TWO MORE CHARACTERIZATIONS OF K-TRIVIALITY

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Abstract. We give two new characterizations of K-triviality. We show that if for all Y such that Ω is Y-random, Ω is $(Y \oplus A)$-random, then A is K-trivial. The other direction was proved by Stephan and Yu, giving us the first titular characterization of K-triviality and answering a question of Yu. We also prove that if A is K-trivial, then for all Y such that Ω is Y-random, $(Y \oplus A) \equiv_{\mathrm{LR}} Y$. This answers a question of Merkle and Yu. The other direction is immediate, so we have the second characterization of K-triviality.

The proof of the first characterization uses a new cupping result. We prove that if $A \not\leq_{\mathrm{LR}} B$, then for every set X there is a B-random set Y such that X is computable from $Y \oplus A$.

1. Preliminaries

We assume that the reader is familiar with basic notions from computability theory and effective randomness. For more information on these topics, we recommend either Nies [12] or Downey and Hirschfeldt [4].

The K-trivial sets have played an important role in the development of effective randomness. A set $A \in 2^\omega$ is *K-trivial* if $K(A \upharpoonright n) \leq^+ K(n)$, where K denotes prefix-free Kolmogorov complexity. Chaitin [1] proved that such sets are always Δ^0_2, while Solovay [16] constructed a noncomputable K-trivial set. Although these results date back to the 1970s, the importance of K-triviality did not become apparent until the 2000s, when several nontrivial characterizations were discovered. In particular:

Theorem 1.1 (Nies [11]; Hirschfeldt, Nies, and Stephan [6]). *The following are equivalent for a set $A \in 2^\omega$:*

(a) *A is K-trivial,*
(b) *A is* low for K: $K^A(n) \geq^+ K(n)$,
(c) *A is* low for randomness: *every random set is A-random,*[1]
(d) *A is a* base for randomness: *there is an A-random set $X \geq_{\mathrm{T}} A$.*

Nies [11] generalized (c) to LR-*reducibility*: we write $A \leq_{\mathrm{LR}} B$ to mean that every B-random set is A-random. In particular, $A \leq_{\mathrm{LR}} \emptyset$ means that A is low for randomness (hence K-trivial).

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[1] Throughout this paper, we consistently use *random* to mean Martin-Löf random.

Much more has been proved about the K-trivial sets, including many other characterizations. We give two more. Our results relate to a weakening of lowness for randomness. If X is random, then we say that Y is *low for* X if X is Y-random. This notion was introduced in [6], where it is shown that a set is K-trivial if and only if it is Δ^0_2 and low for Chaitin's Ω. However, many other sets are low for Ω; for example, every 2-random set.

The following recent result regarding K-triviality and lowness for Ω was used by Stephan and Yu to prove one direction of our first characterization (see the discussion before Proposition 3.2). We will need it in the proof of Lemma 3.4.

Theorem 1.2 (Simpson and Stephan [15, Theorem 3.11]). *If S has PA degree and is low for Ω, then S computes every K-trivial.*

In addition to these facts about the K-trivial sets, we will use several fairly well-known theorems from effective randomness. Van Lambalgen's theorem [17] says that $X \oplus Y$ is random if and only if X is random and Y is X-random. Two applications allow us to show that if X is random and Y is X-random, then X is Y-random. Every set is computable from some random set. Relativizing this to X:

Theorem 1.3 (Kučera [9]; Gács [5]). *For any sets X and C, there is an X-random set Y such that $C \leq_{\mathrm{T}} Y \oplus X$.*

Any random set Turing below a Z-random set is also Z-random. Relativizing this to Y:

Theorem 1.4 (Miller and Yu [10, Theorem 4.3]). *Assume that $X \leq_{\mathrm{T}} W \oplus Y$, X is Y-random, and W is $Z \oplus Y$-random. Then X is $Z \oplus Y$-random.*

Finally, we will use the relativized form of the "randomness preservation" basis theorem:

Theorem 1.5 (Downey, Hirschfeldt, Miller, Nies [3]; Reimann and Slaman [14]). *If W is Y-random and P is a nonempty $\Pi^0_1[Y]$ class, then there is a set $S \in P$ that is low for W.*

2. Cupping with B-random sets

As promised in the abstract, we prove the following cupping result.

Theorem 2.1. *Assume that $A \not\leq_{\mathrm{LR}} B$. Then for any set X, there is a B-random set Y such that $X \leq_{\mathrm{T}} Y \oplus A$ (in fact, we make Y weakly 2-random relative to B).*

This theorem should be compared to the work of Day and Miller [2]. They proved that a set A is *not* K-trivial if and only if there is a random set $Y \not\geq_{\mathrm{T}} \emptyset'$ such that $\emptyset' \leq_{\mathrm{T}} Y \oplus A$. Note that one direction of this follows from Theorem 2.1 by taking $B = \emptyset$ and $X = \emptyset'$. This is because A is not K-trivial if and only if $A \not\leq_{\mathrm{LR}} \emptyset$, and if Y is weakly 2-random, then $Y \not\geq_{\mathrm{T}} \emptyset'$. Day and Miller generalized this basic cupping result by adding requirements to control the degrees of Y' and $Y \oplus A$. Theorem 2.1 offers a different generalization.

Our proof uses a result of Kjos-Hanssen. We state it here in a slightly stronger form than he stated it, though without adding any essential content.

Theorem 2.2 (Kjos-Hanssen [8]). *$A \not\leq_{\mathrm{LR}} B$ if and only if there is a $\Sigma^0_1[A]$ class U of measure less than one that intersects every positive measure $\Pi^0_1[B]$ class. Furthermore, for any $\varepsilon > 0$, we can ensure that $\lambda(U) < \varepsilon$.*

Kjos-Hanssen showed that $A \leq_{\mathrm{LR}} B$ if and only if each $\Pi^0_1[A]$ class of positive measure has a $\Pi^0_1[B]$ subclass of positive measure.[2] Taking the contrapositive: $A \not\leq_{\mathrm{LR}} B$ if and only if there is a $\Pi^0_1[A]$ class T of positive measure that does not have a positive measure $\Pi^0_1[B]$ subclass. So $U = 2^\omega \smallsetminus T$ would be a $\Sigma^0_1[A]$ class of measure less than one that intersects every positive measure $\Pi^0_1[B]$ class.

The fact that U can be taken to have arbitrarily small measure also follows from the work in [8]. We use this fact below, so for completness, we sketch the argument. Assume that $A \not\leq_{\mathrm{LR}} B$. So there is a B-random set X that is not A-random. Let U be a $\Sigma^0_1[A]$ class containing every non-A-random set. We may assume, of course, that the measure of U is as small as we like. Let P be a positive measure $\Pi^0_1[B]$ class. Relativizing a result of Kučera [9], every B-random set has a tail in P, so there is a tail Y of X in P. But Y is not A-random, so $Y \in U$.[3]

We need some basic notation for the proof of Theorem 2.1. If $P \subseteq 2^\omega$ is measurable and $\sigma \in 2^{<\omega}$, let $\lambda(P \mid \sigma)$ denote the *relative measure of P in* $[\sigma]$, i.e., $\lambda(P \cap [\sigma])/\lambda([\sigma])$. If $\sigma \in 2^{<\omega}$ and $W \subseteq 2^{<\omega}$, let $\sigma W = \{\sigma\tau \colon \tau \in W\}$.

Proof of Theorem 2.1. Suppose that $A \not\leq_{\mathrm{LR}} B$. By Theorem 2.2, there is a $\Sigma^0_1[A]$ class U such that $\lambda(U) < 0.1$ and U intersects every positive measure $\Pi^0_1[B]$ class. Let W be an A-c.e. prefix-free set of strings such that $U = [W]^{\prec}$.

Let X be any set. We will construct $Y = X(0)\sigma_0 X(1)\sigma_1 X(2)\sigma_2 \cdots$ such that each $\sigma_i \in W$. In this way, it is clear that $X \leq_{\mathrm{T}} Y \oplus A$. To ensure that Y is weakly 2-random relative to B, we build it inside a nested sequence of $\Pi^0_1[B]$ classes P_n of positive measure such that $\bigcap_{n\in\omega} P_n$ is a subset of every $\Sigma^0_2[B]$ class of measure one. The following claim will let us hit W and code the next bit of X while staying inside the current $\Pi^0_1[B]$ class.

Claim. For any string $\sigma \in 2^{<\omega}$ and any $\Pi^0_1[B]$ class P such that $\lambda(P \mid \sigma) > 0.1$, there is a $\tau \succeq \sigma$ such that $\tau \in \sigma W$ and $\lambda(P \mid \tau) \geq 0.8$.

Proof. We first extend σ to a string ρ that has no prefix in σW and such that $\lambda(P \mid \rho) > 0.9$. Let $Q = 2^\omega \smallsetminus [\sigma W]^{\prec}$. As $\lambda(Q \mid \sigma) > 0.9$ and $\lambda(P \mid \sigma) > 0.1$, we have $\lambda(Q \cap P \mid \sigma) > 0$. By the Lebesgue density theorem, there is a $\rho \succeq \sigma$ such that $\lambda(Q \cap P \mid \rho) > 0.9$. In particular, $\lambda(P \mid \rho) > 0.9$ and $\lambda(Q \mid \rho) > 0.9$; the latter implies that ρ cannot have a prefix in σW.

We now extend ρ to a string τ satisfying the claim: $\tau \in \sigma W$ and $\lambda(P \mid \tau) \geq 0.8$. Consider the $\Pi^0_1(B)$ class $\widetilde{P} = \{X \in P \cap [\rho] \colon (\forall n \geq |\rho|)\ \lambda(P \mid X \upharpoonright n) \geq 0.8\}$. In words, $\widetilde{P}$ is the subclass of $P \cap [\rho]$ in which we remove every basic neighborhood inside $[\rho]$ where the relative measure of P drops below 0.8. It is not hard to show that we remove at most 0.8 from the relative measure of $P \cap [\rho]$ inside $[\rho]$ (consider the antichain of maximal basic neighborhoods that are removed). But $\lambda(P \mid \rho) > 0.9$, so $\lambda(\widetilde{P} \mid \rho) > 0.1$. In particular, $\widetilde{P}$ is a positive measure subclass of $[\sigma]$, so by the choice of $U = [W]^{\prec}$, it must be the case that $[\sigma W]^{\prec}$ intersects $\widetilde{P}$. Take $\tau \in \sigma W$ such that $\widetilde{P} \cap [\tau] \neq \emptyset$. By the definition of $\widetilde{P}$, we have $\lambda(P \mid \tau) \geq 0.8$. $\Diamond$

We are ready to construct Y. We will construct it as the limit of a sequence $\tau_0 \preceq \tau_1 \preceq \tau_2 \preceq \cdots$ of strings, while staying inside a decreasing sequence $P_0 \supseteq$

[2] This partial relativization of [8, Theorem 2.10] is stated in the proof of [8, Theorem 3.2].

[3] In fact, $U \cap P$ has positive measure. Choose $\sigma \in 2^{<\omega}$ such that $Y \in [\sigma] \subseteq U$. Then $\widetilde{P} = P \cap [\sigma] \subseteq P \cap U$ is a $\Pi^0_1[B]$ class. Since it contains Y, which is B-random, it cannot have measure zero.

$P_1 \supseteq P_2 \supseteq \cdots$ of $\Pi^0_1[B]$ classes. Let $P_0 = 2^\omega$ and let τ_0 be the empty string. We start stage n of the construction with a $\Pi^0_1[B]$ class P_n and a string $\tau_n = X(0)\sigma_0 X(1)\cdots X(n-1)\sigma_{n-1}$ such that

$$(\star) \qquad \lambda(P_n \mid \tau_n X(n)) > 0.1.$$

(Note that this is true at stage 0.) First, we want to make progress towards Y being weakly 2-random relative to B. Let $\bigcup_{m\in\omega} R_m$ be the nth $\Sigma^0_2[B]$ class of measure one, where $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ is a nested sequence of $\Pi^0_1[B]$ classes. Pick m large enough that $\lambda(P_n \cap R_m \mid \tau_n X(n)) > 0.1$ and let $P_{n+1} = P_n \cap R_m$. So as long as we ensure that $Y \in P_{n+1}$, we have ensured that Y is in the nth $\Sigma^0_2[B]$ class of measure one. Now apply the claim to get $\tau_{n+1} \succeq \tau_n X(n)$ such that $\lambda(P_{n+1} \mid \tau_{n+1}) \geq 0.8$ and $\tau_{n+1} \in \tau_n X(n) W$. Let σ_n be the string for which $\tau_{n+1} = \tau_n X(n)\sigma_n$; in particular, $\sigma_n \in W$. Note that $\lambda(P_{n+1} \mid \tau_{n+1} X(n+1)) \geq 0.6 > 0.1$, so $(\star)$ holds at stage $n+1$.

Let $Y = \bigcup_{n\in\omega} \tau_n = X(0)\sigma_0 X(1)\sigma_1 X(2)\sigma_2\cdots$. As promised, each σ_i is in W, so $X \leq_{\mathrm{T}} Y \oplus A$. By construction, $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$, and each τ_n can be extended to an element of P_n. Therefore, $Y \in \bigcap_{n\in\omega} P_n$. This ensures that Y is in every $\Sigma^0_2[B]$ class of measure one, so Y is weakly 2-random relative to B. □

3. Low for X preserving

Definition 3.1. Let X be random. A set A is *low for X preserving* if for all Y,

$$Y \text{ is low for } X \implies Y \oplus A \text{ is low for } X.$$

This notion was recently introduced by Yu Liang, who called it *absolutely low for X*. Stephan and Yu proved that every K-trivial is low for Ω preserving (see [7, Fact 1.8]). Yu asked if the converse is true: if a set is low for Ω preserving, is it K-trivial? We show that this holds.

Proposition 3.2. *If X is random, then low for X preserving implies K-triviality.*

Proof. Assume that A is low for X preserving.

First, we claim that $A \leq_{\mathrm{LR}} X$. If not, then Theorem 2.1 gives us an X-random set Y such that $X \leq_{\mathrm{T}} Y \oplus A$. By Van Lambalgen's theorem, X is Y-random. But $X \leq_{\mathrm{T}} Y \oplus A$ implies that X is not $(Y \oplus A)$-random. This contradicts the assumption that A is low for X preserving. Therefore, $A \leq_{\mathrm{LR}} X$.

By Theorem 1.3, there is an X-random set Y such that $A \leq_{\mathrm{T}} Y \oplus X$. By Van Lambalgen's theorem, X is Y-random and because A is low for X preserving, we have that X is $(Y \oplus A)$-random. Furthermore, because Y is X-random and $A \leq_{\mathrm{LR}} X$, we know that Y is A-random. Therefore, by Van Lambalgen's theorem relative to A, $Y \oplus X$ is A-random. But $Y \oplus X$ computes A, so A is a base for randomness. Therefore, it is K-trivial (see Theorem 1.1). □

Together with the result of Stephan and Yu, we get a new characterization of K-triviality.

Theorem 3.3. *A set A is K-trivial if and only if it is low for Ω preserving.*

Our next lemma can be viewed as a slight generalization of Stephan and Yu's result. Assume that A is K-trivial and Y is low for Ω. Stephan and Yu showed that $Y \oplus A$ is also low for Ω. Merkle and Yu [7, Question 1.11] asked if, in fact, $Y \oplus A$ has exactly the same derandomizing power as Y. This is the case:

Lemma 3.4. *If A is K-trivial and Y is low for Ω, then $Y \equiv_{\mathrm{LR}} (Y \oplus A)$.*

Proof. Let A be K-trivial and Y be low for Ω. Let X be any Y-random. By Theorem 1.3, there is a Y-random set W such that both Ω and X are computable from $W \oplus Y$. There is a nonempty $\Pi^0_1[Y]$ class containing only members with PA degree relative to Y. So by Theorem 1.5, there is a low for W set S with PA degree relative to Y. Thus W is S-random and $Y \leq_{\mathrm{T}} S$. By Theorem 1.4, both X and Ω are also S-random. Since S has PA degree and is low for Ω, by Theorem 1.2, S computes every K-trivial. In particular, $A \leq_{\mathrm{T}} S$. Because $Y \oplus A \leq_{\mathrm{T}} S$ and X is S-random, X is $Y \oplus A$-random. But X was any Y-random set, so $Y \equiv_{\mathrm{LR}} Y \oplus A$. $\square$

The converse to Lemma 3.4 is easy, giving us our second characterization of K-triviality.

Theorem 3.5. *A set A is K-trivial if and only if for all Y*

$$Y \text{ is low for } \Omega \implies Y \equiv_{\mathrm{LR}} (Y \oplus A).$$

Proof. One direction is Lemma 3.4. For the other direction, assume that A has the given property. Note Ω is $\emptyset$-random, so $\emptyset \equiv_{\mathrm{LR}} \emptyset \oplus A \equiv_{\mathrm{LR}} A$. In other words, A is low for randomness, hence K-trivial (see Theorem 1.1). $\square$

It is natural to ask if low for X preserving is equivalent to K-triviality for all random X. As we shall see, this is not the case, though it is true for some X.

Proposition 3.6. *If $\Omega \leq_{\mathrm{T}} X$ and X is random, then low for X preserving is equivalent to K-triviality.*

Proof. One direction is given by Proposition 3.2. For the other direction, let A be K-trivial and take any Y such that X is Y-random. By (the unrelativized form of) Theorem 1.4, Ω is also Y-random. By Lemma 3.4, $Y \equiv_{\mathrm{LR}} (Y \oplus A)$. Therefore, X is $(Y \oplus A)$-random. $\square$

For certain other X, low for X preserving is equivalent to being computable.

Proposition 3.7. *If X is Schnorr$[\emptyset']$ random but not 2-random, then only the computable sets are low for X preserving.*

Proof. We prove the contrapositive. Assume that A is not computable. If A is not Δ^0_2, then it is not K-trivial, hence by Proposition 3.2, it is not low for X preserving. So assume that A is Δ^0_2. By Posner–Robinson [13], there is a low set Y such that $Y \oplus A \equiv_{\mathrm{T}} \emptyset'$. Because X is Schnorr$[\emptyset']$ random, it is random relative to any low set,[4] so it is Y-random. But X is not 2-random, so it is not $(Y \oplus A)$-random. Therefore, A is not low for X preserving. $\square$

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