$ *implies* ACA_0.

Proof. This is immediate, given Lemma 6.26. We construct a sequence $\langle X_n \rangle$ of compact spaces and points $x_n \in X_n$, such that if $n \in 0'$ then x_n is a limit point in X_n, and if $n \notin 0'$ then $X_n = \{x_n\}$. Then from a uniform ranking of the X_ns we can uniformly get the rank of each x_n (in X_n) and thus get $0'$. □

Proposition 6.37 (RCA_0)**.** $\mathsf{COMP}(\mathcal{CCS})$ *implies* ACA_0.

Proof. We again show that $0'$ exists by showing how to enumerate infinitely many true stages.

As in the proof of proposition 4.45, we construct a sequence of ordinals $\langle \alpha_s \rangle_{s\in\mathbb{N}}$ such that if s is a true stage, then α_s is a canonical copy of $\omega + 1$, and otherwise α_s is a copy of some $n < \omega$ (where we can tell which is the last element and what is the place of the other elements). We let $\alpha = \sum_{s\in\mathbb{N}} \alpha_s$.

It is easy to see that α is an ordinal (from a decreasing sequence in α we can construct either a decreasing sequence of $s \in \mathbb{N}$ or a decreasing sequence in some α_s). The limit points of α are exactly those last elements of α_s where s is a true stage.

Let $X = \alpha \cdot 2 + 1$, with the order topology. Let Y be a canonical copy of $\omega^2 + 1$. There cannot be an embedding of X into Y. This is because for any embedding, the image of a limit point is a limit point and the image of a limit of limit points is also a limit of limit points; of which X has two but Y only one.

By $\mathsf{COMP}(\mathcal{CCS})$, there is an embedding g of Y into X. Write $X = \alpha_0 + \alpha_1 + 1$ (α_i is a copy of α), and let A be the class of limit points of X. For some $i < 2$, $B_i = A \cap g^{-1}\alpha_i$ is infinite. For such i, $g \restriction B_i$ allows us to enumerate infinitely many limit points of α, and so infinitely many true stages. □

Proposition 6.38 (RCA_0)**.** EQU=$\mathsf{ISO}(\mathcal{CCS})$ *implies* ATR_0.

Proof. Let α and β be ordinals; we will prove that they are comparable. Note that $\delta = \alpha + \beta + \alpha + ...$ and $\gamma = \beta + \alpha + \beta + \alpha + ...$ are equimorphic via continuous embeddings. So, $\omega^\delta + 1$ and $\omega^\gamma + 1$ are also equimorphic by continuous embeddings, and hence equimorphic as compact spaces. By EQU=$\mathsf{ISO}(\mathcal{CCS})$ we have that they are homeomorphic. Then, by Lemma 6.31, δ and γ are isomorphic. It follows that α and β are comparable. □

Proposition 6.39 (RCA_0)**.** $\mathsf{WQO}(\mathcal{CCS})$ *implies* ACA_0.

Proof. As usual, we first work over RCA_2.

We define β^n_t and α_n in a similar way to what is done in the proof of proposition 4.45; in this case, $\beta^n_t = \omega + 1$ exactly when t is the k^{th} true stage for some $k > 2n$; $\alpha_n = \sum_t \beta^n_t$. Let $X_n = \alpha_n + 1$ with the order topology; every X_n is compact. Suppose that $n < m$ and that f is an embedding of X_n into X_m.

Let t_k be the $(2n+k)^{th}$ true stage. For all t, let $a_t = \max \beta^n_t$ and let $b_t = \max \beta^m_t$. For all $k > 0$, a_{t_k} is a limit point of X_n. Consider $f(a_{t_1})$ and $f(a_{t_2})$. At least one of them is in α_m (that is, it is not the last limit point we added to make X_m compact), and in fact, it has to be b_{t_k} for some $k > 2$, say, for example, $k = 7$. From t_7 we can find $t_3, t_4, \dots, t_6$. From $f(a_{t_1}), f(a_{t_2}), \dots, f(a_{t_7})$, at least six are in α_m and are b_{t_k}s for distinct $k > 2$; thus at least one must be b_{t_k} for $k > 7$. Now the process repeats to get all t_k.

The rest (getting $I\Sigma_2$ from $\mathsf{WQO}(\mathcal{CCS})$) is identical to the proof in all the previous sections. That is, we verify in RCA_0 that if $\alpha \preccurlyeq \beta$ then $\mathbb{C}(\alpha) \preccurlyeq \mathbb{C}(\beta)$, and that the analog of fact 4.4.1 holds for compact spaces, and so that $\mathbb{C}(\alpha+1)$ does not embed into $\mathbb{C}(\alpha)$. □

7. Up to equimorphism, hyperarithmetic is recursive

In this section we prove the theorems in subsection 1.6. As we mentioned, the first two are not difficult. They rely on the fact, already mentioned in [AK00] for Boolean algebras and groups, that if X is a hyperarithmetic structure in the well-founded part of any of the classes we considered, its rank is computable. This is simple; we give a general proof: for each class $\mathcal{X}$ we considered, we showed that ATR_0 suffices to prove that each well-founded structure is ranked. By [Sim99, Corollary VII.2.12], we know that there is a β-model M of ATR_0 which consists of hyper-low sets. Each hyperarithmetic structure X in $\mathcal{X}$ is in M; a rank for X, in the sense of M, exists in M; and since M is Σ^1_1-correct, this rank is really an ordinal.

Since the invariant of a compact space determines its isomorphism type, we get Theorem 1.10 immediately. This also yields the result 1.8 for superatomic Boolean algebras mentioned in [AK00]. We also know that the rank of a well-founded tree determines its equimorphism type, so we get that every hyperarithmetic well-founded tree is equimorphic with a recursive one.

Theorem 1.9 follows, because we know that if B is a Boolean algebra which is not superatomic, then it contains a copy of the atomless Boolean algebra, into which every countable Boolean algebra can be embedded. And if T is an ill-founded tree, then every countable tree can be embedded into T.

We turn to the third theorem.

Proof of Theorem 1.11: Let G be a hyperarithmetic p-group. If G is reduced, then (see [AK00, Theorem 8.17]) it has some length $\alpha + n < \omega_1^{CK}$, where α is a limit ordinal and $n < \omega$. By [Bar95, Proposition 4.3] and [BE71, Theorem 4.1], there is a recursive group H of length $\alpha + n$ such that for all $\beta < \alpha$, $U_H(\beta) = \infty$ and for all $m \leqslant n$, $U_H(\alpha+m) = U_G(\alpha+m)$. From Lemma 5.8 and Remark 5.7 we obtain that H and G are equimorphic.

Suppose now that G is not reduced. It can be written as a sum $G_d + G_r$, where G_d is divisible and G_r is reduced (see [Kap69, Theorem 3]). Every countable divisible p-group is of the form $\mathbb{Z}(p^\infty)^m$, for some $m \leqslant \omega$ (see [Kap69, Theorem 4]), and hence has a recursive copy. Again by [AK00, Theorem 8.17], G_r has some length $\alpha \leqslant \omega_1^{CK}$. If G_r has length $\alpha < \omega_1^{CK}$, by the previous argument it is equimorphic to a recursive group, and hence $G = G_d + G_r$ is too. Suppose now that $\alpha = \omega_1^{CK}$. We claim that then, $G_d \cong \mathbb{Z}(p^\infty)^\omega$, and hence G is equimorphic to $\mathbb{Z}(p^\infty)^\omega$. (Note that any countable p-group embeds in $\mathbb{Z}(p^\infty)^\omega$.) Suppose instead, toward a contradiction, that $G_d \cong \mathbb{Z}(p^\infty)^n$ for some $n < \omega$. Note that from Remark 5.7 we get that for all $\beta < \omega_1^{CK}$, $r(p^\beta G) = \infty$. Consider the partial ordering P whose elements are $n+1$ tuples of independent elements of G, and such that $\langle x_0, \dots, x_n\rangle \leqslant \langle x'_0, \dots, x'_n\rangle$ iff there exists some $k \in \mathbb{N}$ such that for every $i \leqslant n$ $p^k x_i = x'_i$. We claim that P is well founded and has rank $\leqslant \omega_1^{CK}$. This will be a contradiction because P is hyperarithmetic. We prove this by defining a rank function on P. Given $\langle x_0, \ldots, x_n \rangle \in T$, first write each x_i as $y_i + z_i$ where $y_i \in G_r$ and $z_i \in G_d$, and then let $g(\langle x_0, \ldots, x_n \rangle) = \min\{\mathrm{rk}_{G_r}(y_i) : i \leqslant n\}$, where $\mathrm{rk}_{G_r}(0) = \infty$. We claim that $g\colon P \to \omega_1^{CK}$ is a rank function. First, we observe that for no $\bar{x} \in P$, $g(\bar{x}) = \infty$: If $g(\langle x_0, \ldots, x_n \rangle) = 0$ then for all $i \leqslant n$, $y_i = 0$, and hence $x_i = z_i \in G_d$. but this cannot be the case because, since $r(G_d) = n$, $\{x_0, \ldots, x_n\}$ cannot be an independent set. Second, we observe that if $\langle x_0, \ldots, x_n \rangle < \langle x'_0, \ldots, x'_n \rangle$, then $g(\langle x_0, \ldots, x_n \rangle) < g(\langle x'_0, \ldots, x'_n \rangle)$: This is because if $g(\langle x_0, \ldots, x_n \rangle) = \mathrm{rk}_{G_r}(x_{i_0})$, then $g(\langle x'_0, \ldots, x'_n \rangle) \leqslant \mathrm{rk}_{G_r}(x'_{i_0}) < \mathrm{rk}_{G_r}(x_{i_0})$. Last, we show that if $\beta < g(\langle x_0, \ldots, x_n \rangle)$, there exists some $\langle x'_0, \ldots, x'_n \rangle < \langle x_0, \ldots, x_n \rangle$ such that $g(\langle x'_0, \ldots, x'_n \rangle) \geqslant \beta$. By definition of rk_{G_r}, for each $i \leqslant n$ there exists x'_i such that $x'_i p = x_i$ and $\mathrm{rk}_{G_r}(x'_i) \geqslant \beta$. Clearly $\langle x'_0, \ldots, x'_n \rangle < \langle x_0, \ldots, x_n \rangle$ and $g(\langle x'_0, \ldots, x'_n \rangle) \geqslant \beta$. We still have to prove that $\langle x'_0, \ldots, x'_n \rangle \in P$. Suppose that $\sum_{i \leqslant n} m_i x'_i = 0$. Then $\sum_{i \leqslant n} m_i x_i = p \sum_{i \leqslant n} m_i x'_i = 0$, and hence $m_i x_i = 0$ for every i. This implies that $p | m_i$ for every i. Then $\sum_{i \leqslant n} (m_i/p) x_i = \sum_{i \leqslant n} m_i x'_i = 0$, and hence $m_i x'_i = (m_i/p) x_i = 0$ for every i. We have proved that $\langle x'_0, \ldots, x'_n \rangle$ is an independent set and hence belongs to P. The fact that P has rank ω_1^{CK} follows from the fact that for all $\beta < \omega_1^{CK}$, $r(p^\beta G) = \infty$. □

References

[AK00] C. J. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[Bar95] Ewan J. Barker. Back and forth relations for reduced abelian p-groups. *Ann. Pure Appl. Logic*, 75(3):223–249, 1995.

[BE71] Jon Barwise and Paul Eklof. Infinitary properties of abelian torsion groups. *Ann. Math. Logic*, 2(1):25–68, 1970/1971.

[Cal04] Wesley Calvert. The isomorphism problem for classes of computable fields. *Arch. Math. Logic*, 43(3):327–336, 2004.

[CCKM] Wesley Calvert, D. Cummins, Julia F. Knight, and Sara Miller. Comparing classes of finite structures. To appear in Algebra and Logic.

[CG01] Riccardo Camerlo and Su Gao. The completeness of the isomorphism relation for countable Boolean algebras. *Trans. Amer. Math. Soc.*, 353(2):491–518 (electronic), 2001.

[CMS04] Peter Cholak, Alberto Marcone, and Reed Solomon. Reverse mathematics and the equivalence of definitions for well and better quasi-orders. *J. Symbolic Logic*, 69(3):683–712, 2004.

[FH90] Harvey M. Friedman and Jeffry L. Hirst. Weak comparability of well orderings and reverse mathematics. *Ann. Pure Appl. Logic*, 47(1):11–29, 1990.

[FH91] Harvey M. Friedman and Jeffry L. Hirst. Reverse mathematics and homeomorphic embeddings. *Ann. Pure Appl. Logic*, 54(3):229–253, 1991.

[Fria] Harvey Friedman. Metamathematics of comparability. Manuscpript dated October 2001.

[Frib] Harvey Friedman. Metamathematics of ulm theory. Manuscpript dated November 2001.

[FSS83] Harvey M. Friedman, Stephen G. Simpson, and Rick L. Smith. Countable algebra and set existence axioms. *Ann. Pure Appl. Logic*, 25(2):141–181, 1983.

[Gao04] Su Gao. The homeomorphism problem for countable topological spaces. *Topology Appl.*, 139(1-3):97–112, 2004.

[GK02] Sergei S. Goncharov and Julia Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[Hir] Jeffry L. Hirst. A survey of the reverse mathematics of ordinal arithmetic. To appear in Reverse Mathematics 2001, edited by S.G. Simpson.

[Hir94] Jeffry L. Hirst. Reverse mathematics and ordinal exponentiation. *Ann. Pure Appl. Logic*, 66(1):1–18, 1994.

[Hir00] Jeffry L. Hirst. Reverse mathematics and rank functions for directed graphs. *Arch. Math. Logic*, 39(8):569–579, 2000.

[HK95] Greg Hjorth and Alexander S. Kechris. Analytic equivalence relations and Ulm-type classifications. *J. Symbolic Logic*, 60(4):1273–1300, 1995.

[HK01] Greg Hjorth and Alexander S. Kechris. Recent developments in the theory of Borel reducibility. *Fund. Math.*, 170(1-2):21–52, 2001. Dedicated to the memory of Jerzy Łoś.

[Kap69] Irving Kaplansky. *Infinite abelian groups*. Revised edition. The University of Michigan Press, Ann Arbor, Mich., 1969.

[Kop89] Sabine Koppelberg. *Handbook of Boolean algebras. Vol. 1*. North-Holland Publishing Co., Amsterdam, 1989. Edited by J. Donald Monk and Robert Bonnet.

[Mon05a] Antonio Montalbán. *Beyond the arithmetic*. PhD thesis, Cornell University, Ithaca, New York, 2005.

[Mon05b] Antonio Montalbán. Up to equimorphism, hyperarithmetic is recursive. *Journal of Symbolic Logic*, 2005. To appear.

[Mun00] James R. Munkres. *Topology*. Prentice-Hall Inc., Englewood Cliffs, N.J., second edition, 2000.

[Sac90] Gerald E. Sacks. *Higher recursion theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

[Sel91] V. L. Selivanov. The fine hierarchy and definable index sets. *Algebra i Logika*, 30(6):705–725, 771, 1991.

[Sho] Richard A. Shore. Invariants, boolean algebras and $\mathtt{ACA}_0^+$. To appear in Trans. Amer. Math. Soc.

[Sho93] Richard A. Shore. On the strength of Fraïssé's conjecture. In *Logical methods (Ithaca, NY, 1992)*, volume 12 of *Progr. Comput. Sci. Appl. Logic*, pages 782–813. Birkhäuser Boston, Boston, MA, 1993.

[Sim99] Stephen G. Simpson. *Subsystems of second order arithmetic*. Springer, 1999.

[Spe55] Clifford Spector. Recursive well-orderings. *J. Symb. Logic*, 20:151–163, 1955.

E-mail address: erlkoenig@nd.edu
URL: www.nd.edu/~∼ngreenbe

Department of Mathematics, Notre Dame University, Notre Dame, IN 46556, USA

E-mail address: antonio@math.cornell.edu
URL: www.math.cornell.edu/∼antonio

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA