COMPUTING FROM PROJECTIONS OF RANDOM POINTS

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Abstract. We study the sets that are computable from both halves of some (Martin-Löf) random sequence, which we call 1/2-*bases*. We show that the collection of such sets forms an ideal in the Turing degrees that is generated by its c.e. elements. It is a proper subideal of the K-trivial sets. We characterise 1/2-bases as the sets computable from both halves of Chaitin's Ω, and as the sets that obey the cost function $\mathbf{c}(x,s) = \sqrt{\Omega_s - \Omega_x}$.

Generalising these results yields a dense hierarchy of subideals in the K-trivial degrees: For $k < n$, let $\mathcal{B}_{k/n}$ be the collection of sets that are below any k out of n columns of some random sequence. As before, this is an ideal generated by its c.e. elements and the random sequence in the definition can always be taken to be Ω. Furthermore, the corresponding cost function characterisation reveals that $\mathcal{B}_{k/n}$ is independent of the particular representation of the rational k/n, and that $\mathcal{B}_p$ is properly contained in $\mathcal{B}_q$ for rational numbers $p < q$. These results are proved using a generalisation of the Loomis–Whitney inequality, which bounds the measure of an open set in terms of the measures of its projections. The generality allows us to analyse arbitrary families of orthogonal projections. As it turns out, these do not give us new subideals of the K-trivial sets; we can calculate from the family which $\mathcal{B}_p$ it characterises.

We finish by studying the union of $\mathcal{B}_p$ for $p < 1$; we prove that this ideal consists of the sets that are robustly computable from some random sequence. This class was previously studied by Hirschfeldt, Jockusch, Kuyper, and Schupp [24], who showed that it is a proper subclass of the K-trivial sets. We prove that all such sets are robustly computable from Ω, and that they form a proper subideal of the sets computable from every (weakly) LR-hard random sequence. We also show that the ideal cannot be characterised by a cost function, giving the first such example of a Σ^0_3 subideal of the K-trivial sets.

Contents

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1. Introduction

If two infinite binary sequences are chosen independently at random, we would expect them not to encode common noncomputable information. This is the case, but Martin-Löf randomness is not strong enough to guarantee this kind of independence. Martin-Löf's notion of randomness is the most widely used in the theory of algorithmic randomness, an area that attempts to label individual binary sequences as "random". For a randomness notion to make sense it must hold of almost every sequence, and it must restrict the behavior of a sequence so that it has natural properties in common with almost every sequence. We want, for example, random sequences to have an equal number of 0s and 1s in the limit. This is a natural property shared by almost every sequence. On the other hand, we do not want to go overboard: an infinite binary sequence $X \in 2^\omega$ always has the unusual propery of being in the singleton set $\{X\}$, a property shared with no other sequence. So if we are to label sequences as "random", we must limit ourselves to natural properties. In practice, we must specify a countable collection of measure zero sets that "cover" the nonrandom sequences. In the case of Martin-Löf randomness, we use the "effective measure zero sets". These are the sets of sequences for which there is an algorithm that takes as input a rational $\varepsilon > 0$ and, as output, generates an open cover of the set with measure less than ε.

Martin-Löf randomness is strong enough to guarantee many of the properties we would want from random sequences. For example, they have an equal number of 0s and 1s in the limit, and in fact, satisfy the law of the iterated logarithm; when viewed as real numbers, they are points of differentiability for every computable function of bounded variation (Demuth [13], see also [9]); and they satisfy Birkhoff's ergodic theorem for computable ergodic systems with respect to effectively closed sets [5, 18]. On the other hand, things get interesting when we look at properties of typical sequences with respect to information content. We already alluded to an example above: independent Martin-Löf random sequences can compute the same noncomputable information. The subject of this paper is to understand exactly how complex such shared information can be. A simpler example of Martin-Löf random sequences having an unusual property was given by Kučera [27] and Gács [20]. They showed that every sequence is computable from some random sequence. Even an incomplete random sequence may be Turing above a noncomputable, computably enumerable (c.e.) set (Kučera [28]). The set of sequences that are Turing above noncomputable c.e. sets has measure zero, but it is not an effective measure zero set.

This latter failure gives rise to a dual question: what kind of c.e. sets can be computed by an incomplete random sequence?[1] The answer to this question is now known [25, 12, 4]: these are the *K-trivial* c.e. sets. The notion of K-triviality was introduced by Solovay [37] as the antithesis of randomness: while random sequences can be characterised as those whose initial segments cannot be compressed beyond their length, the K-trivial sequences are those whose initial segments are maximally compressible and contain no information beyond their length. The K-trivial sets are computationally weak (close to being computable). There are only countably many of them (Chaitin [10]); they are all computable from c.e. K-trivial sets (Nies [33]); and they coincide with the *low for random* sets: the sets A such that

[1]Henceforth in this paper "random" means Martin-Löf random.

every ML-random sequence is ML-random relative to A (Nies [33]). The robustness of K-triviality has been demonstrated through a series of characterisations, including several in terms of notions of weakness. One such notion is being a *base for randomness*: A is K-trivial if and only if it is computable from some sequence which is random relative to A (Hirschfeldt et al. [25]). The class of K-trivial sets induces an ideal in the Turing degrees.

The join $X \oplus Y$ of binary sequences X, Y is the sequence

$$Z = X(0)Y(0)X(1)Y(1)\dots$$

that alternates between X and Y; we call X and Y the "halves" of Z. If two sequences X and Y are random relative to each other and A is computable from both X and Y, then A is K-trivial. Two sequences X and Y are random relative to each other if and only if the pair (X, Y) is random if and only if the join $X \oplus Y$ is random. For this reason we say that A is a 1/2-*base* if there are relatively random sequences X and Y, both of which compute A. Note that if X and Y witness that A is a 1/2-base, then both X and Y are random relative to A. Thus, every 1/2-base is also a base for randomness and hence K-trivial. However, not every K-trivial set is a 1/2-base (Bienvenu et al. [6].) This leads to three questions:

(1) Are there natural witnesses for a set being a 1/2-base?
(2) What structure do the 1/2-base sets induce in the Turing degrees?
(3) In what way can the 1/2-base sets be characterised?

We answer these questions in this paper. To answer questions (1) and (2) we show:

Theorem 1.1.

(1) *Chaitin's halting probability Ω is a universal witness for being a 1/2-base. That is, a set A is a 1/2-base if and only if it is computable from both halves of Ω.*
(2) *The 1/2-base degrees form a Σ^0_3 ideal in the Turing degrees which is generated by its c.e. elements; the two halves of Chaitin's Ω are an exact pair for this ideal.*

What would constitute an answer for the third question? In general, how can we characterise subclasses of the K-trivials in a coherent way? Most of the currently known 15 or so equivalent definitions of K-triviality cannot be modified to distinguish between single K-trivial sequences. K-triviality itself means having the minimal possible initial segment complexity; lowness for randomness means not derandomizing any random sequence—these are extreme properties, and it is not obvious how they can be adapted to yield subclasses of the K-trivial sets. However, there is one characterisation of the K-trivial sets that is amenable to fine-tuning: characterisation by cost functions.

We will explore cost functions in detail in Section 2. Informally, a cost function $\mathbf{c}(x, s)$ tells us how expensive it is for a computable approximation $\langle A_s \rangle$ of a Δ^0_2 set A to change on some value x at some stage s; a set A obeys a cost function if the total cost accrued along some approximation is finite. Obeying a cost function tells us that a set has an approximation with "few" changes and so it is likely to be computationally weak. Varying the cost function allows us to quantify this notion: roughly, the higher the cost, the fewer the changes permitted and so the closer a set obeying the cost function is to being computable. The K-trivial sets themselves are the sets that obey the cost function $\mathbf{c}_\Omega(x, s) = \Omega_s - \Omega_x$ (Nies [35], extending an argument in [33]).

A sufficiently more demanding cost function, i.e., one that makes changes substantially more expensive, will be obeyed by some but not all K-trivial sets. This gives us the flexibility to explore behavior within the ideal of K-trivial sets. The biggest success along these lines was the characterisation of the *strongly jump traceable* sets (which form a proper subideal of the K-trivial sets) using a natural family of cost functions [22, 14].[2] In this paper we answer question (3) by showing that a set is a 1/2-base if and only if it obeys the cost function $\mathbf{c}_{\Omega,1/2}(x,s) = \sqrt{\Omega_s - \Omega_x}$.

The results described so far are a special case. By generalising we will obtain a dense hierarchy of subideals of the K-trivial sets.

Definition 1.2. Let $1 \leqslant k < n$. A set A is a *k/n-base* if there is a random n-tuple $(Z_1, Z_2, \ldots, Z_n)$ such that A is computable from the join of any k of the sets $Z_1, Z_2, \ldots, Z_n$.[3]

For the time being the notation "k/n-base" should not be taken literally as a fraction (but more akin to the expression dy/dx). We will justify this notation by showing that there are subideals of the K-trivial sets $\mathcal{B}_p$ indexed by rational numbers $p \in (0,1)$ which respect order ($p < q$ if and only if $\mathcal{B}_p \subset \mathcal{B}_q$) and such that a set is a k/n-base if and only if it is in $\mathcal{B}_{k/n}$. We note that a priori there is no reason to believe for example that every 1/2-base is also a 2/4-base, but in fact these notions are equivalent.

Theorem 1.3. *Let $1 \leqslant k < n$.*

(1) *The n-columns of Chaitin's Ω (again see Section 2.2) are a universal witness for being a k/n-base: a set A is a k/n-base if and only if it is computable from the join of any k of the n-columns of Ω.*
(2) *A set A is a k/n-base if and only if it obeys the cost function $\mathbf{c}_{\Omega,k/n}(x,s) = (\Omega_s - \Omega_x)^{k/n}$. The collection $\mathcal{B}_{k/n}$ of the k/n-base degrees is a Σ^0_3 ideal in the Turing degrees which is generated by its c.e. elements.*

The proof of Theorem 1.3 relies on the technical notion of weak obedience to cost functions which we introduce in Section 2. We thus delay giving an outline of the proof until the end of that section. We remark that it has already been noticed by Hirschfeldt et al. [24] that each k/n-base is K-trivial; if $(Z_1, \ldots, Z_n)$ witnesses that A is a k/n-base then for all $j \leqslant n$, by van Lambalgen's theorem, Z_j is random relative to $Z_{\neq j}$ (the join of the other Z_i); the latter computes A and so Z_j is random relative to $Z_{\neq j} \oplus A$; using van Lambalgen's theorem relative to A, we see that the tuple $(Z_1, \ldots, Z_n)$ is A-random, and so A is a base for randomness.

The notion of a k/n-base is in fact still not the most general one which we can define. Given a collection $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ we call a set A an *$\mathcal{F}$-base* if there is some random tuple $(Z_1, Z_2, \ldots, Z_n)$ such that A is computable from the join $\bigoplus_{i \in F} Z_i$ for all $F \in \mathcal{F}$. This notion is of independent interest but will actually be needed in the proof of Theorem 1.3 for "degenerate" k/n-bases. We will show that there is a rational number $\|\mathcal{F}\| \geqslant 1$ such that a set A is an $\mathcal{F}$-base if and only

[2] Jump traceability offers another tool to distinguish K-trivial sets: the slower the growth of the trace bound, the smaller the class of sets that is jump traceable with that bound. A sufficiently slow bound guarantees K-triviality. Unfortunately, jump traceability does not seem to be well-suited to our task because it is not even known if K-triviality itself can be characterised using jump traceability with computable bounds.

[3] see Section 2.2 below for a discussion of tuples and their joins.

if it is in the ideal $\mathcal{B}_{1/\|\mathcal{F}\|}$, and that in a weak sense, Ω serves as a universal witness for being an $\mathcal{F}$-base.

The sequence of ideals $\langle \mathcal{B}_p \rangle$ naturally defines two ideals: $\mathcal{B}_{<1} = \bigcup_p \mathcal{B}_p$, and $\mathcal{B}_{>0} = \bigcap_p \mathcal{B}_p$ (where again in both the union and the intersection, p ranges over the rational numbers in the open interval $(0,1)$). Both ideals have interesting properties.

Theorem 1.4. *$\mathcal{B}_{>0}$ is the ideal of sets which are $1/\omega$-bases: the sets which are computable from each Z_n in an infinite random sequence $(Z_1, Z_2, \dots)$.*[4]

The ideal $\mathcal{B}_{<1}$ is related to coarse computability. A *coarse description* of a set $A \in 2^\omega$ is a set $B \in 2^\omega$ such that the density of the symmetric difference $A \triangle B$ is 0. Say that a set A is *robustly computable* from a set Z if A is computable from every coarse description of Z. This notion has been investigated by Hirschfeldt et al. [24], where they show that if A is robustly computable from a random set then it is K-trivial, in fact it is an $(n-1)/n$-base for some n. They also prove that not every K-trivial set is robustly computable from a random set. We show:

Theorem 1.5. *The following are equivalent for a set A:*

(1) *$A \in \mathcal{B}_{<1}$ (that is, A is an $(n-1)/n$-base for some n).*
(2) *A is robustly computable from some random sequence.*
(3) *A is robustly computable from Ω.*
(4) *There is some $\varepsilon > 0$ such that A is computable from all sets B such that the upper density of $B \triangle \Omega$ is below ε.*

We remark that the result mentioned above (that not every K-trivial set is robustly computable from a random sequence) now follows from our characterisation using cost functions; see Section 6.

Finally we show that every set in the ideal $\mathcal{B}_{<1}$ is computable from all LR-hard random sequences. The notion of LR-hardness (equivalent to almost everywhere domination) appears in the investigations into relative computability between random and c.e. sets. If Z is random but is not LR-hard then it is Oberwolfach random [6] and so does not compute the "smart" K-trivial sets [6]. It is open whether this is in fact an equivalence; it is possible that every K-trivial set is computable from all LR-hard random sequences. In Section 7, we show that the collection of K-trivial sets computable from all LR-hard random sequences properly contains the ideal $\mathcal{B}_{<1}$.

We summarise our findings in Fig. 1.

2. Cost functions and the corresponding test notions

Somewhat extending [34, Section 5.3], a *cost function* is a computable function

$$\mathbf{c}\colon \mathbb{N} \times \mathbb{N} \to \{x \in \mathbb{R} : x \geqslant 0\}.$$[5]

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[4] Equivalently, we can just require that each Z_n is random relative to the join of any finite collection of other Z_k's

[5] The definition of cost functions in [34, Section 5.3] requires them to have rational values because in this case it is decidable whether a rational cost bound is satisfied. It is possible to extend the theory of cost functions to admit computable real values instead. One uses rational approximations of computable numbers in the construction of a non computable set obeying a cost function. Alternatively, for this paper we could restrict the values of cost functions to be algebraic numbers.

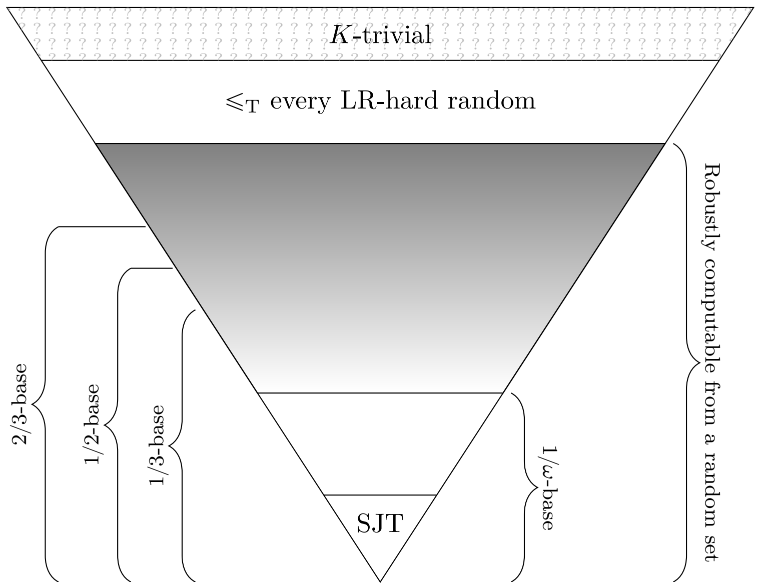

FIGURE 1. Subclasses of the K-trivial degrees.

We say that $\mathbf{c}$ is *monotonic* if $\mathbf{c}(x+1,s) \leqslant \mathbf{c}(x,s) \leqslant \mathbf{c}(x,s+1)$ for each x and s; we also assume that $\mathbf{c}(x,s)=0$ for all $x \geqslant s$. All cost functions in this paper will be monotonic without further mention. As stated above, we view $\mathbf{c}(x,s)$ as the cost of changing at stage s a guess about the value $A(x)$ for some Δ^0_2 set A. Monotonicity means that the cost of a change increases with time and that smaller changes are more costly.

If $\mathbf{c}$ is a cost function, then we let $\underline{\mathbf{c}}(x) = \lim_s \mathbf{c}(x,s)$. We say that $\mathbf{c}$ satisfies the *limit condition* if $\underline{\mathbf{c}}(x)$ is finite for all x, and $\underline{\mathbf{c}}(x) \to 0$ as $x \to \infty$. As with monotonicity, all cost functions mentioned will henceforth satisfy the limit condition.

Definition 2.1 ([34])**.** Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A, and let $\mathbf{c}$ be a cost function. The *total* $\mathbf{c}$*-cost* of the approximation is

$$\sum_{s\in\omega} \{\mathbf{c}(x,s) \,:\, x \text{ is least such that } A_{s-1}(x) \neq A_s(x)\}\,.$$

We say that a Δ^0_2 set A *obeys* $\mathbf{c}$ if the total $\mathbf{c}$-cost of *some* computable approximation of A is finite. We write $A \models \mathbf{c}$.

It is not hard to show that if A is a c.e. set obeying a cost function $\mathbf{c}$, then there is a computable enumeration of A that witnesses this obedience; see [35].

Definition 2.2. For a rational number $p \in (0,1]$ and a left-c.e. real β (equipped with an increasing approximation $\langle \beta_s \rangle$) we let

$$\mathbf{c}_{\beta,p}(x,s) = (\beta_s - \beta_x)^p.$$

(As usual, we let $\mathbf{c}_{\beta,p}(x,s) = 0$ if $x \geqslant s$.)

As mentioned above, a set A is K-trivial if and only if it obeys the cost function $\mathbf{c}_\Omega = \mathbf{c}_{\Omega,1}$. Of course, if $p < q$ then $\mathbf{c}_{\Omega,q} \leqslant \mathbf{c}_{\Omega,p}$, and so if A obeys $\mathbf{c}_{\Omega,p}$, then A obeys $\mathbf{c}_{\Omega,q}$. In particular, if $p \leqslant 1$ and $A \models \mathbf{c}_{\Omega,p}$, then A is K-trivial.

Proposition 2.3. *Let $p \in (0,1]$ be rational. The collection of sets that obey $\mathbf{c}_{\Omega,p}$ is downwards closed in the Turing degrees.*

Key to the proof of Proposition 2.3 will be the fact that K-trivial sets do not help us approximate Chaitin's Ω substantially better than we can with no oracle. Recall that if B is K-trivial, it is low for Martin-Löf randomness, so Ω is Martin-Löf random relative to B.

Barmpalias and Downey [1, Lemma 2.5] proved that if $f \leqslant_T B$ and Ω is Martin-Löf random relative to B (i.e., B is low for Ω), then there is a constant N such that $(\forall k)\ \Omega - \Omega_k < N\left(\Omega - \Omega_{f(k)}\right)$. In other words, B does not allow us to speed up the approximation of Ω by more than a constant. This is exactly what we will need for the proof of Proposition 2.3, but we present a small improvement: in the limit, B does not allow us to speed up the approximation of Ω at all.

Lemma 2.4. *Assume that Ω is Martin-Löf random relative to B and that $f\colon \omega \to \omega$ is a B-computable function. For every $\varepsilon > 0$,*

$$(\forall^\infty k)\ \Omega - \Omega_k < (1+\varepsilon)\left(\Omega - \Omega_{f(k)}\right).$$

In other words,

$$\liminf_{k\to\infty} \frac{\Omega - \Omega_{f(k)}}{\Omega - \Omega_k} \geqslant 1.$$

Proof. Without loss of generality, we may assume that f is strictly increasing, and that ε is rational. Let $n_0 = 0$; given n_s we let $n_{s+1} = f(n_s)$. For each s let $G_s = \left(\Omega_{n_{s+1}}, \Omega_{n_{s+1}} + \frac{1}{\varepsilon}\left(\Omega_{n_{s+1}} - \Omega_{n_{s-1}}\right)\right)$. Note that $\sum_{s\in\omega}\mu(G_s) \leqslant \frac{2}{\varepsilon}\Omega$, so $\langle G_s\rangle$ is a B-Solovay test. By assumption, Ω cannot be captured by this test, so there is an s^* such that if $s \geqslant s^*$, then $\Omega \notin G_s$. But $\Omega \geqslant \Omega_{n_{s+1}}$, so it must be the case that $\Omega > \Omega_{n_{s+1}} + \frac{1}{\varepsilon}(\Omega_{n_{s+1}} - \Omega_{n_{s-1}})$. Rearranging:

$$(\forall s \geqslant s^*)\ \varepsilon\left(\Omega - \Omega_{n_{s+1}}\right) > \Omega_{n_{s+1}} - \Omega_{n_{s-1}}.$$

Now let $k \geqslant n_{s^*}$. Fix $s > s^*$ such that $k \in [n_{s-1}, n_s)$. Since $f(k) < f(n_s) = n_{s+1}$,

$$\begin{aligned}(1+\varepsilon)\left(\Omega - \Omega_{f(k)}\right) &> (1+\varepsilon)\left(\Omega - \Omega_{n_{s+1}}\right)\\ &> \Omega - \Omega_{n_{s+1}} + \Omega_{n_{s+1}} - \Omega_{n_{s-1}}\\ &= \Omega - \Omega_{n_{s-1}} \geqslant \Omega - \Omega_k. \qquad \square\end{aligned}$$

Proof of Proposition 2.3. Suppose that $A \leqslant_{\mathrm{T}} B$ and $B \models \mathbf{c}_{\Omega,p}$. Let Φ be a Turing functional such that $\Phi(B) = A$. Let φ be the use function for the reduction. Since B is K-trivial and φ is B-computable, we can apply Lemma 2.4 (or Lemma 2.5 of Barmpalias and Downey [1]) to get an $N > 0$ such that

$$(\forall k)\ \Omega - \Omega_k < N\left(\Omega - \Omega_{\varphi(k)}\right).$$

The idea of the proof is to use this inequality to bound the cost of an A-change in terms of the cost of a corresponding B-change, where we take an approximation of A induced by a given approximation of B.

Let $\langle B_s\rangle$ be an approximation of B that witnesses that B obeys $\mathbf{c}_{\Omega,p}$. At stage s, for all $n \in \operatorname{dom}\Phi_s(B_s)$, let $\varphi_s(n)$ be the Φ_s-use of the stage s computation. We define a computable increasing sequence of stages $s(0) < s(1) < \cdots$ as follows. Let $s(0) = 0$. Given $s(i-1)$, let $s(i) > s(i-1)$ be least such that

$$(\forall k \leqslant i)\ \Omega_{i+1} - \Omega_k < N\left(\Omega_{s(i)} - \Omega_{\varphi_{s(i)}(k)}\right).$$

Included in this condition is the assumption that $k \in \operatorname{dom} \Phi_{s(i)}(B_{s(i)})$ for all $k \leqslant i$. The choice of N guarantees that such an $s(i)$ will be found. For $i \geqslant 0$, let $A_i = \Phi_{s(i)}(B_{s(i)}) \upharpoonright i+1$. We claim that the approximation $\langle A_i \rangle$ witnesses that A obeys $\mathbf{c}_{\Omega,p}$.

Let $i \geqslant 0$ and let k be least such that $A_{i+1}(k) \neq A_i(k)$. If $k > i$, then the $\mathbf{c}_{\Omega,p}$-cost accrued by the approximation $\langle A_i \rangle$ at stage $i+1$ is 0. If $k \leqslant i$, then the $\mathbf{c}_{\Omega,p}$-cost is $(\Omega_{i+1} - \Omega_k)^p$, which is bounded by $N^p(\Omega_{s(i)} - \Omega_{\varphi_{s(i)}(k)})^p$. Let $v = \varphi_{s(i)}(k)$. The change in A corresponds to a change in B; specifically, there must be a stage $t \in (s(i), s(i+1)]$ such that $B_t \upharpoonright v \neq B_{t-1} \upharpoonright v$. The $\mathbf{c}_{\Omega,p}$-cost accrued by the approximation $\langle B_s \rangle$ at stage t is at least $(\Omega_t - \Omega_v)^p$, which in turn is at least $(\Omega_{s(i)} - \Omega_{\varphi_{s(i)}(k)})^p$. It follows that the total cost for $\langle A_j \rangle$ is bounded by $N^p \cdot$ (the total cost for $\langle B_s \rangle$), hence it is finite. □

Definition 2.5. Let $\mathcal{B}_p$ be the collection of sets that obey $\mathbf{c}_{\Omega,p}$.

We have seen that $\mathcal{B}_p$ is closed downward under Turing reduction and only contains K-trivial sets. Nies [35] proved several general results about the class of sets obeying a cost function that are helpful in understanding $\mathcal{B}_p$. For example, $\mathcal{B}_p$ is closed under join, which along with downward closure means that it induces an ideal in the Turing degrees. Further, every member of $\mathcal{B}_p$ is bounded by a c.e. member of $\mathcal{B}_p$, and the index set of c.e. members is Σ^0_3. As we have already mentioned, if $p < q$, then $\mathcal{B}_p \subseteq \mathcal{B}_q$.

We will use the following, which is Theorem 3.4 of [35]. Here and below we write $g \leqslant^\times h$ to mean that $g \leqslant ch$ for some constant $c > 0$.

Proposition 2.6. *The following are equivalent for two cost functions* $\mathbf{c}$ *and* $\mathbf{d}$*:*

(1) *Every set obeying* $\mathbf{c}$ *also obeys* $\mathbf{d}$*;*
(2) $\underline{\mathbf{d}} \leqslant^\times \underline{\mathbf{c}}$.

Suppose that $p < q$. Since $\underline{\mathbf{c}}_{\Omega,p}$ is not bounded by any constant multiple of $\underline{\mathbf{c}}_{\Omega,q}$, the ideal $\mathcal{B}_p$ is properly contained in $\mathcal{B}_q$. While Proposition 2.6 only produces a Δ^0_2 set in $\mathcal{B}_q - \mathcal{B}_p$, in our case this difference between the ideals can be witnessed by a c.e. set. For take $V \in \mathcal{B}_q - \mathcal{B}_p$. Since $\mathcal{B}_q$ is characterised by obedience to a cost function, there is a c.e. set $A \geqslant_T V$ in $\mathcal{B}_q$. Since $\mathcal{B}_p$ is downward closed, we have $A \notin \mathcal{B}_p$.

2.1. **Coherent tests, and tests bounded by cost functions.** A Π^0_2 class (i.e., an effective G_δ set) is the intersection $\bigcap_n V_n$ of a nested sequence $V_0 \supseteq V_1 \supseteq \cdots$ of uniformly c.e. open sets. Nesting ensures that the class is null if and only if $\mu(V_n) \to 0$. Such null classes characterise weak 2-randomness. If we assume that $\mu(V_n)$ is bounded by a computable function tending to 0, then we have a Martin-Löf test. Randomness notions in between Martin-Löf randomness and weak 2-randomness can be introduced by taking a suitable noncomputable witness to the fact that $\mu(V_n) \to 0$. For example, Oberwolfach randomness [6] can be characterised using tests satisfying $\mu(V_n) \leqslant \beta - \beta_n$, where $\langle \beta_n \rangle$ is a computable increasing sequence of approximations limiting to a left-c.e. real β. In general, cost functions can be used to gauge the rate that $\mu(V_n)$ converges to 0. This generalises the previous example because $\mathbf{c}(n,s) = \beta_s - \beta_n$ is a cost function (these are the *additive* cost functions in the sense of [35]).

Definition 2.7 ([6, Def. 2.13])**.** Let $\mathbf{c}$ be a cost function. A nested sequence $\langle V_n \rangle$ of uniformly c.e. open sets is a $\mathbf{c}$-*bounded test* if $\mu(V_n) \leqslant^\times \underline{\mathbf{c}}(n)$ for all n.[6]

The limit condition for $\mathbf{c}$ ensures that $\mu(V_n) \to 0$, so $\bigcap_n V_n$ is indeed null. If $\langle V_n \rangle$ is a $\mathbf{c}$-bounded test, then there is a (uniform) enumeration $\langle V_{n,s} \rangle$ of the sets V_n such that for all n and s, $V_{n+1,s} \subseteq V_{n,s}$ and $\mu(V_{n,s}) \leqslant^\times \mathbf{c}(n,s)$ (so in particular, $V_{n,n} = \varnothing$).

Tests bounded by additive cost functions as described above (also known as "Auckland tests") define the same null sets as "Oberwolfach tests" [6], which are coherent restrictions of balanced tests [17]. The general context here is Demuth's framework for defining null sets using components that can be reset. We consider nested tests $\langle V_n \rangle$ where $V_n = W_{f(n)}$ for some Δ^0_2 function f. (Here $\langle W_e \rangle$ is an effective enumeration of all effectively open sets.) We require that $\mu(V_n) \leqslant 2^{-n}$. If $\langle f_s \rangle$ is a computable approximation for f, then $V_n\langle s \rangle = W_{f_s(n)}$ is the stage s approximation of the components of the test. The informal idea is that when building such a test we start covering some reals, but at a later stage we change our minds ($f_s(n) \neq f_{s-1}(n)$); we empty some of the components of the test and restart them. A *balanced test* is such a test for which the approximation for $f(n)$ changes $O(2^n)$ times. An Oberwolfach test (a coherent balanced test) requires the changes to be coordinated across the levels of the test: if s and t are successive stages at which $f_s(n) \neq f_{s-1}(n)$ and $f_t(n) \neq f_{t-1}(n)$, then either $f_s(n-1) \neq f_{s-1}(n-1)$ or $f_t(n-1) \neq f_{t-1}(n-1)$. That is, every two changes in $V_n\langle s \rangle$ prompt a change to $V_{n-1}\langle s \rangle$. If we further assume that V_0 never changes, then this is equivalent to the existence of a system of (uniformly c.e. open) components G_σ for $\sigma \in 2^{<\omega}$ and a left-c.e. real $\alpha \in 2^\omega$ such that $\mu(G_\sigma) \leqslant 2^{-|\sigma|}$ and $V_n = G_{\alpha \upharpoonright n}$.

Definition 2.8. Let $p \in (0,1]$ be rational. A *p-Oberwolfach test* consists of a left-c.e. binary sequence $\alpha \in 2^\omega$ and a uniformly c.e. open array $\langle G_\sigma \rangle_{\sigma \in 2^{<\omega}}$ such that:

- For all $\sigma \in 2^{<\omega}$ and $i < 2$, $G_{\sigma\hat{\ }i} \subseteq G_\sigma$;
- For all $\sigma \in 2^{<\omega}$, $\mu(G_\sigma) \leqslant^\times 2^{-p|\sigma|}$.

The null set defined by the test is $\bigcap_n G_{\alpha \upharpoonright n}$.

We say that a test Q *covers* a test P if the null set defined by P is a subset of the null set defined by Q. The following generalises one direction of the equivalence of Auckland and Oberwolfach tests; the proof however required modification.

Proposition 2.9. *Let $p \in (0,1]$ be rational. Every p-Oberwolfach test can be covered by a $\mathbf{c}_{\Omega,p}$-bounded test.*

Proof. Let $(\langle G_\sigma \rangle, \alpha)$ be a p-Oberwolfach test. Let $\beta = \alpha + 1$ and let $\beta_s = \alpha_s + (1 - 2^{-s})$ be the associated increasing approximation; so $\beta - \beta_s = (\alpha - \alpha_s) + 2^{-s}$. We first show that the test $(\langle G_\sigma \rangle, \alpha)$ can be covered by a $\mathbf{c}_{\beta,p}$-bounded test $\langle V_n \rangle$. We let $V_n = \bigcup_{s>n} G_{\alpha_s \upharpoonright n}$. Certainly $\bigcap_n G_{\alpha \upharpoonright n} \subseteq \bigcap_n V_n$; we need to show that $\mu(V_n) \leqslant^\times (\beta - \beta_n)^p$. Let k be the natural number such that $2^{-k-1} \leqslant (\alpha - \alpha_n) < 2^{-k}$.

[6] We note that the concept of $\mathbf{c}$-test in [6] was defined without the linear constant.

Then there are at most two strings of the form $\alpha_s \upharpoonright k$ for $s > n$. If $k \geqslant n$ then there are at most two strings of the form $\alpha_s \upharpoonright n$ for $s > n$, and so $\mu(V_n) \leqslant^\times 2 \cdot 2^{-pn}$; because $\beta - \beta_n \geqslant 2^{-n}$ we get $\mu(V_n) \leqslant^\times (\beta - \beta_n)^p$. If $k < n$ then we use the fact that $\langle G_\sigma \rangle$ is nested: in that case $V_n \subseteq \bigcup_{s>n} G_{\alpha_s \upharpoonright k}$ and so $\mu(V_n) \leqslant^\times 2 \cdot 2^{-pk}$, while $(\beta - \beta_n)^p \geqslant (\alpha - \alpha_n)^p \geqslant 1/2^p \cdot 2^{-pk}$, whence $\mu(V_n) \leqslant^\times 2^{p+1}(\beta - \beta_n)^p$ as required.

Next we cover $\langle V_n \rangle$ by a $\mathbf{c}_{\Omega,p}$-bounded test $\langle U_n \rangle$. For this we use the fact that Ω is Solovay complete; there is some increasing computable function f such that $\beta - \beta_{f(n)} \leqslant^\times \Omega - \Omega_n$. So we let $U_n = V_{f(n)}$. ☐

It is also the case that every $\mathbf{c}_{\Omega,p}$-bounded test can be covered by a p-Oberwolfach test. However, we do not need this fact and do not include a proof.

2.2. **Capturing the columns of Ω.** We will work with the computable probability space $(2^\omega)^n$ for various $n < \omega$, and in fact with computable probability spaces $(2^\omega)^F$ where $F \subseteq \{1, 2, \dots n\}$; the latter is immediately identified with $(2^\omega)^{|F|}$ by using the increasing enumeration of F. Elements of $(2^\omega)^F$ will be denoted by uppercase Roman letters. If $Z \in (2^\omega)^F$ and $i \in F$ then Z_i is the i^{th} component of Z. We identify n with $\{1, 2, \dots, n\}$ so each $Z \in (2^\omega)^n$ is the tuple $(Z_1, Z_2, \dots, Z_n)$.

For each $n < \omega$, the computable probability space $(2^\omega)^n$ is computably isomorphic to 2^ω via a measure-preserving map (and so the map preserves both Turing degree and ML-randomness). There are several such maps and for most applications it does not matter which one we take. However at times it is important that we use the canonical map which distributes bits evenly: for $X \in 2^\omega$ we define $j_n(X) = (X_1, X_2, \dots, X_n) \in (2^\omega)^n$ by letting $X_{j+1}(k) = X(nk + j)$. The sequences $X_1, \dots, X_n$ are called the *n-columns* of X. We also write $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$. We sometimes abuse notation and write X for $j_n(X)$. However, we will denote the n-columns of Ω by $\bar{\Omega}_1, \dots, \bar{\Omega}_n$, so as to not confuse them with Ω_s, the stage-s approximation for Ω.

We introduce further notation which will be useful here and later. Let $F \subseteq \{1, 2, \dots, n\}$. We define the projection $\pi_F \colon (2^\omega)^n \to (2^\omega)^F$ by erasing the entries with indices outside F. For clarity, for $Z \in (2^\omega)^n$ we also denote $\pi_F(Z)$ by Z_F. Using this notation we can rephrase Definition 1.2:

Definition 2.10. A set A is a *k/n-base* if there is a random tuple $Z \in (2^\omega)^n$ such that A is computable from Z_F for every $F \subseteq \{1, 2, \dots, n\}$ of size k.

The tuple Z is called a *witness* for A being a k/n-base. The abuse of notation mentioned above results in us sometimes calling $\bigoplus_{i \leqslant n} Z_i$ a witness as well. As promised, we will show that Chaitin's Ω (which of course can be taken to be any left-c.e. random sequence) is a witness for every k/n-base. The following proposition is the first step toward that result.

Proposition 2.11. *Let $n \geqslant 1$ and $F \subseteq \{1, 2, \dots, n\}$. Let $\bar{\Omega}_1, \bar{\Omega}_2, \dots, \bar{\Omega}_n$ be the n-columns of Ω. Then $(\bar{\Omega}_j)_{j \in F} = \Omega_F$ is captured by a $\mathbf{c}_{\Omega,|F|/n}$-bounded test.*

Proof. For $\sigma \in 2^{<\omega}$, let $G_\sigma = \{X_F : X \in 2^\omega \ \& \ \sigma \prec X\}$, where again by X_F we really mean $\pi_F(j_n(X))$. The test $(\langle G_\sigma \rangle, \Omega)$ captures Ω_F. If n divides $|\sigma|$, then $\mu(G_\sigma)$ precisely equals $2^{-(|F|/n) \cdot |\sigma|}$ (as we specify precisely $(|F|/n)|\sigma|$ many bits). So in general $\mu(G_\sigma)$ is bounded by $2^n \cdot 2^{-|F|/n \cdot |\sigma|}$ and is thus an $|F|/n$-Oberwolfach test. The result follows from Proposition 2.9. ☐

Note that we used the fact that we are dealing with the canonical "bits evenly distributed" isomorphism between 2^ω and $(2^\omega)^n$. We cannot capture an arbitrary computable split of Ω by a $\mathbf{c}_{\Omega,1/2}$-test. As a result, it is not the case that for any computable splitting of Ω into two parts, both parts compute every 1/2-base.[7]

2.3. **Analysis of c.e. k/n bases.** We can now sketch the proof of Theorem 1.3 in the special case that A is c.e. The main technical result is Proposition 2.16, which says that if A is a k/n-base, then A obeys the cost function $\mathbf{c}_{\Omega,k/n}$. Another important fact (see for example [6]) is that if a set A obeys a cost function $\mathbf{c}$, then it is computable from any random sequence that is captured by a $\mathbf{c}$-bounded test (thus for example, any random that is not Oberwolfach random computes all K-trivial sets). Proposition 2.11 says that any k-tuple of distinct n-columns of Ω can by captured by a $\mathbf{c}_{\Omega,k/n}$-bounded test, and so:

- each such k-tuple computes every k/n-base (since these obey $\mathbf{c}_{\Omega,k/n}$);
- any set obeying the cost function $\mathbf{c}_{\Omega,k/n}$ is a k/n-base.

2.4. **Weak obedience to cost functions.** We do not know how to show directly that, in general, every k/n-base obeys $\mathbf{c}_{\Omega,k/n}$. To overcome this we introduce a weakening of the notion of obedience.

Definition 2.12. Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A. An *n-stage* for this approximation is a stage s at which $A_s \upharpoonright n = A_{s-1} \upharpoonright n$, $A_s(n) \neq A_{s-1}(n)$, and $A_s \upharpoonright n+1 = A \upharpoonright n+1$. Note that there is not necessarily an n-stage for every n, but there are n-stages for infinitely many n.

Let $\mathbf{c}$ be a cost function. The *weak total $\mathbf{c}$-cost* of the approximation $\langle A_s \rangle$ is

$$\sum \{\mathbf{c}(n,s) \,:\, s \text{ is the last } n\text{-stage}\}\,.$$

The approximation $\langle A_s \rangle$ witnesses that A *weakly obeys* the cost function $\mathbf{c}$ if the weak total $\mathbf{c}$-cost of the approximation is finite.

If A obeys $\mathbf{c}$ then it also weakly obeys it; the converse fails by the following, combined with the fact that a K-trivial is never Turing complete.

Proposition 2.13. *There is a c.e. set $A \equiv_T \varnothing'$ that weakly obeys $\mathbf{c}_\Omega$.*

Proof. We enumerate A as follows. For each $n \notin \varnothing'_s$, we have a marker $\gamma_s(n) \geqslant n$. If n enters $\varnothing'$ at stage s, then we enumerate the marker $\gamma_s(n)$ into A_s and initialise all markers $\gamma_{s+1}(m)$ for $m > n$ to be greater than s. A k-stage is a "true stage" in the enumeration of A, equivalently of $\varnothing'$. If s is a k-stage of the enumeration $\langle A_s \rangle$, then $k = \gamma_s(n)$ for some n, and every number that enters A after stage s is greater than s. Thus the set of intervals $I = \{[k,s) \,:\, s \text{ is a } k\text{-stage of } \langle A_s \rangle\}$ is pairwise disjoint. The weak total $\mathbf{c}_\Omega$-cost of this enumeration of A is the sum of $\Omega_s - \Omega_k$, where $[k,s)$ is an interval in I. Therefore, it is bounded by Ω. $\square$

Weak obedience is not very useful when we try to build our own sets. However it suffices for the following.

Proposition 2.14. *If A weakly obeys $\mathbf{c}$, then A is computable from any A-random sequence captured by a $\mathbf{c}$-bounded test.*

[7]For example, consider the 3-columns $\bar\Omega_1, \bar\Omega_2, \bar\Omega_3$ of Ω. By considering each splitting $\bar\Omega_i, \bar\Omega_j \oplus \bar\Omega_k$ (where $\{i,j,k\} = \{1,2,3\}$), we see that if for every computable splitting of Ω into two parts, both parts compute A, then A is a 1/3-base. However, not every 1/2-base is a 1/3-base.

Proof. Let $\langle A_s\rangle$ be an approximation witnessing that A weakly obeys $\mathbf{c}$; let $\langle V_n\rangle$ be a $\mathbf{c}$-bounded test. Being an n-stage for the approximation is recognisable by A. For $n < \omega$, let $G_n = \varnothing$ if there is no n-stage; otherwise, let $G_n = V_{n,s}$, where s is the last n-stage. In other words, $G_n = \bigcup V_{n,s}$ as s ranges over all n-stages. Then the sequence $\langle G_n\rangle$ is uniformly A-c.e. Since $\mu(V_{n,s}) \leqslant \mathbf{c}(n,s)$, the sequence $\langle G_n\rangle$ is an A-Solovay test.

Suppose that $Z \in \bigcap_n V_n$ is not captured by $\langle G_n\rangle$; let r be the last stage at which Z enters any G_n. Suppose that Z enters V_n at stage $s > r$; we claim that $A_s \upharpoonright n+1 = A \upharpoonright n+1$, in fact that $A_t \upharpoonright n+1 = A_s \upharpoonright n+1$ for all $t \geqslant s$. Let m be least such that for some $t > s$, $A_t(m) \neq A_{t-1}(m)$; let t be the last such stage. Then t is an m-stage. Since $t > r$, $Z \notin V_{m,t}$. Since $Z \in V_{n,t}$ and the sets $\langle V_{k,t}\rangle$ are nested, it must be that $m > n$. Hence Z computes A. □

A refinement. We will require a technical refinement. Let $\mathcal{I} = \langle i_0, i_1, \ldots\rangle$ be a strictly increasing computable sequence. Let $\langle A_s\rangle$ be a computable approximation of a Δ^0_2 set A. An *$\mathcal{I}$-n-stage* of the approximation is a stage s at which: $A_s \upharpoonright i_n = A_{s-1} \upharpoonright i_n$, $A_s \upharpoonright i_{n+1} \neq A_{s-1} \upharpoonright i_{n+1}$, and $A_s \upharpoonright i_{n+1} = A \upharpoonright i_{n+1}$. If $\mathbf{c}$ is a cost function, then the *total $\mathcal{I}$-weak cost* of the approximation is the sum of all $\mathbf{c}(n,s)$ where n is the last $\mathcal{I}$-n-stage. The approximation witnesses that A *$\mathcal{I}$-weakly obeys* $\mathbf{c}$ if the total $\mathcal{I}$-weak cost of the approximation is finite. Weak obedience is $\mathcal{I}$-weak obedience for $\mathcal{I}$ being the identity sequence $i_n = n$.

The proof of Proposition 2.14 gives its $\mathcal{I}$-analogue: if A $\mathcal{I}$-weakly obeys $\mathbf{c}$ then A is computable from any A-random set captured by a $\mathbf{c}$-bounded test. (Equivalently, we could generalise the theory of obedience and weak obedience to computably bounded elements of Baire space.)

Lemma 2.15. *Let $\mathcal{I}$ be an increasing computable sequence. Suppose that A is K-trivial and $\mathcal{I}$-weakly obeys $\mathbf{c}_{\Omega,k/n}$. Then A is a k/n-base; in fact, the n-columns of Ω witness that A is a k/n-base.*

Proof. Since A is low for random, any k-tuple of distinct n-columns of Ω is A-random. So Proposition 2.11 and the $\mathcal{I}$-version of Proposition 2.14 show that any such join computes A. □

2.5. **The proof of Theorem 1.3.** The two main results that we show later are the following. The first was mentioned above.

Proposition 2.16. *Every c.e. k/n-base obeys $\mathbf{c}_{\Omega,k/n}$.*

Proposition 2.17. *Every k/n-base weakly obeys $\mathbf{c}_{\Omega,k/n}$. In fact, if $\langle A_s\rangle$ is any computable approximation of a k/n-base A, then there is a sub-approximation $\langle A_{s(n)}\rangle$ which witnesses that A weakly obeys $\mathbf{c}_{\Omega,k/n}$.*

These suffice to give a proof of Theorem 1.3. To prove that every k/n-base obeys $\mathbf{c}_{\Omega,k/n}$ we will show that each k/n-base is bounded by a c.e. k/n-base, and use Propositions 2.3 and 2.16.

Theorem 2.18. *Let $1 \leqslant k < n$. The following are equivalent for a set A:*

(1) *A is a k/n-base;*
(2) *The n-columns of Ω witness that A is a k/n-base;*
(3) *A obeys $\mathbf{c}_{\Omega,k/n}$;*
(4) *A is K-trivial and weakly obeys $\mathbf{c}_{\Omega,k/n}$.*

Proof. (2)→(1) is immediate. (3)→(4) holds because obedience implies weak obedience; and if A obeys $\mathbf{c}_{\Omega,p}$, then it obeys $\mathbf{c}_\Omega$ and so is K-trivial. (4)→(2) follows from Lemma 2.15.

It remains to show that (1) implies (3). Suppose that A is a k/n-base. As mentioned in the introduction, we know that A is K-trivial [24]; let $\langle \bar{A}_t \rangle$ be an approximation that witnesses that A obeys $\mathbf{c}_\Omega$. By Proposition 2.17, there is a sub-approximation $\langle A_s \rangle = \langle \bar{A}_{t(s)} \rangle$ of A that witnesses that A weakly obeys $\mathbf{c}_{\Omega,k/n}$. This sub-approximation also witnesses that A (fully) obeys $\mathbf{c}_\Omega$. As a consequence, the approximation $\langle A_s \rangle$ is an ω-computable approximation ([35, Fact 2.12]): there is a computable function h bounding the number of changes of $A_s(n)$. Using h, we can devise a "reasonable" *change-set* C for the approximation $\langle A_s \rangle$. Let $i_n = \sum_{m\leqslant n} h(m)$ (and let $\mathcal{I} = \langle i_n \rangle$). We define the c.e. set C as follows: if $A_s(n) \neq A_{s-1}(n)$ and s is the j^{th} stage at which we saw a change in this approximation, then we enumerate $i_{n-1} + j$ into C_s. Then:

- the (full) total $\mathbf{c}_\Omega$-cost of the enumeration $\langle C_s \rangle$ is bounded by the total $\mathbf{c}_\Omega$-cost of the approximation $\langle A_s \rangle$, and so C is K-trivial;
- the $\mathcal{I}$-weak total $\mathbf{c}_{\Omega,k/n}$-cost of the enumeration $\langle C_s \rangle$ equals the (normal) weak total $\mathbf{c}_{\Omega,k/n}$-cost of the approximation $\langle A_s \rangle$, and so C $\mathcal{I}$-weakly obeys $\mathbf{c}_{\Omega,k/n}$.

By Lemma 2.15, C is a k/n-base. Since C is c.e., Proposition 2.16 says that C fully obeys $\mathbf{c}_{\Omega,k/n}$. By Proposition 2.3, A also obeys this cost function. □

3. 1/2-bases and ravenous sets

We first prove Propositions 2.16 and 2.17 for the special case $k = 1$ and $n = 2$. This allows us to describe the dynamics of the construction while suppressing the geometric considerations that appear in the general case.

3.1. Adapting the hungry sets construction.

The proofs of Propositions 2.16 and 2.17 are inspired by the "hungry sets" construction from [25], which was used to show that every set that is a base for randomness is K-trivial (or low for K). That argument can be transformed into a direct argument showing that every c.e. set A that is a base for randomness obeys the cost function $\mathbf{c}_\Omega$. It may be instructive to sketch that argument.

Sketch of a hungry sets construction for cost function obedience. Let $\langle A_s \rangle$ be an enumeration of a c.e. set A (what we use is the fact that it is an increasing approximation of a left-c.e. real). Suppose that Z is an A-random that computes A; let Ψ be a functional such that $\Psi(Z) = A$.

We have a separate "ε-construction" for every dyadic rational $\varepsilon > 0$. One of these constructions will give us the speed-up of the enumeration of A that witnesses that A obeys the cost function $\mathbf{c}_\Omega$. If an ε-construction fails to do so, then it produces an A-effectively open set U_ε of measure at most ε that contains Z; so if every such construction fails, we can build an A-Solovay test capturing Z.

Fix $\varepsilon > 0$. For each string τ we define an open set G_τ. This is the "hungry set" used to certify τ. It is a subset of $\Psi^{-1}[\tau] = \{X \in 2^\omega : \Psi(X) \succcurlyeq \tau\}$. We require that the sets G_τ be pairwise disjoint. The set G_τ is satiated if its measure is precisely $\varepsilon \cdot \left(\Omega_{|\tau|+1} - \Omega_{|\tau|}\right)$; this is the "goal" for G_τ. When the set reaches its goal, we declare τ to be *confirmed*. If τ' is an immediate successor of τ, we start filling the set $G_{\tau'}$ only after τ is confirmed; we fill $G_{\tau'}$ by clopen subsets of $\Psi^{-1}[\tau']$ disjoint from $\bigcup_{\sigma\preccurlyeq\tau} G_\sigma$.

Let $U_\varepsilon = \bigcup_{n<\omega} G_{A\restriction n}$. Then $\mu(U_\varepsilon) \leqslant \sum \varepsilon\cdot(\Omega_{n+1}-\Omega_n) \leqslant \varepsilon\cdot\Omega \leqslant \varepsilon$. If there is an n such that $A\restriction n$ is never confirmed, then $Z \in U_\varepsilon$ because $\Psi^{-1}[A\restriction n] \subseteq \bigcup_{m\leqslant n} G_{A\restriction m}$.

Suppose that every initial segment of A is confirmed at some stage. Define an increasing sequence $s_0 < s_1 < s_2 < \cdots$ of stages such that at stage s_n the string $A_{s_n}\restriction n$ is confirmed. We claim that the total $\mathbf{c}_\Omega$ cost of the approximation $\langle A_{s_n}\rangle$ is finite. Let $n \geqslant 0$ and consider the cost paid at stage s_{n+1}; it is $\Omega_{n+1}-\Omega_k$ where k is the least such that $A_{s_{n+1}}(k) \neq A_{s_n}(k)$. Consider the set $V_n = \bigcup_{m\in(k,n]} G_{A_{s_n}\restriction m}$. The sets $G_{A_{s_n}\restriction m}$ are full and pairwise disjoint, so $\mu(V_n) = \varepsilon\cdot(\Omega_{n+1}-\Omega_k)$. And further, because the approximation is left-c.e., the sets V_n are pairwise disjoint, so the total $\mathbf{c}_\Omega$ cost is bounded by $(1/\varepsilon)\cdot\mu(\bigcup_n V_n)$, which is of course finite. $\square$

Now consider a 1/2-base A witnessed by a pair Z_1 and Z_2; say $\Psi_i(Z_i) = A$ for $i = 1, 2$. We attempt a similar construction. Again fixing ε, as before the aim is to certify strings τ by filling hungry sets G_τ. If certification does not happen then we want to capture the pair (Z_1, Z_2) by $U = \bigcup_{\tau<A} G_\tau$, so certification happens by letting $G_\tau \subseteq \Psi_1^{-1}[\tau] \times \Psi_2^{-1}[\tau]$. The main idea is that if a string τ is certified with weight $\delta = \varepsilon\cdot\left(\Omega_{|\tau|+1}-\Omega_{|\tau|}\right)$ and is then discovered to be wrong, then either the projection $\pi_1[G_\tau]$ to the first coordinate or its projection $\pi_2[G_\tau]$ to the second coordinate has to have measure at least $\sqrt{\delta}$: G_τ is contained in the product of the two projections. So we plan to charge the $\mathbf{c}_{\Omega,1/2}$-cost of the change away from τ to one of these projections.

The problem is that while we can keep the hungry sets G_τ pairwise disjoint, this is not true of their projections. Consider Fig. 2. The two rectangles represent G_τ and $G_{\tau'}$ for an extension τ' of τ. If we required $\pi_2[G_{\tau'}]$ to be disjoint from $\pi_2[G_\tau]$ then we would not be able to capture the pair (Z_1, Z_2) in either G_τ or $G_{\tau'}$.

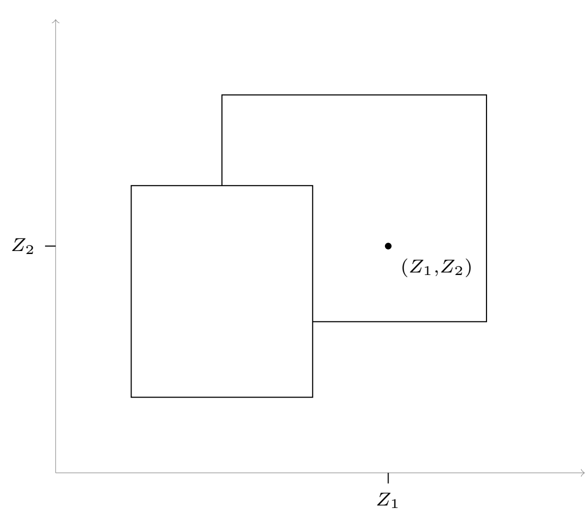

FIGURE 2. Overlapping projections of hungry sets

The fact that the projections of the hungry sets are not disjoint is a serious obstacle to the plan to charge the cost of changes to the measures of these projections. In the situation described, we may first discover that τ' is incorrect and charge against the projection $\pi_1[G_{\tau'}]$ say; and later discover that τ is incorrect and charge against $\pi_1[G_\tau]$.

To overcome this problem, when we discover that τ' is incorrect and want to confirm an incomparable extension τ'' of τ, for our measure calculations we ignore the oracles that map to τ'. That is, from the point of view of τ'', mass has been removed from G_τ upon discovering that τ' was incorrect, and τ'' wants to first "refill" G_τ before filling $G_{\tau''}$. This may seem like a bad idea because then the telescopic sum calculation bounding the size of U is now violated. But it is not violated if instead of an A-Solovay test we construct a *difference test*, where we are actually allowed to throw out that part of G_τ that mapped to τ' and not count it toward the measure of U. Now however, we need to explain why the pair (Z_1, Z_2) cannot be captured by such a test. For this we need the concept of measure-theoretic density.

3.2. **Density points and difference tests.** Restricting ourselves to binary density in Cantor space, for a measurable set $C \subseteq 2^\omega$ and a sequence $X \in 2^\omega$, the *density* of C at X is defined by

$$\varrho(C \,|\, X) = \liminf_{n\to\infty} \mu(C \,|\, X \upharpoonright n)$$

where for a string $\sigma \in 2^{<\omega}$, the conditional measure $\mu(C \,|\, \sigma)$ is $\mu(C \cap [\sigma])/\mu([\sigma])$. The Lebesgue density theorem says that for almost all $X \in C$, $\varrho(C \,|\, X) = 1$. A sequence X is a *density one point* if $\varrho(C \,|\, X) = 1$ for every effectively closed (Π^0_1) set C containing X. It is a *positive density point* if $\varrho(C \,|\, X) > 0$ for every effectively closed set C containing X. A random set X is a positive density point if and only if it is incomplete (Bienvenu et al. [7]). There is an incomplete random set that is not a density one point (Day and Miller [12]). Every Oberwolfach random set is a density one point [6], so every random set that is not a density one point computes every K-trivial set.

The concept of dyadic density is extended to the spaces $(2^\omega)^n$ (and so $(2^\omega)^F$) using the standard "evenly distributed bits" isomorphisms j_n (see Section 2.2). It is not difficult to see that, for example, a point (X, Y) in the plane $(2^\omega)^2$ is a density one point if and only if for every effectively closed set $C \subseteq (2^\omega)^2$ containing (X, Y), $\liminf_{n\to\infty} \mu\left(C \,|\, [X \upharpoonright n] \times [Y \upharpoonright n]\right) = 1$.

In our investigation of 1/2-bases we will use *product classes*, effectively closed subsets of the "Cantor plane" $(2^\omega)^2$ of the form $C_1 \times C_2$, where both $C_i \subseteq 2^\omega$ are effectively closed.

Proposition 3.1. *Suppose that (Z_1, Z_2) is a random pair that does not form a minimal pair (i.e., there is a noncomputable set reducible to both Z_1 and Z_2, so that the pair (Z_1, Z_2) witnesses that some noncomputable set is a 1/2-base). Then the pair (Z_1, Z_2) has positive density in every effectively closed product class $C_1 \times C_2$ containing it.*

Proof. Let $C_1 \times C_2$ be the product of two effectively closed sets; suppose that $Z_1 \in C_1$, $Z_2 \in C_2$, and that $\varrho(C_1 \times C_2 \,|\, (Z_1, Z_2)) = 0$. Since $\varrho(C_1 \times C_2 \,|\, (Z_1, Z_2)) \geqslant \varrho(C_1 \,|\, Z_1) \cdot \varrho(C_2 \,|\, Z_2)$, either $\varrho(C_1 \,|\, Z_1) = 0$ or $\varrho(C_2 \,|\, Z_2) = 0$. By [7], either Z_1 or Z_2

computes $\varnothing'$, say Z_1. But then Z_2 is 2-random, so it cannot compute any Δ^0_2 set, and in particular, no 1/2-base. ☐

Remark 3.2. In fact, the assumption of Proposition 3.1 implies that the pair (Z_1, Z_2) is a density one point in product classes. If $\varrho(C_1 \times C_2 \,|\, (Z_1, Z_2)) < 1$ then either Z_1 or Z_2 is not a density one point, say Z_1. By [7], Z_1 is almost everywhere dominating, which implies that it is LR-hard [26]. So again, Z_2 is 2-random and forms a minimal pair with Z_1.

When we discuss k/n-bases, we will need a generalisation of this fact (Lemma 4.10). But Proposition 3.1 suffices in the simple case of 1/2-bases.

The equivalence between positive density and incompleteness for random sequences passes through the notion of difference randomness. A *difference test* is a sequence $\langle P \cap G_n \rangle$, where P is a fixed effectively closed set, $\langle G_n \rangle$ is uniformly effectively open and nested, and $\mu(P \cap G_n) \leqslant 2^{-n}$; the null set defined is $P \cap \bigcap_n G_n$. Franklin and Ng [19] showed that a random sequence Z computes $\varnothing'$ if and only if it is captured by some difference test. Bienvenu et al. [7, Lemma 3.3] showed the following:

Lemma 3.3. *The following are equivalent for a random sequence Z and an effectively closed set P containing Z:*

(1) $\varrho(P \,|\, Z) = 0$*;*
(2) *Some difference test of the form $\langle P \cap G_n \rangle$ captures Z.*

Thus, Lemma 3.3 and Proposition 3.1 together show that if a pair (Z_1, Z_2) witnesses that some noncomputable set is a 1/2-base, then this pair cannot be captured by a difference test whose effectively closed component is a product class.

3.3. **The c.e. construction.** We can now provide the proof of Proposition 2.16 in the case of 1/2-bases:

Proposition 3.4. *Every c.e. 1/2-base obeys* $\mathbf{c}_{\Omega,1/2}$.

Proof. Let A be a c.e. 1/2-base, witnessed by the pair (Z_1, Z_2). Let Ψ_1 and Ψ_2 be functionals such that $\Psi_i(Z_i) = A$. Let $\langle A_s \rangle$ be an enumeration of A. For $i = 1, 2$, we let P^i_s be the collection of oracles X such that $\Psi_{i,s}(X)$ does not lie strictly to the left of A_s. We let $P^i = \bigcap_s P^i_s$, $P_s = P^1_s \times P^2_s$, and $P = \bigcap_s P_s = P^1 \times P^2$.

Let $\varepsilon > 0$ be a dyadic rational. We describe the ε-construction. All sets defined henceforth depend on ε, and we omit mentioning this parameter.

At stage $s < \omega$, we define for all strings τ clopen sets $G_{\tau,s} \subseteq \Psi^{-1}_{1,s}[\tau] \times \Psi^{-1}_{2,s}[\tau]$. These sets are increasing in s. We refill $G_{\tau,s}$ as parts of it exit P_s. The new measure could come from $G_{\sigma,s-1}$ where σ extends τ, so the sets $G_{\tau,s}$ will not be pairwise disjoint.

We start with $G_{\tau,0} = \varnothing$ for all τ. At stage s, we only add mass to $G_{\tau,s}$ for $\tau \prec A_s$, and decide whether such strings τ are *confirmed* at stage s. This is done by induction on $|\tau|$. We start with $G_{\langle\rangle,s} = \varnothing$; the empty string is always confirmed. Let $\tau \prec A_s$ be nonempty and suppose that we have already defined $G_{\tau',s}$ for all proper initial segments τ' of τ, and that all these initial segments are confirmed at stage s. We let $G_{\prec\tau,s} = \bigcup_{\tau' \precneqq \tau} G_{\tau',s}$.

We ensure that for all s,

$$\mu\left((G_{\tau,s} \setminus G_{\prec\tau,s}) \cap P_s\right) \leqslant \varepsilon \cdot (\Omega_{|\tau|} - \Omega_{|\tau|-1}).$$

Note that this implies that $\mu(G_{\preccurlyeq\tau,s} \cap P_s) \leqslant \varepsilon \cdot \Omega_{|\tau|}$. To define $G_{\tau,s}$, we add mass from $\Psi^{-1}_{1,s}[\tau] \times \Psi^{-1}_{2,s}[\tau]$ to $G_{\tau,s-1}$, being careful to maintain the bound. Note that the bound has not already been violated because $P_{s-1} \supseteq P_s$ and $G_{\prec\tau,s-1} \subseteq G_{\prec\tau,s}$. If sufficient mass is found so that equality is obtained, then we declare τ to be confirmed at stage s and move on to the next initial segment of A_s. Otherwise, we declare τ (and all of the longer initial segments of A_s) unconfirmed at stage s; we let $G_{\tau',s} = G_{\tau',s-1}$ for every $\tau' \not\prec \tau$, and move to stage $s+1$. Observe that if τ is confirmed at stage s, then $\mu(G_{\preccurlyeq\tau,s} \cap P_s) = \varepsilon \cdot \Omega_{|\tau|}$.

For all τ, let $G_\tau = \bigcup_s G_{\tau,s}$ and $G_{\preccurlyeq\tau} = \bigcup_s G_{\preccurlyeq\tau,s} = \bigcup_{\tau'\preccurlyeq\tau} G_{\tau'}$. Then $G_{\preccurlyeq\tau} \cap P = \bigcup_s (G_{\preccurlyeq\tau,s} \cap P)$. For each τ and s, $G_{\preccurlyeq\tau,s} \cap P \subseteq G_{\preccurlyeq\tau,s} \cap P_s$ and so $\mu(G_{\preccurlyeq\tau,s} \cap P) \leqslant \varepsilon \cdot \Omega_{|\tau|}$. It follows that $\mu(G_{\preccurlyeq\tau} \cap P) \leqslant \varepsilon \cdot \Omega_{|\tau|}$.

Let $G = \bigcup_{\tau\prec A} G_\tau$. Then $G \cap P$ is the increasing union of the sets $G_{\preccurlyeq\tau} \cap P$ for $\tau \prec A$, all of measure bounded by ε, and so $\mu(G \cap P) \leqslant \varepsilon$.

Suppose that there is some n such that the string $A \upharpoonright n$ is confirmed during only finitely many stages; let n be the least such. Let t be sufficiently large so that $A_t \upharpoonright n \prec A$; $A_t \upharpoonright n-1$ is confirmed at stage t; and for both $i \leqslant 2$, $Z_i \in \Psi^{-1}_{i,t}[A \upharpoonright n]$. Then the fact that $A \upharpoonright n$ is not confirmed at stage t implies that $G_{\preccurlyeq A\upharpoonright n,t} \supseteq \Psi^{-1}_{1,t}[A \upharpoonright n] \times \Psi^{-1}_{2,t}[A \upharpoonright n]$ and so $(Z_1, Z_2) \in G$.

So there must be some ε for which every initial segment of A is confirmed infinitely often. For suppose otherwise. Then the sets G_n given by the 2^{-n}-constructions together with P form an A-difference test $\langle P \cap G_n \rangle$ which captures (Z_1, Z_2). Relativising Lemma 3.3 to A and using the fact that (Z_1, Z_2) is A-random (as A is K-trivial), we conclude that $\varrho(P \,|\, (Z_1, Z_2)) = 0$. However P is a product class; this contradicts Proposition 3.1.

Fix ε for which every initial segment of A is confirmed infinitely often. Define an increasing sequence of stages $s_0 < s_1 < \cdots$ such that at stage s_k the string $A_{s_k} \upharpoonright (k+1)$ is confirmed. We claim that the total $\mathbf{c}_{\Omega,1/2}$-cost of the enumeration $\langle A_{s_k} \rangle$ is finite. Let $k \geqslant 0$; let $x = x_k$ be the least such that $A_{s_{k+1}}(x) \neq A_{s_k}(x)$; assume that $x \leqslant k$. The incurred cost between s_k and s_{k+1} is $\sqrt{\Omega_{k+1} - \Omega_x}$. Let

$$D_k = \{A_{s_k} \upharpoonright (x+1), A_{s_k} \upharpoonright (x+2), \dots, A_{s_k} \upharpoonright (k+1)\}$$

and let

$$V_k = P_{s_k} \cap \bigcup_{\tau \in D_k} G_{\tau,s_k}.$$

The fact that every string in D_k is confirmed at stage s_k implies that $\mu(V_k) = \varepsilon \cdot (\Omega_{k+1} - \Omega_x)$. Hence either $\pi_1[V_k]$ or $\pi_2[V_k]$ has measure at least $\sqrt{\varepsilon}\sqrt{\Omega_{k+1} - \Omega_x}$.

Suppose that $k < k'$. Then $\pi_1[V_k]$ and $\pi_1[V_{k'}]$ are disjoint (and the same holds for π_2). The reason is that $\pi_1[V_k] \subseteq \Psi_1^{-1}[A_{s_k} \upharpoonright x_k + 1]$, which is disjoint from $P^1_{s_{k+1}}$ and so from $P^1_{s_{k'}}$, whereas $\pi_1[V_{k'}] \subseteq P^1_{s_{k'}}$. Overall, we see that the total cost paid is bounded by $2/\sqrt{\varepsilon}$. ☐

3.4. **The general 1/2-base construction.** Weak obedience is very much weaker than full obedience. The main problem above, which drove us to use difference tests, is no longer a problem: the notion was designed so that we always charge for incompatible strings. We can therefore return to the basic hungry sets construction.

Proposition 3.5. *Every 1/2-base weakly obeys $\mathbf{c}_{\Omega,1/2}$. In fact, if $\langle A_s \rangle$ is a computable approximation of a 1/2-base A, then there is a computable subapproximation $\langle A_{s_k} \rangle$ that witnesses that A weakly obeys $\mathbf{c}_{\Omega,1/2}$.*

Proof. We simplify the proof of Proposition 3.4. Fix a witness (Z_1, Z_2) and functionals Ψ_1 and Ψ_2 as above.

Fix a dyadic rational $\varepsilon > 0$. As before, we enumerate clopen sets $G_{\tau,s} \subseteq \Psi_{1,s}^{-1}[\tau] \times \Psi_{2,s}^{-1}[\tau]$. We ensure that for all $\tau \neq \varnothing$ and all s,

$$\mu\left(G_{\preccurlyeq\tau,s}\right) \leqslant \varepsilon \cdot (\Omega_{|\tau|} - \Omega_{|\tau|-1}).$$

In this construction, the sets G_τ will be pairwise disjoint. A string τ is *confirmed* at stage s if equality holds. We start with $G_{\tau,0} = \varnothing$ for all τ. Let $s > 0$ be a stage. The empty string is always confirmed and $G_{\langle\rangle,s} = \varnothing$. Let τ be a string, and suppose that at stage s, its immediate predecessor τ^- is already confirmed, but that τ is not yet confirmed. We enumerate mass from $\Psi_{1,s}^{-1}[\tau] \times \Psi_{2,s}^{-1}[\tau]$ into $G_{\tau,s}$, ensuring that we do not overshoot the bound $\varepsilon \cdot (\Omega_{|\tau|} - \Omega_{|\tau|-1})$. If the bound is met, we declare that τ is confirmed (currently and at all future stages), and go on to deal with the two immediate successors of τ. If not, then for every proper extension σ of τ we let $G_{\sigma,s} = G_{\sigma,s-1} = \varnothing$.

As before we let $G_\tau = \bigcup_s G_{\tau,s}$, and $G = \bigcup_{\tau\prec A} G_\tau$; so $\mu(G) \leqslant \varepsilon$. If some initial segment τ of A is never confirmed, then $(Z_1, Z_2) \in G$. If this holds for every $\varepsilon > 0$, then (Z_1, Z_2) is captured by an A-ML test. However, since A is a 1/2-base, it is K-trivial, and so low for random, and so (Z_1, Z_2) is A-random and cannot be captured by such a test. Thus, fix some $\varepsilon > 0$ such that in the ε-construction, every initial segment of A is eventually confirmed.

Again define an increasing sequence $s_0 < s_1 < \cdots$ such that $A_{s_k} \upharpoonright (k+1)$ is confirmed at stage s_k. We show that the weak total $\mathbf{c}_{\Omega,1/2}$-cost of the enumeration $\langle A_{s_k}\rangle$ is finite. Let N be the set of n for which the approximation $\langle A_{s_k}\rangle$ has an n-stage; for $n \in N$, let $k(n)$ be the last n-stage for this approximation. The weak total cost is $\sum_{n\in N} \sqrt{\Omega_{k(n)} - \Omega_n}$.

For $n \in N$, for brevity let $t(n) = s_{k(n)-1}$ and $\sigma_n = A_{t(n)} \upharpoonright k(n)$. Let

$$V_n = \bigcup_{m=n+1}^{k(n)} G_{\sigma_n \upharpoonright m, t(n)}$$

Then

$$V_n \subseteq \Psi_1^{-1}[\sigma_n \upharpoonright (n+1)] \times \Psi_2^{-1}[\sigma_n \upharpoonright (n+1)],$$

and $\mu(V_n) = \varepsilon \cdot (\Omega_{k(n)} - \Omega_n)$, so $\mu(\pi_i[V_n]) \geqslant \sqrt{\varepsilon}\sqrt{\Omega_{k(n)} - \Omega_n}$ for at least one $i \leqslant 2$.

Let $n < n'$ be two elements of N. Then

$$A \upharpoonright (n+1) = A_{t(n')} \upharpoonright (n+1) = A_{s_{k(n)}} \upharpoonright (n+1)$$

is incomparable with $\sigma_n \upharpoonright (n+1)$, and so $\sigma_n \upharpoonright (n+1)$ and $\sigma_{n'} \upharpoonright (n'+1)$ are incomparable. For each $i \leqslant 2$, $\pi_i[V_n] \subseteq \Psi_i^{-1}[\sigma_n \upharpoonright (n+1)]$ and the same holds for n', and so $\pi_i[V_n]$ and $\pi_i[V_{n'}]$ are disjoint. This shows that the weak total cost of the approximation $\langle A_{s_k}\rangle$ is bounded by $2/\sqrt{\varepsilon}$. ☐

4. $\mathcal{F}$-bases

We want to adapt the proof from the previous section to the case of k/n-bases. In fact, it is useful and convenient to work in more generality. Recall the notation introduced in Section 2.2: if $F \subseteq \{1, 2, \ldots, n\}$, then $\pi_F \colon (2^\omega)^n \to (2^\omega)^F$ is the projection map erasing entries not indexed by elements of F. For $Z \in (2^\omega)^n$ we also write Z_F for $\pi_F(Z)$.

Definition 4.1. Let $\mathcal{F}$ be a nonempty family of subsets of $\{1,\dots,n\}$. We say that A is an $\mathcal{F}$-*base* if there is a random tuple $Z \in (2^\omega)^n$ such that $A \leqslant_{\mathrm{T}} Z_F$ for all $F \in \mathcal{F}$.

We will prove that a set is an $\mathcal{F}$-base if and only if it obeys $\mathbf{c}_{\Omega,p}$ for the appropriate choice of p. The power p that corresponds to $\mathcal{F}$ will be $1/\|\mathcal{F}\|$, where we define $\|\mathcal{F}\|$ using a linear optimisation problem.

4.1. **The norm of $\mathcal{F}$.** Fix a nonempty $\mathcal{F} \subseteq \mathcal{P}(\{1,2,\dots,n\})$. We attempt to quantify the "amount of disjointness" present in $\mathcal{F}$.

- An assignment $\langle x_F\rangle_{F\in\mathcal{F}}$ is a *normalised weighting of the sets in* $\mathcal{F}$ if for all $F \in \mathcal{F}$, x_F is a nonnegative real number, and for all $i \in \{1,\dots,n\}$, $\sum\{x_F \,:\, F \in \mathcal{F} \text{ and } i \in F\} \leqslant 1$.

We let

$$\|\mathcal{F}\| = \sup\left\{\sum_{F\in\mathcal{F}} x_F \,:\, \langle x_F\rangle \text{ is a normalised weighting of the sets in } \mathcal{F}\right\}.$$

For example, if $\varnothing \notin \mathcal{F}$ and the sets in $\mathcal{F}$ are pairwise disjoint, then $\|\mathcal{F}\| = |\mathcal{F}|$. On the other hand, if $\bigcap\mathcal{F} \neq \varnothing$ then $\|\mathcal{F}\| = 1$. Since we have assumed that $\mathcal{F} \neq \varnothing$, we always have $\|\mathcal{F}\| \geqslant 1$. If $\mathcal{F}$ contains the empty set, then $\|\mathcal{F}\| = \infty$. Otherwise each weight in a normalised weighting $\langle x_F\rangle$ is bounded by 1, and so $\|\mathcal{F}\| \leqslant |\mathcal{F}|$. It is also the case that $\|\mathcal{F}\| \leqslant n$; to see this, note that if $\varnothing \notin \mathcal{F}$, then $\sum_{F\in\mathcal{F}} x_F \leqslant \sum_{i\leqslant n}\sum\{x_F \,:\, F\in\mathcal{F} \text{ and } i \in F\} \leqslant \sum_{i\leqslant n} 1 = n$.

The norm $\|\mathcal{F}\|$ is the solution to a linear optimisation (linear programming) problem in standard form: let M be the $n \times |\mathcal{F}|$-incidence matrix (the (i,F)-entry is 1 if $i \in F$, 0 otherwise). The problem is to maximise $\sum_F x_F$ under the constraints $\langle x_F\rangle \geqslant \mathbb{0}$ and $M \cdot \langle x_F\rangle \leqslant \mathbb{1}$ (where we think of $\langle x_F\rangle$ as a column). This problem is *feasible* (the constraints are not contradictory), as is witnessed by the zero vector (or by the constant weighting $(1/|\mathcal{F}|)\cdot\mathbb{1}$, which witnesses that $\|\mathcal{F}\| \geqslant 1$). Further, if $\mathcal{F}$ does not contain the empty set, then the problem is *bounded* since the feasible solutions all have entries between 0 and 1. This implies that if $\mathcal{F}$ does not contain the empty set, then the problem has an optimal solution, that is, there is a normalised weighting $\langle x_F\rangle$ such that $\sum_F x_F = \|\mathcal{F}\|$. Moreover, since the problem is defined with rational coefficients, $\|\mathcal{F}\|$ is rational and an optimal solution can be taken to consist of rational numbers; this is because $\langle x_F\rangle$ can be taken to be a *basic feasible solution*, i.e., a vertex of the convex region defined by the constraints. For more information on linear programming, see for example [3].

A linear optimisation problem in standard form has a *dual problem.* The dual for the problem defining $\|\mathcal{F}\|$ is to minimise $\sum_{i\leqslant n} y_i$ where $\langle y_i\rangle \geqslant \mathbb{0}$ and $\langle y_i\rangle \cdot M \geqslant \mathbb{1}$ (where we think of $\langle y_i\rangle$ as a row). Conceptually:

- an n-tuple of nonnegative real numbers $\langle y_i\rangle$ is a *weighting of coordinates, normalised for* $\mathcal{F}$ if for all $F \in \mathcal{F}$, $\sum_{i\in F} y_i \geqslant 1$.

The dual problem is to minimise the sum of the weights of such a weighting. The strong duality theorem (see for example [3, Theorem 4.4]) says that as long as a linear optimisation problem has an optimal solution, then its dual problem also has an optimal solution, and the optimal values are the same. This gives us an alternate expression for $\|\mathcal{F}\|$ in the case that $\mathcal{F}$ does not contain the empty set:

$$\|\mathcal{F}\| = \min\left\{\sum_{i\leqslant n} y_i \;:\; \langle y_i\rangle \text{ is a weighting of coordinates normalised for } \mathcal{F}\right\}.$$

Example 4.2. Let $\mathcal{F}$ be the collection of all k-element subsets of $\{1, 2, \dots, n\}$. We can use both expressions for $\|\mathcal{F}\|$ to show that $\|\mathcal{F}\| = n/k$.

In one direction, let $\langle x_F\rangle$ be the constant weighting of sets $x_F = \frac{n}{k}/\binom{n}{k}$. Each $i \leqslant n$ is an element of precisely $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$ many of the sets in $\mathcal{F}$ and so $\langle x_F\rangle$ is indeed a normalised weighting of the sets in $\mathcal{F}$. This weighting witnesses that $\|\mathcal{F}\| \geqslant n/k$. On the other hand, the constant weighting $y_i = 1/k$ of coordinates is normalised for $\mathcal{F}$, so the dual expression shows that $\|\mathcal{F}\| \leqslant n/k$.

4.2. ***$\mathcal{F}$*-bases: the easy direction.** As promised above, we will show that a set is an $\mathcal{F}$-base if and only if it obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$. Actually this is not strictly true; it fails when $\|\mathcal{F}\| = 1$. So we first discuss the extreme values. We note that for a nonempty family $\mathcal{F}$ of subsets of $\{1, 2, \dots, n\}$,

- $\|\mathcal{F}\| = 1$ if and only if $\bigcap\mathcal{F} \neq \varnothing$. If $\|\mathcal{F}\| > 1$ then $\|\mathcal{F}\| \geqslant n/(n-1)$.
- $\|\mathcal{F}\| = \infty$ if and only if $\varnothing \in \mathcal{F}$.

If $\|\mathcal{F}\| = 1$, then every set is an $\mathcal{F}$-base (this follows from the Kučera–Gács theorem), so no obedience of any cost function can be deduced. On the other hand, if $\|\mathcal{F}\| = \infty$, then the equivalence between being a base and obeying the cost function holds trivially: in that case being an $\mathcal{F}$-base is the same as being computable, and the cost function $\mathbf{c}_{\Omega,0}$ fails the limit condition and any set obeying it is computable.

Henceforth we restrict ourselves to nonempty families $\mathcal{F}$ satisfying $1 < \|\mathcal{F}\| < \infty$. Note that for such a family, every $\mathcal{F}$-base is an $(n-1)/n$-base. Fix n and let $\mathcal{F}$ be such a family.

Lemma 4.3. *Let $\mathcal{I}$ be an increasing computable sequence. Suppose that A is K-trivial and $\mathcal{I}$-weakly obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$. Then A is an $\mathcal{F}$-base.*

Proof. Let $\langle y_i\rangle$ be an optimal solution for the dual problem defining $\|\mathcal{F}\|$ (an optimal normalised weighting of coordinates). We may assume that each y_i is rational. For each $i \leqslant n$, we let Z_i be a $y_i/\|\mathcal{F}\|$-part of Ω, with the parts chosen disjointly; since

$$\sum_{i<n} y_i/\|\mathcal{F}\| = 1,$$

Ω is exhausted when it is distributed between the sequences Z_i in this way. (Formally, we let m be a common denominator of the fractions $y_i/\|\mathcal{F}\|$ and let Z_i be the join of $m \cdot y_i/\|\mathcal{F}\|$-many of the m-columns of Ω.) If $y_i = 0$, then we let Z_i be random relative to Ω (joined with the other Z_j for which $y_j = 0$).

Now consider $F \in \mathcal{F}$. Since $\sum_{i\in F} y_i \geqslant 1$, Z_F accounts for at least $1/\|\mathcal{F}\|$ of Ω, and so Proposition 2.11 and the $\mathcal{I}$-version of Proposition 2.14 show that Z_F computes A. Therefore, Z witnesses that A is an $\mathcal{F}$-base. □

We note that if y_i is positive for all i, then the distribution of the bits of Ω as performed in the proof of Lemma 4.3 gives a measure-preserving, computable isomorphism $j\colon 2^\omega \to (2^\omega)^n$ such that $j(\Omega)$ witnesses that A is an $\mathcal{F}$-base.

We will prove the following generalisations of Propositions 2.16 and 2.17.

Proposition 4.4. *Every c.e. $\mathcal{F}$-base obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$.*

Proposition 4.5. *Every $\mathcal{F}$-base weakly obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$. In fact, if $\langle A_s \rangle$ is any computable approximation of an $\mathcal{F}$-base A, then there is a sub-approximation $\langle A_{s(n)} \rangle$ that witnesses that A weakly obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$.*

Example 4.2 shows that these propositions imply Propositions 2.16 and 2.17. We get an analogue of Theorem 2.18.

Theorem 4.6. *The following are equivalent for a set A:*

1. *A is an $\mathcal{F}$-base;*
2. *A obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$;*
3. *A is K-trivial and weakly obeys $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$.*

The proof of Theorem 4.6 is identical to the proof of Theorem 2.18, except that we use Propositions 4.4 and 4.5 and replace Lemma 2.15 by Lemma 4.3.

As discussed above, if $y_i > 0$ for all i then Ω is a universal witness for $\mathcal{F}$-bases, but only in a weak sense: if A is an $\mathcal{F}$-base then we can divide the digits of Ω effectively into n many parts that together witness that A is an $\mathcal{F}$-base. However, unlike the case of k/n-bases, we cannot require that this is the even division of bits which gives the standard isomorphism between 2^ω and $(2^\omega)^n$. For a very simple example, let $n = 3$ and $\mathcal{F} = \big\{\{1,2\},\{3\}\big\}$. Then $\|\mathcal{F}\| = 2$, so Theorem 4.6 says that being an $\mathcal{F}$-base is the same as being a 1/2-base. A 1/2-base may not be computable from any of the 3-columns of Ω; but if $\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3, \bar{\Omega}_4$ are the four 4-columns of Ω then the triple $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3 \oplus \bar{\Omega}_4)$ witnesses that every 1/2-base is an $\mathcal{F}$-base.

4.3. **A generalisation of the Loomis–Whitney inequality.** We need a geometric lemma for the proofs of Propositions 4.4 and 4.5. In the proofs of Propositions 3.4 and 3.5, we relied on the fact that the area of a 2-dimensional set is bounded by the product of the measures of its projections into each dimension. To generalise to $\mathcal{F}$-bases, we need to bound the volume of an n-dimensional set in terms of its projections onto the subspaces corresponding to members of $\mathcal{F}$.

We again fix $n \geqslant 1$ and a family $\mathcal{F}$ of subsets of $\{1, \dots, n\}$. Recall the definition of a normalised weighting of the sets in $\mathcal{F}$ that was used for the definition of $\|\mathcal{F}\|$: a sequence $\langle x_F \rangle_{F \in \mathcal{F}}$ of nonnegative real numbers such that $\sum_{\{F \,:\, i \in F\}} x_F \leqslant 1$ for all $i \leqslant n$.

Lemma 4.7. *Suppose that $\langle x_F \rangle$ is a normalised weighting of the sets in $\mathcal{F}$. If $U \subseteq (2^\omega)^n$ is Borel, then*

$$\mu(U) \leqslant \prod_{F \in \mathcal{F}} \mu(\pi_F[U])^{x_F}.$$

As we will see below, this result is a generalisation of the Loomis–Whitney inequality [29]. It is not new; it seems to first appear in 1995 in Bollobás and Thomason [8] as an application of their "box theorem", where it is stated for sufficiently nice compact subsets of $\mathbb{R}^n$, though the restriction is not essential. To prove their main result, they (apparently independently) reprove a weaker generalisation of the Loomis–Whitney inequality that (in its discrete form) is due to Shearer in 1978. Shearer's result first appeared in [11], where it is proved from an inequality on entropies. This connection between entropy and combinatorics has become common; a recent paper of Madiman, Marcus, and Tetali [31] starts with:

> It is well known in certain circles that there appears to exist an informal parallelism between entropy inequalities on the one hand, and set cardinality inequalities on the other.

In their paper, in fact, the authors derive the discrete version of Lemma 4.7 (i.e., Claim 4.7.1 below) from one of their main results, a corresponding inequality on entropy. An earlier derivation, with entropy replaced by Kolmogorov complexity, was given by Romashchenko, Shen, and Vereshchagin [36]. Claim 4.7.1 is an immediate consequence of their Theorems 1 and 2.

We provide a proof for completeness.

Proof. We will reduce the statement of the lemma to a statement of finite combinatorics. Before we do so, we make two simplifications:

(1) It suffices to prove the lemma for clopen sets U.
(2) We may assume that the weighting $\langle x_F \rangle$ is *tight*: $\sum_{\{F\,:\,i\in F\}} x_F = 1$ for all $i \leqslant n$.

For (1), note that the desired property of the set U is certainly closed under taking countable increasing unions. If U is closed, then $U = \bigcap_k U_k$, where $\langle U_k \rangle$ is a decreasing sequence of clopen sets. The compactness of $(2^\omega)^n$ implies that $\pi_F[U] = \bigcap_k \pi_F[U_k]$, and so taking limits gives the desired inequality for all closed sets, and hence for all F_σ sets. We then use the regularity of Lebesgue measure to replace an arbitrary Borel U by an F_σ subset of the same measure. The projections of this subset may be smaller, but of course, this makes the inequality stronger.

For (2), we add "slack sets". Let $\hat{\mathcal{F}} = \mathcal{F} \cup \{\{i\} : i \leqslant n\}$. For every $F \in \mathcal{F}$ that is not a singleton, let $\hat{x}_F = x_F$. For each $i \leqslant n$ such that $\{i\} \notin \mathcal{F}$, let $\hat{x}_{\{i\}} = 1 - \sum_{\{F\,:\,i\in F\}} x_F$; if $\{i\} \in \mathcal{F}$, then let $\hat{x}_{\{i\}} = x_{\{i\}} + (1 - \sum_{\{F\,:\,i\in F\}} x_F)$. Then $\langle \hat{x}_F \rangle$ is a tight weighting of the sets in $\hat{\mathcal{F}}$. If the lemma holds for U and for $\hat{\mathcal{F}}$, then it also holds for U and for $\mathcal{F}$, as $x_F \leqslant \hat{x}_F$ for all $F \in \mathcal{F}$, $\mu(\pi_F[U]) \leqslant 1$, and $\mu(\pi_{\{i\}}[U])^{\hat{x}_{\{i\}}} \leqslant 1$ for all $\{i\} \in \hat{\mathcal{F}} \setminus \mathcal{F}$.

So we suppose that U is clopen and that $\langle x_F \rangle$ is tight. Fix $m < \omega$ sufficiently large so that 2^{-m} is smaller than the granularity of U. This means that U is the union of basic clopen sets of the form $\prod_{i\leqslant n}[\sigma_i]$ where each σ_i is a binary string of length m.[8] For $i \leqslant n$, let X_i be the set 2^m of binary strings of length m. Define a relation $R \subseteq \prod_{i\leqslant n} X_i$ by letting $\bar{\sigma} = (\sigma_1, \dots, \sigma_n) \in R$ if $[\bar{\sigma}] = \prod_i [\sigma_i] \subseteq U$ (if $\bar{\sigma} \notin R$ then $[\bar{\sigma}] \cap U = \varnothing$). The measure of U is the *relative size* $d(R)$ of the relation R, the number of n-tuples in R divided by the number of possible tuples, i.e., the size of $\prod_i X_i$; in this case the relative size is $|R|/2^{mn}$.

Extend the notation π_F to give the obvious function from $\prod_{i\leqslant n} X_i$ to $\prod_{i\in F} X_i$ (erasing entries). For $F \in \mathcal{F}$, the forward image $\pi_F[R]$ is the relation associated with $\pi_F[U]$: for $\bar{\sigma} \in \prod_{i\in F} X_i$, if $\bar{\sigma} \in \pi_F[R]$ then $[\bar{\sigma}] \subseteq \pi_F[U]$; otherwise $[\bar{\sigma}]$ and $\pi_F[U]$ are disjoint. Thus $\mu(\pi_F[U])$ is the density of the relation $\pi_F[R]$, namely $d(\pi_F[R]) = |\pi_F[R]|/|\prod_{i\in F} X_i|$. The desired inequality for U follows from a combinatorial statement:

[8] Note that X_i does not depend on i here, but we need to index these sets since we soon deal with projections of $\prod_i X_i$.

Claim 4.7.1. Let $\langle X_i \rangle_{i \leqslant n}$ be a sequence of nonempty finite sets. Let $R \subseteq \prod_{i \leqslant n} X_i$ be a relation. Then

$$d(R) \leqslant \prod_{F \in \mathcal{F}} d(\pi_F[R])^{x_F}.$$

To visualise the situation, consider the example that defined 2/3-bases, i.e., $n = 3$ and $\mathcal{F} = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$. We use the optimal weighting $x_F = 1/2$ for all $F \in \mathcal{F}$, which witnesses the fact that $\|\mathcal{F}\| = 3/2$. The measure-theoretic inequality we need bounds $\mu(U)$ by the product of the square roots of the measures of its projections onto the three orthogonal 2-dimensional planes. The corresponding combinatorial statement involves a tripartite graph with three vertex sets X_1, X_2 and X_3: for $F = \{i,j\} \in \mathcal{F}$, $\pi_F[R]$ is a set of edges between X_i and X_j; the combinatorial lemma bounds the relative size of the triangle relation in the graph.

We prove Claim 4.7.1 by induction on $\sum_{i \leqslant n} |X_i|$ (in the example above, by induction on the number of vertices in the tripartite graph). The base case is $|X_i| = 1$ for all i, in which case either R is empty and has density 0, or $R = \prod_i X_i$ and each $\pi_F[R] = \prod_{i \in F} X_i$ has density 1. For the induction step, choose some $i^* \leqslant n$ such that $|X_{i^*}| > 1$. Partition X_{i^*} into two nonempty sets Y_0 and Y_1. For both $j = 0,1$, we apply the induction hypothesis to R_j, the restriction of R to those tuples whose $(i^*)^{\text{th}}$ entry lies in Y_j. Note that $R = R_0 \cup R_1$ is a disjoint union. We let:

- for $j < 2$, $y_j = |Y_j|$ and $r_j = |R_j|$;
- for $F \in \mathcal{F}$ and $j < 2$, $s_{F,j} = |\pi_F[R_j]|$;
- for nonempty $F \subseteq \{1,\dots,n\}$, $z_F = \prod_{i \in F \smallsetminus \{i^*\}} |X_i|$;
- and for brevity, $z^* = z_{\{1,\dots,n\}} = \prod_{i \neq i^*} |X_i|$.

Let $\mathcal{F}^* = \{F \in \mathcal{F} : i^* \in F\}$. The induction hypothesis yields, for both $j = 0,1$,

$$\frac{r_j}{y_j z^*} \leqslant \prod_{F \in \mathcal{F}^*} \left(\frac{s_{F,j}}{y_j z_F}\right)^{x_F} \cdot \prod_{F \in \mathcal{F} \smallsetminus \mathcal{F}^*} \left(\frac{s_{F,j}}{z_F}\right)^{x_F}.$$

The assumption that $\sum_{F \in \mathcal{F}^*} x_F = 1$ means that the occurrences of y_j cancel out. Let

$$q = z^* \cdot \prod_{F \in \mathcal{F}^*} \frac{1}{{z_F}^{x_F}} \cdot \prod_{F \in \mathcal{F} \smallsetminus \mathcal{F}^*} \left(\frac{|\pi_F[R]|}{z_F}\right)^{x_F}.$$

For all $F \in \mathcal{F}$ and $j < 2$, $s_{F,j} \leqslant |\pi_F[R]|$ and $x_F \geqslant 0$, so

$$r_j \leqslant q \cdot \prod_{F \in \mathcal{F}^*} {s_{F,j}}^{x_F}. \tag{4.1}$$

We observe that if $i^* \in F$, then $\pi_F(R) = \pi_F[R_0] \cup \pi_F[R_1]$ is a disjoint union. So to complete the induction step, we need to show that

$$\frac{r_0 + r_1}{(y_0 + y_1) z^*} \leqslant \prod_{F \in \mathcal{F}^*} \left(\frac{s_{F,0} + s_{F,1}}{(y_0 + y_1) z_F}\right)^{x_F} \prod_{F \in \mathcal{F} \smallsetminus \mathcal{F}^*} \left(\frac{|\pi_F[R]|}{z_F}\right)^{x_F}.$$

Equivalently, we need to show that

$$(r_0 + r_1) \leqslant q \cdot \prod_{F \in \mathcal{F}^*} (s_{F,0} + s_{F,1})^{x_F}.$$

This follows from Eq. (4.1) and the inequality

$$\prod_{F \in \mathcal{F}^*} {s_{F,0}}^{x_F} + \prod_{F \in \mathcal{F}^*} {s_{F,1}}^{x_F} \leqslant \prod_{F \in \mathcal{F}^*} (s_{F,0} + s_{F,1})^{x_F},$$

which follows from the weighted arithmetic mean–geometric mean inequality using the assumption that $\sum_{F\in\mathcal{F}^*} x_F = 1$. □

Lemma 4.7 can be seen as a generalisation of the Loomis–Whitney inequality [29], which bounds the measure of an n-dimensional set using the measures of its $(n-1)$-dimensional projections. To see the connection, we give a proof of the Loomis–Whitney result from ours.

Corollary 4.8 (Loomis and Whitney [29])**.** *Let $n \geqslant 1$. For $j \leqslant n$, let $\pi_j = \pi_{\{1,\dots,n\}\smallsetminus\{j\}}$ be the projection from $[0,1]^n$ to the $(n-1)$-dimensional orthogonal subspace $x_j = 0$. If $U \subset [0,1]^n$ is Borel, then*

$$\mu(U)^{n-1} \leqslant \prod_{j\leqslant n} \mu(\pi_j[U]).$$

Proof. Apply Lemma 4.7 with $\mathcal{F}$ being the set of subsets of $\{1,2,\dots,n\}$ of size $n-1$, and use the optimal weighting $x_F = 1/(n-1)$. □

Lemma 4.7 gives an upper bound for the size of a set in terms of its projections. We will use it in the reverse direction, to give a lower bound for the size of one of the projections in terms of the size of the set. This can be stated in a clean, sharp form using $\|\mathcal{F}\|$. As usual, fix a family $\mathcal{F}$.

Lemma 4.9. *Let $U \subseteq (2^\omega)^n$ be Borel. There is an $F \in \mathcal{F}$ such that*

$$\mu(\pi_F[U]) \geqslant \mu(U)^{1/\|\mathcal{F}\|}.$$

Moreover, this cannot be improved: there is a $U \subseteq (2^\omega)^n$, of arbitrary measure $\leqslant 1$, such that for all $F \in \mathcal{F}$, $\mu(\pi_F[U]) \leqslant \mu(U)^{1/\|\mathcal{F}\|}$.

Proof. Let U be Borel and assume that $\mu(\pi_F[U]) < \mu(U)^{1/\|\mathcal{F}\|}$ for all $F \in \mathcal{F}$. Let $\langle x_F\rangle$ be an optimal solution for the definition of $\|\mathcal{F}\|$ (a normalised weighting of the sets in $\mathcal{F}$ such that $\sum x_F = \|\mathcal{F}\|$). Apply Lemma 4.7 using $\langle x_F\rangle$ to get

$$\mu(U) \leqslant \prod_{F\in\mathcal{F}} \mu(\pi_F[U])^{x_F} < \prod_{F\in\mathcal{F}} \mu(U)^{x_F/\|\mathcal{F}\|} = \mu(U),$$

where the strict inequality follows from the fact that $\|\mathcal{F}\| > 0$ and so $x_F > 0$ for some F. This is a contradiction, so there must be some $F \in \mathcal{F}$ such that $\mu(\pi_F[U]) \geqslant \mu(U)^{1/\|\mathcal{F}\|}$.

To prove sharpness, fix a measure c (which must be in $[0,1]$). Let $y \in \mathbb{R}^n$ be an optimal solution for the dual problem defining $\|\mathcal{F}\|$: a normalised weighting of the coordinates such that $\sum_i y_i = \|\mathcal{F}\|$. Let $U = \prod_{i\leqslant n}\left[0, c^{y_i/\|\mathcal{F}\|}\right]$. Note that $\mu(U) = \prod_{i\leqslant n} c^{y_i/\|\mathcal{F}\|} = c$. Fix any $F \in \mathcal{F}$. Since $\sum_{i\in F} y_i \geqslant 1$ and $c \leqslant 1$,

$$\mu(\pi_F[U]) = \prod_{i\in F} c^{y_i/\|\mathcal{F}\|} = c^{\sum_{i\in F} y_i/\|\mathcal{F}\|} \leqslant c^{1/\|\mathcal{F}\|} = \mu(U)^{1/\|\mathcal{F}\|}.$$ □

4.4. **The proofs of Propositions 4.4 and 4.5.** We extend the proofs of Propositions 3.4 and 3.5. Say Z witnesses that A is an $\mathcal{F}$-base; we fix functionals Ψ_F for $F \in \mathcal{F}$ such that $\Psi_F(Z_F) = A$. The hungry sets G_τ only contain tuples $X \in (2^\omega)^n$ such that $\Psi_F(X_F) \succcurlyeq \tau$ for all $F \in \mathcal{F}$. If every ε-construction failed, then in the c.e. case we would capture Z by a difference test based on the effectively closed class $P = \bigcap_{F\in\mathcal{F}} \pi_F^{-1}[P_F]$, where $P_F \subseteq (2^\omega)^F$ is the class of oracles $Y \in (2^\omega)^F$ for which $\Psi_F(Y)$ does not lie to the left of A. We need to show that this is impossible. As in the case of 1/2-bases we use the concept of Lebesgue density; again by Lemma 3.3 due to [7], it suffices to show that the density $\varrho(P \,|\, Z)$ is positive.

We show that *if we choose the functionals cleverly*, then the density $\varrho(P \,|\, Z)$ is actually 1. We will want to show that $\varrho(P_F \mid Z_F) = 1$ for all $F \in \mathcal{F}$; this implies that $\varrho(\pi_F^{-1}[P_F] \mid Z) = 1$, from which $\varrho(P \mid Z) = 1$ follows. However, consider the redundant case $n = 2$ and $\mathcal{F} = \big\{\{1\}, \{2\}, \{1, 2\}\big\}$. In general, for any F, we could choose Ψ_F in such a way that P_F is contained in an arbitrary effectively closed subset of $(2^\omega)^F$.[9] For $F = \{1, 2\}$ and $Z = \Omega$, we have $Z_F = \Omega$, which is complete, and so we could choose P_F so that $\varrho(P_F \mid \Omega) = 0$. However, Z does witness that 1/2-bases are $\mathcal{F}$-bases too. The problem occurs because the reduction Ψ_F does not need to consult both parts of the oracle. We prove:

Lemma 4.10. *Suppose that A is an $(n-1)/n$-base as witnessed by Z. If $F \subseteq \{1, \dots, n\}$ is minimal such that $A \leqslant_{\mathrm{T}} Z_F$, then Z_F is a density-one point for effectively closed classes in $(2^\omega)^F$.*

Given the lemma, for each $F \in \mathcal{F}$ we choose some minimal $\hat{F} \subseteq F$ such that $A \leqslant_{\mathrm{T}} Z_{\hat{F}}$; for Ψ_F we choose a functional that only looks at the columns indexed by elements of $\hat{F}$. We then have that $P_F = Q \times (2^\omega)^{F \smallsetminus \hat{F}}$ where $Q \subseteq (2^\omega)^{\hat{F}}$ is effectively closed. Lemma 4.10 says that $\varrho(Q \,|\, Z_{\hat{F}}) = 1$, from which it follows that $\varrho(P_F \mid Z_F) = 1$, as required. As mentioned earlier, the fact that $\|\mathcal{F}\| > 1$ means that Z witnesses that A is a $(n-1)/n$-base, so the lemma applies.

To prove Lemma 4.10, we use a weak van-Lambalgen-type property for Lebesgue density:

Lemma 4.11. *Let $X_0, X_1 \in 2^\omega$. Suppose that X_0 is a density one point relative to X_1, and X_1 is a density one point relative to X_0. Then $X = (X_0, X_1)$ is a density one point.*

Proof. Let $C \subseteq 2^\omega \times 2^\omega$ be an effectively closed set such that $X \in C$. For $Z \in 2^\omega$, let $C_Z = \{Y \in 2^\omega : (Z, Y) \in C\}$. Let $\varepsilon > 0$. Since $X_1 \in C_{X_0}$ and C_{X_0} is effectively closed relative to X_0, there is an n^* such that for all $m \geqslant n^*$, $\mu(C_{X_0} \,|\, X_1 \upharpoonright m) \geqslant 1-\varepsilon$. Now let

$$P = \{Z \in 2^\omega : (\forall m \geqslant n^*)\; \mu(C_Z \,|\, X_1 \upharpoonright m) \geqslant 1 - \varepsilon\}.$$

The set P is effectively closed relative to X_1, so there is an $n^{**} \geqslant n^*$ such that for all $m \geqslant n^{**}$, $\mu(P \,|\, X_0 \upharpoonright m) \geqslant 1-\varepsilon$. So if $m \geqslant n^{**}$, then $\mu(C \,|\, X \upharpoonright 2m) \geqslant (1-\varepsilon)^2$. $\square$

Proof of Lemma 4.10. By permuting, we may assume that $F = \{1, 2, \dots, k\}$ for some $k < n$. Let $W = (Z_{k+1}, \dots, Z_n)$. For $i \leqslant k$ let $Y_i = (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_k)$.

First we see that for all $i \leqslant k$, Z_i is a density 1 point relative to Y_i. Suppose not. By [7], a random set X that is not a density 1 point is *LR-hard*: $\varnothing' \leqslant_{LR} X$, which means that every X-random set is 2-random. Relativising, we see that $(Z_i, Y_i) \geqslant_{LR} Y_i'$. Now W is random relative to $Z_F = (Z_i, Y_i)$ and so is 2-random relative to Y_i. Every weakly 2-random set forms a minimal pair with $\varnothing'$. Relativising to Y_i, every set that is computable from both (W, Y_i) and Y_i' is Y_i-computable. Since (W, Y_i) consists of $n-1$ many columns of Z, it computes A. Also A is Δ^0_2, so it certainly is Y_i'-computable. Hence $A \leqslant_{\mathrm{T}} Y_i$, contradicting the minimality of F.

[9] If $Q \subseteq (2^\omega)^F$ is effectively closed and $\Gamma(Z_F) = A$, then we modify Γ so that when we discover that σ drops out of Q at stage s, we map all $X \succcurlyeq \sigma$ to $\Gamma_s(X)\hat{\ }0^\infty$, which would lie to the left of A.

Now by induction on $i \leqslant k$, we see that $(Z_1, \dots, Z_i)$ is a density 1 point relative to $(Z_{i+1}, \dots, Z_k)$. This is already established for $i = 1$. Let $i > 1$ and suppose that $(Z_1, \dots, Z_{i-1})$ is a density 1-point relative to $(Z_i, \dots, Z_k)$. We use Lemma 4.11 relativised to $(Z_{i+1}, \dots, Z_k)$ (and the fact that Z_i is a density 1 point relative to Y_i) to obtain the desired result. □

For the benefit of the reader, we sketch the proof of Proposition 4.4 (Proposition 4.5 is again easier).

Sketch of the proof of Proposition 4.4. We explain how to modify the proof of Proposition 3.4. Let A be a c.e. $\mathcal{F}$-base, witnessed by the tuple $Z = (Z_1, Z_2, \dots, Z_n)$. For $F \in \mathcal{F}$ wisely choose a functional Ψ_F such that $\Psi_F(Z_F) = A$, as discussed after the statement of Lemma 4.10; it only looks at oracles for a minimal $\hat{F} \subseteq F$. For each $F \in \mathcal{F}$ and $s < \omega$ we let $P_{F,s}$ be the set of $X \in (2^\omega)^F$ such that $\Psi_{F,s}(X)$ does not lie to the left of A_s. We let $P_s = \bigcap_{F \in \mathcal{F}} \pi_F^{-1}[P_{F,s}]$.

Again we fix a dyadic rational $\varepsilon > 0$, and enumerate clopen sets $G_{\tau,s} \subseteq (2^\omega)^n$, with $\Psi_{F,s}(\pi_F(X)) \succcurlyeq \tau$ for all $F \in \mathcal{F}$ and $X \in G_{\tau,s}$. The goal $\varepsilon \cdot (\Omega_{|\tau|-\Omega_{|\tau|-1}})$ for $(G_\tau \setminus G_{\tau^-}) \cap P$ is the same, as well as the confirmation process and the instructions of how to increase each $G_{\tau,s}$.

The definitions, at the end of the construction, of P and G are the same, as well as the argument that $\mu(G \cap P) \leqslant \varepsilon$. Similar also is the argument that if some initial segment of A is confirmed at only finitely many stages then $Z \in G \cap P$. If this happens for every ε, then Z is captured by the A-difference test $\langle P \cap G_\varepsilon \rangle$. As before it follows that $\varrho(P \mid Z) = 0$. As described above, for each $F \in \mathcal{F}$, since $Z_{\hat{F}} \in P_{\hat{F}}$, $\varrho(P_{\hat{F}} \mid Z_{\hat{F}}) = 1$ (Lemma 4.10), and so $\varrho(P_F \mid Z_F) = 1$, so $\varrho(\pi_F^{-1}[P_F] \mid Z) = 1$, so $\varrho(P \mid Z) = 1$.

We again choose ε such that in the ε-construction, every initial segment of A is confirmed infinitely often. As above, we define the increasing computable sequence $\langle s_k \rangle$ so that $A_{s(k)} \upharpoonright (k+1)$ is confirmed at stage s_k. We also define V_k exactly as above. Again the fact that every string $A_{s_k} \upharpoonright x+1, \dots, A_{s_k} \upharpoonright (k+1)$ is confirmed implies that $\mu(V_k) = \varepsilon \cdot (\Omega_{k+1} - \Omega_x)$. For every $F \in \mathcal{F}$ and every k, $\pi_F[V_k] \subseteq \Psi_F^{-1}[A_{s_k} \upharpoonright (x_k+1)]$, which is disjoint from $P_{F,s_{k'}}$ for all $k' > k$, whereas $\pi_F[V_k] \subseteq P_{F,s_k}$ for all k. Hence for $k < k'$ we get $\pi_F[V_k] \cap \pi_F[V_{k'}] = \varnothing$ for all $F \in \mathcal{F}$. Finally, Lemma 4.9 shows that for every k there is some $F \in \mathcal{F}$ such that

$$\mu(\pi_F[V_k]) \geqslant (\varepsilon \cdot (\Omega_{k+1} - \Omega_x))^{1/\|\mathcal{F}\|}.$$

This shows that the total $\mathbf{c}_{\Omega,1/\|\mathcal{F}\|}$-cost of this enumeration is bounded by $|\mathcal{F}|/\varepsilon^{1/\|\mathcal{F}\|}$. □

5. Consequences of the characterisation of $\mathcal{F}$-bases

The generality of the development in the previous section allows us to prove a number of interesting results. The first was already mentioned, namely the characterisation of k/n-bases: Example 4.2 shows that Propositions 4.4 and 4.5 imply Propositions 2.16 and 2.17 and so complete our proof of Theorems 1.3 and 2.18.

5.1. **Cyclic k/n-bases.** Note that $\binom{n}{k}$ can be quite large compared to n, especially if $k \approx n/2$. This makes the definition of a k/n-base look very demanding, as it requires a set to be computable from a large number of different random tuples. It turns out that we can get away with a weaker hypothesis. Fix natural numbers $0 < k < n$. For each $i \leqslant n$, let $F_i = \{i, i+1 \pmod n, \dots, i+k-1 \pmod n\}$. Let

$\mathcal{F} = \{F_i\}_{i<n}$. We call a set A a *cyclic k/n-base* if it an $\mathcal{F}$-base. Note that this definition only requires A to be computable from n distinct tuples. And yet, it is enough to capture that A is a k/n-base.

Proposition 5.1. *A set is a cyclic k/n-base if and only if it is a k/n-base.*

Proof. Clearly, every k/n-base is a cyclic k/n-base. So assume that A is a cyclic k/n-base. Let $\mathcal{F}$ be as above. So A is an $\mathcal{F}$-base. We show that $\|\mathcal{F}\| = n/k$; Theorems 2.18 and 4.6 imply that A is a k/n-base.

Again we use the duality in the definition of $\|\mathcal{F}\|$. To bound the norm from below consider the constant weighting $x_F = 1/k$ for all $F \in \mathcal{F}$. This is normalised since every $i \leqslant n$ is an element of precisely k many sets in $\mathcal{F}$. Hence $\|\mathcal{F}\| \geqslant n/k$. From above, consider the weighting $y_i = 1/k$; each set in $\mathcal{F}$ has size k and so $\langle y_i \rangle$ is normalised for $\mathcal{F}$. Hence $\|\mathcal{F}\| \leqslant n/k$. ☐

Since every k/n-base is witnessed by the n-columns of Ω, so is every cyclic k/n-base.

5.2. **Degenerate k/n-bases.** Assume that $1 < k < n$. We call A a *degenerate* k/n-base if there is a random tuple Z that witnesses that A is a k/n-base but this is not tight: there is some $G \subseteq \{1, \dots, n\}$ such that $|G| < k$ and $A \leqslant_T Z_G$. We show that degenerate k/n-bases must obey cost functions that are stronger than $\mathbf{c}_{\Omega,k/n}$.

Proposition 5.2. *Let $p = \max\left\{\frac{k}{n+1}, \frac{k-1}{n-1}\right\}$. A set A is a degenerate k/n-base if and only if it is a p-base.*

Proof. Let $\mathcal{F}$ consist of all k-element subsets of $\{1, \dots, n\}$ along with $G = \{1, \dots, k-1\}$. Note that a set is a degenerate k/n-base if and only if it is an $\mathcal{F}$-base, so by Theorem 4.6, all we have to do is prove that $\|\mathcal{F}\| = 1/p$. Let M be the matrix from the definition of $\|\mathcal{F}\|$. There are two cases.

Case 1: $2k - 1 \leqslant n$. In this case, it is easy to check that $p = k/(n+1)$. Consider the following vector $x \in \mathbb{R}^{|\mathcal{F}|}$. We will indicate the coordinate of x corresponding to $F \in \mathcal{F}$ by x_F. Let $x_G = 1$. If F is a k-element subsets of $\{k-1, \dots, n-1\}$, let $x_F = \frac{n-k+1}{k\binom{n-k+1}{k}}$ (note that such sets exists because $n - k + 1 \geqslant k$). Let the other coordinates of x be 0. We claim that $Mx = \mathbb{1}$. If $i \in \{0, \dots, k-2\}$, then $Mx(i) = \sum\{x_F : F \in \mathcal{F} \text{ and } i \in F\} = x_G = 1$. If $i \in \{k-1, \dots, n-1\}$, then i is in a fraction of $k/(n-k+1)$ of the k-element subsets of $\{k-1, \dots, n-1\}$. There are $\binom{n-k+1}{k}$ such sets F, each with $x_F = \frac{n-k+1}{k\binom{n-k+1}{k}}$, so $Mx(i) = \sum\{x_F : F \in \mathcal{F} \text{ and } i \in F\} = 1$. This proves that $Mx = \mathbb{1}$, hence

$$\|\mathcal{F}\| \geqslant \sum_{F \in \mathcal{F}} x_F = 1 + \binom{n-k+1}{k} \frac{n-k+1}{k\binom{n-k+1}{k}} = \frac{n+1}{k} = 1/p.$$

Next consider the following vector $y \in R^n$. For $i \in \{0, \dots, k-2\}$, let $y_i = 1/(k-1)$. For $i \in \{k-1, \dots, n-1\}$, let $y_i = 1/k$. We claim that $M^T y \geqslant \mathbb{1}$. Again, we use elements of $\mathcal{F}$ to index the corresponding dimensions. Note that $M^T y(G) = \sum_{i \in G} y_i = (k-1)\frac{1}{k-1} = 1$. For any other $F \in \mathcal{F}$ we have $M^T y(F) = \sum_{i \in F} y_i \geqslant \sum_{i \in F} \frac{1}{k} = k\frac{1}{k} = 1$. This proves that $M^T y \geqslant \mathbb{1}$, hence

$$\|\mathcal{F}\| \leqslant \sum_{i<n} y_i = (k-1)\frac{1}{k-1} + (n-k+1)\frac{1}{k} = \frac{n+1}{k} = 1/p.$$

Therefore, $\|\mathcal{F}\| = 1/p$.

Case 2: $2k-1 > n$. In this case, it is easy to check that $p = (k-1)/(n-1)$. Consider the following vector $x \in \mathbb{R}^{|\mathcal{F}|}$. Let $x_G = (n-k)/(k-1)$. If F is a k-element subset of $\{0,\dots,n-1\}$ that contains $\{k-1,\dots,n-1\}$, let $x_F = 1/\binom{k-1}{2k-1-n}$ (note that such sets exist because $k > n-k+1$). Let the other coordinates of x be 0. We claim that $Mx = \mathbb{1}$. If $i \in \{0,\dots,k-2\}$, then i is in G and in a fraction of $(2k-1-n)/(k-1)$ of the k-element subsets of $\{0,\dots,n-1\}$ that contain $\{k-1,\dots,n-1\}$. There are $\binom{k-1}{2k-1-n}$ such sets. Therefore, $Mx(i) = \sum\{x_F : F \in \mathcal{F} \text{ and } i \in F\} = (n-k)/(k-1) + (2k-1-n)/(k-1) = 1$. On the other hand, if $i \in \{k-1,\dots,n-1\}$, then i is not in G but is in every k-element subset of $\{0,\dots,n-1\}$ that contains $\{k-1,\dots,n-1\}$. So $Mx(i) = \sum\{x_F : F \in \mathcal{F} \text{ and } i \in F\} = 1$. This proves that $Mx = \mathbb{1}$, hence

$$\|\mathcal{F}\| \geqslant \sum_{F\in\mathcal{F}} x_F = \frac{n-k}{k-1} + \binom{k-1}{2k-1-n}\frac{1}{\binom{k-1}{2k-1-n}} = \frac{n-1}{k-1} = 1/p.$$

Next consider the following vector $y \in R^n$. For $i \in \{0,\dots,k-2\}$, let $y_i = 1/(k-1)$. For $i \in \{k-1,\dots,n-1\}$, let $y_i = \frac{n-k}{(n-k+1)(k-1)}$. We claim that $M^T y \geqslant \mathbb{1}$. As in Case 1, $M^T y(G) = \sum_{i\in G} y_i = (k-1)\frac{1}{k-1} = 1$. Consider any other $F \in \mathcal{F}$. At least $k-(n-k+1) = 2k-n-1$ coordinates in F are from $\{0,\dots,k-2\}$. Therefore, $M^T y(F) = \sum_{i\in F} y_i \geqslant (2k-n-1)\frac{1}{k-1} + (n-k+1)\frac{n-k}{(n-k+1)(k-1)} = 1$. This proves that $M^T y \geqslant \mathbb{1}$, hence

$$\|\mathcal{F}\| \leqslant \sum_{i<n} y_i = (k-1)\frac{1}{k-1} + (n-k+1)\frac{n-k}{(n-k+1)(k-1)} = \frac{n-1}{k-1} = 1/p.$$

Therefore, $\|\mathcal{F}\| = 1/p$. □

Corollary 5.3. *There is a (c.e.) k/n-base that is not a degenerate k/n-base.*

Proof. It is easy to see that both $k/(n+1)$ and $(k-1)/(n-1)$ are less than k/n. Therefore, by Proposition 2.6, there is a c.e. set A that obeys $\mathbf{c}_{\Omega,k/n}$ but not $\mathbf{c}_{\Omega,p}$ for $p = \max\left\{\frac{k}{n+1}, \frac{k-1}{n-1}\right\}$. By Proposition 5.2, A is not a degenerate k/n-base. □

5.3. **$1/\omega$-bases.** A $1/n$-base is computable from each of the n coordinates of some Martin-Löf random $(Z_1,\dots,Z_n)$. One can generalise this to infinite sequences. We now work in the computable probability space $(2^\omega)^\omega$. It is effectively isomorphic to 2^ω via a measure-preserving map. Such a map is determined by a computable partition of ω into infinitely many computable sets (“columns”). Below we will use the fact that this can be done in such a way that the density of each column is positive.

Definition 5.4. A set A is a $1/\omega$-*base* if there is a Martin-Löf random sequence $(Z_1, Z_2, \dots)$ such that $(\forall i)\ A \leqslant_{\mathrm{T}} Z_i$.

Such bases are now easy to characterise.

Proposition 5.5. *A set is a $1/\omega$-base iff it is a p-base for every rational $p > 0$.*

Proof. Assume that A is a $1/\omega$-base witnessed by $(Z_1, Z_2, \dots)$. For each n, the sequence $(Z_1,\dots,Z_n)$ witnesses that A is a $1/n$-base. This implies that A is a p-base for every rational $p > 0$

Now assume that A is a p-base for every rational $p > 0$. Consider breaking Ω up into countably many sequences $\{\bar{\Omega}_n\}_{n\in\omega}$ such that $\Omega = \Omega_1 \oplus (\Omega_2 \oplus (\Omega_3 \oplus (\cdots)))$, where here $\oplus$ is the usual split into evens and odds. In other words, $\bar{\Omega}_n$ is a 2^{-n}-part of Ω. For each n, we know that A is a 2^{-n}-base. Hence by Theorem 2.18, $A \leqslant_T \bar{\Omega}_n$. Therefore, A is a $1/\omega$-base as witnessed by the sequence $(\bar{\Omega}_1, \bar{\Omega}_2, \dots)$. □

The proof shows that every $1/\omega$-base is witnessed by a single Martin-Löf random sequence $(\bar{\Omega}_1, \bar{\Omega}_2, \dots)$ that arises from a computable partition of Ω. Again we remark that the proof used a partition of ω into columns of positive density; if we use Gödel's pairing function (as is commonly done), then each column has density 0 and the proof will not work. This distinction is only important when we consider the ways in which Ω can be considered as a universal witness for being a $1/\omega$-base; it does not affect the definition of being a $1/\omega$-base, in that a set A is a $1/\omega$-base if and only for some, or any, effective measure-preserving isomorphism $j\colon 2^\omega \to (2^\omega)^\omega$ there is a random sequence $Z \in 2^\omega$ such that A is computable from each coordinate of $j(Z)$.

As mentioned in the introduction, the notion of a $1/\omega$-base could theoretically be weakened, but we obtain an equivalent notion. The proof of the first direction of Proposition 5.5 shows:

Proposition 5.6. *A set A is a $1/\omega$-base if and only if there is a countable infinite set $Q \subset 2^\omega$ such that: (a) every $Z \in Q$ computes A; and (b) the join of any finitely many elements of Q is random.*

5.3.1. *$1/\omega$-bases and strong jump-traceability.* Recall that a set A is ω-c.a. if it can be computably approximated with a computably bounded number of changes; equivalently, $A \leqslant_{wtt} \varnothing'$. A set is *strongly jump-traceable* (SJT) if it is h-jump traceable for every order function h; see [23] for a survey. Every strongly jump traceable set is a $1/\omega$-base. For, by [21] and [14] together, any SJT set is computable from every ω-c.a. random sequence; the columns of Ω are ω-c.a.

On the other hand, there is an $1/\omega$-base that is not SJT. To see this, let $\mathbf{c} = \sum_n 2^{-n}\mathbf{c}_{\Omega,2^{-n}}$. Then any set obeying $\mathbf{c}$ is an $1/\omega$-base, and $\mathbf{c}$ is a benign cost function in the sense of [22]. Thus there is a computable order h such that every h-jump traceable obeys $\mathbf{c}$ (ibid.). But by work of Ng [32], h-jump traceability is strictly weaker than SJT.

We conjecture that the $1/\omega$ bases form a Π^0_4 complete ideal.

6. Robust computability from random sequences

Recall that a set is *robustly computable* from a sequence Z if it is computable from every Y such that the upper density of $Y \triangle Z$ is 0 (such a Y is called a "coarse description" of Z). This notion was investigated in [24], where it is shown that every set that is robustly computable from a random sequence is K-trivial, and in fact, is a $(k-1)/k$-base for some k.

In this section we provide the proof of the converse, Theorem 1.5, which states that the following are equivalent for a set A:

(1) $A \in \mathcal{B}_{<1}$ (that is, A is a p-base for some $p < 1$).
(2) A is robustly computable from some random sequence.
(3) A is robustly computable from Ω.
(4) There is a $\delta > 0$ such that A is computable from all sets Z such that the upper density of $Z \triangle \Omega$ is less than δ.

Proof. (4)→(3)→(2) are trivial. As mentioned, the implication (2)→(1) is in the proof of [24, Thm. 3.2].

It remains to show (1) → (4). Recall that for strings σ, τ of the same length n, we let

$$d(\sigma,\tau) = \frac{|\{i : \sigma(i) \neq \tau(i)\}|}{n}$$

and that for $X, Y \in 2^\omega$ we let $d(X,Y) = \limsup_n d(X \upharpoonright n, Y \upharpoonright n)$. For all strings σ and all $q \in [0,1]$, we let $B(\sigma, q) = \{\tau : |\tau| = |\sigma| \ \& \ d(\sigma,\tau) \leqslant q\}$. A well-known estimate gives $|B(\sigma,q)| \leqslant 2^{H(q)n}$ when $q \leqslant 1/2$, where H is binary entropy: $H(q) = -q\log_2(q) - (1-q)\log_2(1-q)$ (see for example [30, Cor.9,p.310]). Now choose δ small enough so that $H(2\delta) < 1-p$ (so if p is close to 1, then δ will be small; if p is very small, then δ can approach $1/4$).

Let A be a p-base; take $Z \in 2^\omega$ such that $d(\Omega, Z) < \delta$. We show that Z computes A. By our proof of Theorem 2.18, we may assume that A is c.e. For every string τ, let $G_\tau = \bigcup\{[\rho] : \rho \in B(\tau, 2\delta)\}$. Then

$$\mu(G_\tau) = 2^{-|\tau|}|B(\tau,2\delta)| \leqslant 2^{-|\tau|}2^{(1-p)|\tau|} = 2^{-p|\tau|}.$$

Fix $\sigma \prec Z$ such that for all $m \geqslant |\sigma|$, $d(Z \upharpoonright m, \Omega \upharpoonright m) \leqslant \delta$. We define a functional Γ using an approximation $\langle A_s \rangle$ of A that witnesses that A obeys $\mathbf{c}_{\Omega,p}$. We also use an approximation of Ω such that $\Omega_{s+1} - \Omega_s \geqslant 2^{-s-1}$. We define Γ as follows: for every $n \geqslant |\sigma|$, for every string τ of length n extending σ, we set $\Gamma(\tau, n) = A_s(n)$ where s is the least stage $s > n$ such that for all $m \in [|\sigma|, n]$, $d(\tau \upharpoonright m, \Omega_s \upharpoonright m) \leqslant \delta$ (note δ and not 2δ). If there is no such stage s, then $\Gamma(\tau, n)\uparrow$.

The assumption on Z implies that $\Gamma(Z, n)\downarrow$ for all $n \geqslant |\sigma|$. Let $s(n)$ be the stage at which the computation $\Gamma(Z \upharpoonright n, n)$ is made. We need to show that for all but finitely many $n \in A$, n enters A by stage $s(n)$.

We enumerate open sets U_n for $n \geqslant |\sigma|$. If $n \notin A$ then $U_n = \varnothing$. If $n \in A$, let $t = t(n)$ be the stage at which n is enumerated into A (i.e., $n \in A_t \setminus A_{t-1}$). If $t(n) \leqslant n$ then $U_n = \varnothing$. Suppose that $n < t(n)$. Resembling the proof of Proposition 2.9, let $k = k_t$ be the unique k such that $2^{-k-1} \leqslant \Omega_t - \Omega_n < 2^{-k}$. Note that by our choice of approximation to Ω, we have $k \leqslant n$. We let $U_n = \bigcup_{s=n}^{t(n)} G_{\Omega_s \upharpoonright k}$. Again, there are at most two values ρ for $\Omega_s \upharpoonright k$ for $s \in [n,t]$. Thus our calculation above shows that $\mu(U_n) \leqslant 2 \cdot 2^{-pk} \leqslant 2 \cdot 2^p \cdot (\Omega_t - \Omega_n)^p$. Recall that the $\mathbf{c}_{\Omega,p}$-cost of enumerating n into A is exactly $(\Omega_t - \Omega_n)^p$. Since A obeys $\mathbf{c}_{\Omega,p}$, we see that $\sum_n \mu(U_n)$ is finite, that is, $\langle U_n \rangle$ is a Solovay test. Thus, $\Omega \in U_n$ for only finitely many n.

Since $\Omega_s \to \Omega$, for all but finitely many n, $\Omega - \Omega_n < 2^{-|\sigma|}$, which shows that for all but finitely many $n \in A$, $k_{t(n)} \geqslant |\sigma|$. Let $n \geqslant |\sigma|$ and suppose that $\Gamma(Z,n) \neq A(n)$; so $n \in A$ and $s(n) < t(n)$. Suppose that $k_{t(n)} \geqslant |\sigma|$. We show that $\Omega \in U_n$. Note that $n < s(n)$ so $n < t(n)$. Let $k = k_{t(n)}$. Then $G_{\Omega_{s(n)} \upharpoonright k} \subseteq U_n$, and since $k \in [|\sigma|, n]$, we have $d(Z \upharpoonright k, \Omega_{s(n)} \upharpoonright k) \leqslant \delta$. But by assumption on Z, $d(Z \upharpoonright k, \Omega \upharpoonright k) \leqslant \delta$. So $d(\Omega_{s(n)} \upharpoonright k, \Omega \upharpoonright k) \leqslant 2\delta$, i.e., $\Omega \upharpoonright k \in B(\Omega_{s(n)} \upharpoonright k, 2\delta)$, so $\Omega \in G_{\Omega_{s(n)} \upharpoonright k}$. ☐

We remark that (2)→(4) above is implied by [24, Thm. 3.7], which is more general; the proof is more elaborate.

Theorem 3.19 of [24] states that not every K-trivial set is robustly computable from a random sequence. This fact can now be established using cost functions. It is not difficult to construct a cost function $\mathbf{c}$ such that $\underline{\mathbf{c}} >^\times \underline{\mathbf{c}}_\Omega$ but for all $p < 1$, $\underline{\mathbf{c}} <^\times \underline{\mathbf{c}}_{\Omega,p}$. By Proposition 2.6 there is a set A obeying $\mathbf{c}_\Omega$ but not $\mathbf{c}$. That set is

K-trivial but not a p-base for any $p < 1$, hence not robustly reducible to a random. We extend this result a little. Say that an ideal $\mathcal{I} \subseteq \Delta^0_2$ is characterised by a cost function $\mathbf{c}$ if $\mathcal{I}$ is the collection of sets that obey $\mathbf{c}$.

Proposition 6.1. *The ideal $\mathcal{B}_{<1}$ is not characterised by a cost function.*

In particular, $\mathcal{B}_{<1}$ is not the ideal of K-trivial sets, as the latter is characterised by $\mathbf{c}_\Omega$. Proposition 6.1 gives the first example of a Σ^0_3 subideal of the K-trivial sets that is not characterised by any cost function.[10] The next lemma, which is key to the proof of Proposition 6.1, essentially says that there is no greatest lower bound for a strictly descending uniform sequence of cost functions.

Lemma 6.2. *Let $\langle \mathbf{d}_n \rangle$ be a computable sequence of cost functions such that $(\forall n)$ $\mathbf{d}_{n+1} \leqslant \mathbf{d}_n$. Let $\mathbf{e}$ be a cost function such that $(\forall n)$ $\underline{\mathbf{e}} \leqslant^\times \underline{\mathbf{d}}_n$ and $\underline{\mathbf{d}}_n \nleqslant^\times \underline{\mathbf{e}}$. Then there is a cost function $\mathbf{c} \geqslant \mathbf{e}$ such that $(\forall n)$ $\underline{\mathbf{c}} \leqslant^\times \underline{\mathbf{d}}_n$ and $\underline{\mathbf{c}} \nleqslant^\times \underline{\mathbf{e}}$.*

Proof. We define $\mathbf{c}(x, s)$ by induction on s, starting with $\mathbf{c}(x, s) = 0$ for all $x \geqslant s$. At stage s we let, for each $n < s$, $k_s(n)$ be the least k such that $n \cdot \mathbf{e}(k, s) \leqslant \mathbf{d}_n(k, s)$ (for all s and n, $k_s(n) \leqslant s$ by one of our assumptions on cost functions). We then define, for $x < s$,

$$\mathbf{c}(x, s) = \begin{cases} \max\{\mathbf{c}(x, s-1), n \cdot \mathbf{e}(x, s)\} & \text{if } n \text{ is greatest such that } x = k_s(n); \\ \max\{\mathbf{c}(x, s-1), \mathbf{e}(x, s)\} & \text{if } x \neq k_s(n) \text{ for all } n < s. \end{cases}$$

The point is that for each n, the set $\{k_s(n) : s < \omega\}$ is bounded: since $\underline{\mathbf{d}}_n \nleqslant^\times \underline{\mathbf{e}}$ there is some k such that for almost all s, $n \cdot \mathbf{e}(k, s) < \mathbf{d}_n(k, s)$. If $k = k_s(n)$ for infinitely many s then $n \cdot \underline{\mathbf{e}}(k) \leqslant \underline{\mathbf{c}}(k)$, so $\underline{\mathbf{c}} \nleqslant^\times \underline{\mathbf{e}}$. But this also shows that for all n, for almost all k, $\underline{\mathbf{c}}(k) \leqslant \max\{\underline{\mathbf{e}}(k), \underline{\mathbf{d}}_n(k)\}$, so $\underline{\mathbf{c}} \leqslant^\times \underline{\mathbf{d}}_n$ follows from $\underline{\mathbf{e}} \leqslant^\times \underline{\mathbf{d}}_n$. □

Proof of Proposition 6.1. Suppose, for a contradiction, that $\mathcal{B}_{<1}$ is characterised by the cost function $\mathbf{e}$. Proposition 2.6 implies that for all p, $\underline{\mathbf{e}} \leqslant^\times \underline{\mathbf{c}}_{\Omega,p}$. Apply Lemma 6.2 for $\mathbf{e}$ and $\mathbf{d}_n = \mathbf{c}_{\Omega,(n-1)/n}$ to obtain a cost function $\mathbf{c}$. Then $\underline{\mathbf{c}} \leqslant^\times \underline{\mathbf{d}}_n$ for each n implies that $\mathbf{c}$ characterises a class containing $\mathcal{B}_{<1}$. However $\underline{\mathbf{c}} \nleqslant^\times \underline{\mathbf{e}}$, so by Proposition 2.6 again, there is a set obeying $\underline{\mathbf{e}}$ but not $\underline{\mathbf{c}}$, which is a contradiction. □

7. Being computable from all weakly LR-hard randoms

This section provides further evidence that the ideal $\mathcal{B}_{<1}$ is, in a sense, much smaller than the ideal of K-trivials. We show that it is properly contained in the ideal of degrees which lie below every so-called weakly LR-hard sequence; the latter ideal itself is properly contained in the K-trivial degrees.

As mentioned earlier, a set X is *LR-hard* if every X-random sequence is 2-random.

Proposition 7.1. *If a set A is computable from all LR-hard random sequences, then it is K-trivial.*

Proof. Day and Miller [12] showed that if A is not K-trivial, then there is a random X that is not a density 1 point and yet does not compute A. Such a random must be LR-hard [7]. □

[10] Every K-trivial set is ω-c.a. and there is a uniform listing of all such sets (e.g., [34, 1.4.5]). By a Σ^0_3-ideal we mean that the collection of sets in the ideal is a Σ^0_3 subset of Cantor space, or equivalently, that the collection of ω-c.a. indices of the elements of the ideal is Σ^0_3.

It is still unknown whether every K-trivial is computable from every LR-hard random sequence. We say that X is *weakly LR-hard* if every X-random sequence is Schnorr random relative to $\varnothing'$.

Proposition 7.2. *There is a K-trivial set that is not computable from all weakly LR-hard randoms.*

Proof. Barmpalias, Miller, and Nies [2] have shown that X is weakly LR-hard if and only if $\varnothing'$ is c.e. traceable by X: there is a computable bound h such that each function $f \leqslant_{\mathrm{T}} \varnothing'$ has an h-bounded trace c.e. in X. A c.e. set is array computable if and only if it is c.e. traceable, and it is known that such a set can be properly low_2. Hence, by pseudojump inversion for ML-random sets, there is a weakly LR-hard ML-random Δ^0_2 set X that is properly high_2.

Every random set Turing above every K-trivial is not Oberwolfach random in the sense of [6]. Hence it is LR-hard, and in particular high. So not every K-trivial is computable from all weakly LR-hard randoms. □

For background, there are several results characterising sub-ideals of the K-trivials as those degrees computable from all random elements of some null Σ^0_3 class. One example is the class of strongly jump-traceable sets; they are precisely the sets computable from all superhigh random sequences [21, 23]. Theorem 2.18 implies that the ideal of k/n-bases is such a class: the collection of sets computable from the n-columns of Ω. This notion is closely related to that of a *diamond class*: the class of *c.e.* sets computable from all random sequences is some null Σ^0_3 class. This restriction to c.e. sets is often immaterial, since the ideals under discussion are generated by their c.e. elements. However, at times we need to work harder to show one implication for general sets. For example, proving Proposition 7.1 for c.e. sets A does not require the work of Day and Miller; we can use the existence of an incomplete LR-hard random, which follows from pseudo-jump inversion for randoms.

We also remark that the ideal of sets computable from every JT-hard random (studied implicitly in [22] and in more detail in [34, Section 8.5]) contains the ideal of sets below every weakly LR-hard random; the former though is not yet known to be properly contained in the K-trivials.

Remark 7.3. Recall that X is LK-hard if $(\forall y)\ K^X(y) \leqslant^+ K^{\varnothing'}(y)$. A computable measure machine is a prefix free machine M such that $\mu[\mathrm{dom}\, M]^{\prec}$ (the measure of its domain) is a computable real [34, 3.5.14]. We say that X is *weakly LK-hard* if $(\forall y)\ K^X(y) \leqslant^+ K_M(y)$ for each computable measure machine M relative to $\varnothing'$.

Kjos-Hanssen et al. [26] have proved that every LR-hard set is LK-hard. An adaptation of their argument, available in [16, Section 2], shows that every weakly LR-hard set is weakly LK-hard.

Theorem 7.4.

(1) *Every set in $\mathcal{B}_{<1}$ is computable from all weakly LR-hard random sets.*
(2) *Some set not in $\mathcal{B}_{<1}$ is computable from all weakly LR-hard random sets.*

Proof. (1) Fix $p < 1$. We show that every p-base is computable from all weakly LR-hard random sets. Let Z be weakly LR-hard.

We will build a discrete measure ν such that $\nu(m)$ is Δ^0_2 uniformly in m and $\sum_m \nu(m)$ is a computable real. Let α be the universal uniform left-c.e. discrete measure, namely, $\alpha^Z(n)$ is the chance that the standard universal prefix-free machine with oracle Z prints out n. Since Z is weakly LK-hard, by Remark 7.3, $\alpha^Z \geqslant^\times \nu$; this uses a slight adaptation of the Coding Theorem (e.g., [34, Thm. 2.2.25] or [15, Thm. 3.9.4]).

We view α as a function of two variables, and let

$$\alpha_s^X(\omega) = \sum_{n \in \omega} \alpha_s^X(n) = \Omega_s^X;$$

Let μ denote Lebesgue measure and c the counting measure on ω. By Fubini's Theorem

$$I_s = \int \alpha_s(X, n) d(\mu \times c) = \int \alpha_s^X(\omega) d\mu$$

So $I_s \leqslant 1$ for each s.

Let $\gamma_n = 2^{(p-1)n}$. The point is that $\sum \gamma_n$ is finite and computable, and $2^{-pn}\gamma_n = 2^{-n}$. We define $\nu(m)$ as a Δ_2^0 real uniformly in m. We view m as a code for a pair of numbers. The algorithm for defining ν is as follows:

> If $I_s \in (k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]$, let $\nu_s(n, t) = \gamma_n$, and $\nu_s(n, t') = 0$ for all $t' \neq t$, where t is the least stage such that $I_t > k \cdot 2^{-n}$.

The total weight of ν is $\sum \gamma_n < \infty$.

Now fix a constant d such that $d \cdot \alpha^Z \geqslant \nu$. We define a p-OW test that succeeds on all the oracles X such that $d \cdot \alpha^X \geqslant \nu$. If $I_s \in (k2^{-n}, (k+1)2^{-n}]$ then the k^{th} version of U_n at stage s is the collection of oracles X such that $d \cdot \alpha_s^X(n, t) \geqslant \gamma_n$, where t is the least stage such that $I_t > k \cdot 2^{-n}$.

The measure of each version of U_n is at most $d \cdot 2^{-n}/\gamma_n = d \cdot 2^{-pn}$. This is because by convention, for all X, $\alpha_t^X(n, t) = 0$; so if $I_s \in (k2^{-n}, (k+1)2^{-n}]$ then $\int \alpha_s^X(n, t)\, d\mu \leqslant I_s - I_t \leqslant 2^{-n}$.

If A is a p-base, then A obeys $\mathbf{c}_{\Omega,p}$, and hence $A \leqslant_{\mathrm{T}} Z$ (Proposition 2.9 and Proposition 2.14).

(2) We modify the construction for (1). We choose a non-decreasing computable function $h\colon [0,1] \to \mathbb{R}$ such that $h(x) \geqslant x$ and:

- $\sum_n 2^{-n}/h(2^{-n}) < \infty$;
- For all $p < 1$, $h(x) \leqslant^\times x^p$; and
- For all $M > 0$, $h(Mx) \leqslant^\times h(x)$.

For example we can choose $h(x) = x(\log x)^2$.

We carry out the construction above with $\gamma_n = 2^{-n}/h(2^{-n})$. This tells us that every LR-hard set can be captured by an h-OW test, namely a test (G_σ, α) as in Definition 2.8 but such that $\mu(G_\sigma) \leqslant^\times h(2^{-|\sigma|})$. We then follow the proof of Proposition 2.9 to see that every such test can be covered by a $\mathbf{c}_{\Omega,h}$-test, namely a test $\langle V_n \rangle$ such that $\mu(V_n) \leqslant^\times h(\Omega - \Omega_n)$. So every set that obeys $\mathbf{c}_{\Omega,h}$ is computable from all weakly LR-hard random sequences. Since $\mathbf{c}_{\Omega,h} \leqslant^\times \mathbf{c}_{\Omega,p}$ for all $p < 1$, Proposition 6.1 implies that there is a set that obeys $\mathbf{c}_{\Omega,h}$ but is not in $\mathcal{B}_{<1}$. $\square$

References

[1] George Barmpalias and Rod G. Downey. Exact pairs for the ideal of the K-trivial sequences in the Turing degrees. *J. Symb. Log.*, 79(3):676–692, 2014.

[2] George Barmpalias, Joseph S. Miller, and André Nies. Randomness notions and partial relativization. *Israel J. Math.*, 191(2):791–816, 2012.

[3] Dimitris Bertsimas and John Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific, 1st edition, 1997.

[4] Laurent Bienvenu, Adam R. Day, Noam Greenberg, Antonín Kučera, Joseph S. Miller, André Nies, and Daniel Turetsky. Computing K-trivial sets by incomplete random sets. *Bull. Symb. Log.*, 20(1):80–90, 2014.

[5] Laurent Bienvenu, Adam R. Day, Mathieu Hoyrup, Ilya Mezhirov, and Alexander Shen. A constructive version of Birkhoff's ergodic theorem for Martin-Löf random points. *Inform. and Comput.*, 210:21–30, 2012.

[6] Laurent Bienvenu, Noam Greenberg, Antonín Kučera, André Nies, and Dan Turetsky. Coherent randomness tests and computing the K-trivial sets. *J. Eur. Math. Soc. (JEMS)*, 18(4):773–812, 2016.

[7] Laurent Bienvenu, Rupert Hölzl, Joseph S. Miller, and André Nies. Denjoy, Demuth and density. *J. Math. Log.*, 14(1):1450004, 35, 2014.

[8] Béla Bollobás and Andrew Thomason. Projections of bodies and hereditary properties of hypergraphs. *Bull. London Math. Soc.*, 27(5):417–424, 1995.

[9] Vasco Brattka, Joseph S. Miller, and André Nies. Randomness and differentiability. *Trans. Amer. Math. Soc.*, 368(1):581–605, 2016.

[10] Gregory J. Chaitin. Nonrecursive infinite strings with simple initial segments. *IBM Journal of Research and Development*, 21:350–359, 1977.

[11] F. R. K. Chung, R. L. Graham, P. Frankl, and J. B. Shearer. Some intersection theorems for ordered sets and graphs. *J. Combin. Theory Ser. A*, 43(1):23–37, 1986.

[12] Adam R. Day and Joseph S. Miller. Density, forcing, and the covering problem. *Math. Res. Lett.*, 22(3):719–727, 2015.

[13] Osvald Demuth. The differentiability of constructive functions of weakly bounded variation on pseudo numbers. *Comment. Math. Univ. Carolinae*, 16(3):583–599, 1975.

[14] David Diamondstone, Noam Greenberg, and Daniel Turetsky. Inherent enumerability of strong jump-traceability. *Trans. Amer. Math. Soc.*, 367(3):1771–1796, 2015.

[15] Rod G. Downey and Denis Hirschfeldt. *Algorithmic randomness and complexity*. Springer-Verlag, Berlin, 2010. 855 pages.

[16] Andre Nies (editor). Logic Blog 2015. Available at http://arxiv.org/abs/1602.04432, 2015.

[17] Santiago Figueira, Denis R. Hirschfeldt, Joseph S. Miller, Keng Meng Ng, and André Nies. Counting the changes of random Δ^0_2 sets. *J. Logic Computation*, 25:1073–1089, 2015. Journal version of paper at CiE 2010.

[18] Johanna N. Y. Franklin, Noam Greenberg, Joseph S. Miller, and Keng Meng Ng. Martin-Löf random points satisfy Birkhoff's ergodic theorem for effectively closed sets. *Proc. Amer. Math. Soc.*, 140(10):3623–3628, 2012.

[19] Johanna N. Y. Franklin and Keng Meng Ng. Difference randomness. *Proc. Amer. Math. Soc.*, 139(1):345–360, 2011.

[20] Péter Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70(2-3):186–192, 1986.

[21] Noam Greenberg, Denis R. Hirschfeldt, and André Nies. Characterizing the strongly jump-traceable sets via randomness. *Adv. Math.*, 231(3-4):2252–2293, 2012.

[22] Noam Greenberg and André Nies. Benign cost functions and lowness properties. *J. Symbolic Logic*, 76(1):289–312, 2011.

[23] Noam Greenberg and Dan Turetsky. Strong jump-traceability. *Bulletin of Symbolic Logic*, 24(2):147–164, 2018.

[24] Denis R Hirschfeldt, Carl G Jockusch, Rutger Kuyper, and Paul E Schupp. Coarse reducibility and algorithmic randomness. *The Journal of Symbolic Logic*, 81(3):1028–1046, 2016.

[25] Denis R. Hirschfeldt, André Nies, and Frank Stephan. Using random sets as oracles. *J. Lond. Math. Soc. (2)*, 75(3):610–622, 2007.

[26] Bjørn Kjos-Hanssen, Joseph S. Miller, and Reed Solomon. Lowness notions, measure, and domination. *J. London Math. Soc.*, 84, 2011.

[27] Antonín Kučera. Measure, Π^0_1-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[28] Antonín Kučera. An alternative, priority-free, solution to Post's problem. In *Mathematical foundations of computer science, 1986 (Bratislava, 1986)*, volume 233 of *Lecture Notes in Comput. Sci.*, pages 493–500. Springer, Berlin, 1986.

[29] Lynn H. Loomis and Hassler Whitney. An inequality related to the isoperimetric inequality. *Bull. Amer. Math. Soc*, 55:961–962, 1949.

[30] F. J. MacWilliams and N. J. A. Sloane. *The theory of error-correcting codes. I.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. North-Holland Mathematical Library, Vol. 16.
[31] Mokshay Madiman, Adam W. Marcus, and Prasad Tetali. Entropy and set cardinality inequalities for partition-determined functions. *Random Structures Algorithms*, 40(4):399–424, 2012.
[32] Keng Meng Ng. On strongly jump traceable reals. *Ann. Pure Appl. Logic*, 154(1):51–69, 2008.
[33] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.
[34] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009.
[35] André Nies. Calculus of cost functions. In *The Incomputable, Cooper, S. Barry and Soskova, Mariya I., eds.*, pages 183–216. Springer, 2017.
[36] A. Romashchenko, A. Shen, and N. Vereshchagin. Combinatorial interpretation of Kolmogorov complexity. *Theoret. Comput. Sci.*, 271(1-2):111–123, 2002. Kolmogorov complexity.
[37] Robert M. Solovay. Draft of paper (or series of papers) related to Chaitin's work. IBM Thomas J. Watson Research Center, Yorktown Heights, NY, 215 pages, 1975.

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