A VAN LAMBALGEN THEOREM FOR DEMUTH RANDOMNESS

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Abstract. If $\mathcal{R}$ is a relativizable notion of randomness, we say that van Lambalgen's theorem holds for $\mathcal{R}$ if for all $A, B \in 2^\omega$, we have $A \oplus B \in \mathcal{R}$ if and only if $A \in \mathcal{R} \wedge B \in \mathcal{R}^A$. Van Lambalgen proved that this holds for Martin-Löf randomness. We show that van Lambalgen's theorem fails for Demuth randomness, but holds for the partial relativization $\text{Demuth}_{\text{BLR}}$.

1. Introduction

1.1. **Partial relativization vs. full relativization.** Studies in algorithmic randomness have identified a hierarchy of effective randomness notions, of which the best known is Martin-Löf's. A notion of randomness is determined by a collection of statistical tests; these formalize the notion that a sequence is random if it lacks patterns which can be discerned in some sufficiently effective way. Formally, tests are null sets of reals, and a randomness notion is defined by specifying a countable collection of tests, whose union is the resulting collection of non-random reals (which we here identify, via binary expansion, with infinite binary sequences). For example, a Martin-Löf test is the intersection of a nested sequence of sets $\langle U_n \rangle$ which are uniformly effectively open and which satisfy $\mu(U_n) \leqslant \epsilon(n)$, where μ is the usual Lebesgue measure and $\epsilon(n)$ is a computable sequence of rational numbers tending to 0. In general, the less effectivity we require, the larger the collections of tests and the stronger the resulting notion of randomness.

A particular way of expanding the collection of tests is by appealing to an oracle, a "black box" containing non-computable information. An oracle (such as the halting problem) may be sufficiently powerful to detect patterns in sequences, which cannot be found effectively. This process gives rise to the notion of *relative randomness*. Full relativization is the process of replacing all effective aspects of the definition of a statistical test by concepts which appeal to an oracle. To give an example, given an oracle A, an A-Martin-Löf test is a nested sequence $\langle U_n \rangle$ of sets which are A-effectively open (their basic open subsets can be enumerated with oracle A), such that $\mu(U_n) \leqslant \epsilon(n)$, where now ϵ is an A-computable sequence of rational numbers tending to 0.

Nies has pointed out that sometimes full relativization is not desirable. While trying to convert lowness notions to weak reducibilities, transitivity is usually obtained by letting only some of the computable processes allowed by the definition

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to appeal to an oracle, while barring others from doing so. The main example is the generalization of lowness for Martin-Löf randomness to obtain LR-reducibility: $A \leqslant_{\mathrm{LR}} B$ if every B-random sequence is also A-random, whereas full relativization to B would require every B-random sequence to be $A \oplus B$-random. Here only partial relativization results in a transitive relation.

When applied to randomness notions, often partial relativization does not give new notions. For example, in relativizing Martin-Löf randomness, one can require the measure-bounding function ϵ to be computable rather than A-computable, without changing the resulting notion of A-randomness. This is not always so.

The partial relativization of a randomness notion was first utilized by Franklin and Stephan [4] when studying lowness for Schnorr randomness, although this is not the context in which it interests us. An important basic tool in the study of relative randomness is van Lambalgen's theorem, which states that for every pair A, B of sets of natural numbers, the join $A \oplus B$ is Martin-Löf random if and only if A is Martin-Löf random, and B is Martin-Löf random relative to A. Liang Yu [12] proved that this theorem fails when Martin-Löf randomness is replaced by Schnorr randomness. Franklin and Stephan defined an alternate relativization of Schnorr randomness (*truth-table Schnorr randomness*), and Miyabe [8] (later corrected [9]) showed that van Lambalgen's theorem holds for this notion of relative Schnorr randomness.

This prompts a general question: given a notion of randomness, is there one best way of defining its relativization to an oracle (which may not be necessarily the full relativization)? If so, what are the criteria for relativizations which are better than others? Alternately, should one think of different relativizations of a randomness notion as distinct notions, which happen to coincide when no oracles are present? In this context, Miyabe suggested that van Lambalgen's theorem should be a criterion for the "proper" relativization of a randomness notion.

Another example where a partially relativized randomness notion is better behaved than the full randomness notion comes from a paper of Bienvenu, Downey, Greenberg, Nies, and Turetsky [2]. They were interested in characterizing lowness for Demuth randomness—identifying when being Demuth random relative to a set A is equivalent to simply being Demuth random. In order to find their characterization, they were motivated to identify a partial relativization of Demuth randomness, $\mathrm{Demuth}_{\mathrm{BLR}}$, which was easier to work with than the full relativization. Demuth randomness is defined using *Demuth tests*, which are Martin-Löf tests where the index of the nth member of the test can change a computably bounded number of times.

Definition 1. Given a set $W \subseteq 2^{<\omega}$, $[W]^{\prec}$ denotes the set of reals

$$\{Z \in 2^{\omega} : \exists \sigma \in W\, [\sigma \prec Z]\}.$$

A *Demuth test* is a sequence of c.e. open sets $\langle V_n \rangle$ such that $\mu(V_n) \leqslant 2^{-n}$ for all n, and there is an ω-c.e. function f such that $V_n = [W_{f(n)}]^{\prec}$.

A real Z *passes* the test $\langle V_n \rangle$ if Z is contained in only finitely many of the test elements. A real Z is *Demuth random* if it passes every Demuth test.

The full relativization of this definition to an oracle A replaces the c.e. open sets with open sets which are c.e. in A, and the ω-c.e. function by a function which is ω-c.e. relative to A; that is, both the approximation and the bound on the number of changes are A-computable. In contrast, a $\mathrm{Demuth}_{\mathrm{BLR}}(A)$ test is a Demuth test relative to A where the function f is *ω-c.e. by* A—the approximation is A computable, but the bound on the number of changes is actually *computable*. The passing notion is the same for $\text{Demuth}_{\text{BLR}}$ tests and for Demuth tests: a real Z passes the test $\langle V_n \rangle$ if it is contained in only finitely many of the test elements.

By applying the partial relativization $\text{Demuth}_{\text{BLR}}$, Bienvenu et al. were able to characterize lowness for Demuth randomness. This is an example where understanding the partial relativization aids in understanding the full relativization. Bienvenu et al. write that the characterization of lowness for $\text{Demuth}_{\text{BLR}}$ is "the fundamental one", and the characterization of lowness for Demuth randomness is simply a corollary of that result and a related result of Downey and Ng [3].

In the present work, we show that van Lambalgen's theorem holds for $\text{Demuth}_{\text{BLR}}$, but not for Demuth randomness. If one accepts Miyabe's thesis, this would imply that $\text{Demuth}_{\text{BLR}}$ is the correct relativization of Demuth randomness, rather than the full relativization. That the characterization of lowness for Demuth randomness [2] goes through $\text{Demuth}_{\text{BLR}}$, which is described as the more fundamental result, gives further evidence that Miyabe's thesis is correct, at least in the case of Demuth randomness.

1.2. **Survey of van Lambalgen's theorem for various randomness notions.** If $\mathcal{R}$ is a relativizable notion of randomness, we say that van Lambalgen's theorem holds for $\mathcal{R}$ if for every pair A, B, we have

$$A \oplus B \in \mathcal{R} \iff A \in \mathcal{R} \wedge B \in \mathcal{R}^A.$$

Van Lambalgen's theorem has been investigated for notions of randomness including Schnorr and computable randomness, n-randomness, and weak 1-randomness. Perhaps surprisingly, the right-to-left or "hard" direction of this equivalence, though harder to prove in the case of Martin-Löf randomness, holds for nearly all of the most-studied randomness notions. On the other hand, the easier to prove (in the case of Martin-Löf randomness) left-to-right direction is the one that is known to fail for several important randomness notions. We summarize the situation in Table 1.

We remark that in many of the cases where the "hard" direction is known to hold, the proof is a straightforward modification of the proof for Martin-Löf randomness. For Kolmogorov-Loveland randomness, it is not known whether the hard direction of van Lambalgen's theorem holds, but Merkle, Miller, Nies, Reimann, and Stephan [7] proved the weaker statement

$$A \oplus B \in \mathcal{KLR} \iff A \in \mathcal{KLR}^B \wedge B \in \mathcal{KLR}^A.$$

TABLE 1. Van Lambalgen's Theorem

Randomness notion	"easy" direction	"hard" direction
weak 1-randomness	false	true
Schnorr	false [7] [12]	true [5] [9]
computable	false [7] [12]	?
Kolmogorov–Loveland	true [7]	?
Martin-Löf	true [11]	true [11]
weak 2-randomness	false [1]	true [1]
n-randomness	true [11]	true [11]
Π^1_1	true [6]	true [6]

1.3. **Notation.** Notation is generally standard and follows Nies [10]. We use μ to denote the Lebesgue measure on 2^ω, and write $\mu(U|V)$ to denote the relative measure $\frac{\mu(U\cap V)}{\mu(V)}$ of U inside V. For $W \subseteq 2^{<\omega}$, we use $\mu(W)$ as an abbreviation for $\mu([W]^{<})$.

2. A van Lambalgen theorem for Demuth randomness

Theorem 1. *$A \oplus B$ is Demuth random if and only if A is Demuth random and B is* $\text{Demuth}_{\text{BLR}}(A)$ *random.*

We will prove the two directions separately.

Lemma 2. *If $A \oplus B$ is Demuth random, then A is Demuth random and B is* $\text{Demuth}_{\text{BLR}}(A)$ *random.*

Proof. By contraposition. If A is not Demuth random, it is clear that $A \oplus B$ is not Demuth random. Suppose $\langle [W_{g(n)}]^{<} \rangle$ is a $\text{Demuth}_{\text{BLR}}(A)$ test that B fails, where g has an A-computable approximation $\Phi^A(n,s)$ with mind-changes bounded by a computable function f. We can assume Φ is total and has mind-changes bounded by f on all oracles X, as we are only interested in the limit, and we can also assume that for all n, s, X, the measure $\mu(W_{\Phi^X(n,s)})$ is bounded by 2^{-n}.

What we want is to say that $A \oplus B$ fails the test

$$\left\langle \{X \oplus Y : Y \in [W_{\lim_s \Phi^X(n,s)}]^{<}\} \right\rangle_n.$$

This object is well-defined; the limit $\lim_s \Phi^X(n,s)$ always exists because Φ is total and has mind-changes bounded by f on all oracles. Furthermore, it must have measure at most 2^{-n} by Fubini's theorem, and captures $A\oplus B$ since $B \in [W_{\lim_s \Phi^A(n,s)}]^{<}$ for all n. However, it is not a Demuth test, because the natural approximation to it changes whenever $\Phi^X(n,s)$ changes, for *any* X, so potentially infinitely often, and certainly not bounded by f. To fix this, the idea is to enlarge each test element slightly by absorbing changes (increasing the measure by at most some set amount), until $\Phi^X(n,s)$ has changed at least once for some large measure of oracles, and only then changing our approximation to the nth test element.

To be more precise, for each n we define (uniformly in n) a (hopefully finite) sequence of *n-stages* $s_0 < s_1 < \dots$ by $s_0 = 0$, and s_i is the least stage t after s_{i-1} such that

$$\mu\left(\bigcup_{s=s_{i-1}}^{t} \{X \oplus Y : Y \in [W_{\Phi^X(n,s)}]^{<}\} \right) > 2 \cdot 2^{-n}.$$

For all n, t, let $p_n(t)$ be the greatest *n-stage* $t' \leqslant t$, and let $s_n(t)$ be the least *n-stage* $t' > t$ (or $s_n(t) = \infty$ if there is no *n-stage* after t). Then we can define

$$V_n[t] = \bigcup_{p_n(t)\leqslant s< s_n(t)} \{X \oplus Y : Y \in [W_{\Phi^X(n,s)}]^{<}\},$$

and will have $\mu(V_n[t]) < 2 \cdot 2^{-n}$ for all t. As $V_n[t]$ is a c.e. open set (uniformly in n and t) and changes only at *n-stages*, $\langle \lim_t V_n[t] \rangle_n$ will be a Demuth test, provided we can exhibit a computable bound on the number of *n-stages*. If this is the case, $\lim_t V_n[t]$ will contain $\{X\oplus Y : Y \in [W_{\lim_s \Phi^X(n,s)}]^{<}\}$, and hence will contain $A\oplus B$.

We will show the number of *n-stages* is bounded by $2^n f(n)$. By the definition of s_i, we have

$$\mu\left(\bigcup_{s=s_{i-1}}^{s_i} \{X \oplus Y : Y \in [W_{\Phi^X(n,s)}]^{\prec}\}\right) > 2 \cdot 2^{-n}.$$

Furthermore,

$$\bigcup_{s=s_{i-1}}^{s_i} \{X \oplus Y : Y \in [W_{\Phi^X(n,s)}]^{\prec}\}$$
$$\subseteq \{X \oplus Y : Y \in [W_{\Phi^X(n,s_{i-1})}]^{\prec}\} \cup \{X \oplus Y : \Phi^X(n,\cdot)|_{[s_{i-1},s_i]} \text{ non-constant}\}.$$

Thus

$$\mu\{X : \Phi^X(n,\cdot)|_{[s_{i-1},s_i]} \text{ non-constant}\} > 2^{-n}.$$

Suppose there are more than $2^n f(n)$ *n-stages*, and for $1 \leqslant i \leqslant 2^n f(n)$, let

$$S_i = \{X : \Phi^X(n,\cdot)|_{[s_{i-1},s_i]} \text{ non-constant}\}.$$

Since $\mu(S_i) > 2^{-n}$ for each i, by the pigeonhole principle there must be some X such that $X \in S_i$ for more than $f(n)$ distinct i, which implies that $\Phi^X(n,\cdot)$ changes more than $f(n)$ times. This contradicts the definition of Φ. □

The other direction is very similar to the proof of van Lambalgen's theorem for Martin-Löf randomness. In that proof, given a Martin-Löf test $\langle V_n \rangle$, two tests $\langle \hat{V}_n \rangle$ and $\langle U_n^X \rangle$ are constructed, the second being an oracle test, such that if $A \oplus B$ fails $\langle V_n \rangle$, then either A fails $\langle \hat{V}_n \rangle$ or B fails $\langle U_n^A \rangle$. Here we follow the same strategy, and observe that if the original test $\langle V_n \rangle$ was a Demuth test, then $\langle \hat{V}_n \rangle$ is also a Demuth test, and $\langle U_n^X \rangle$ is a $\text{Demuth}_{\text{BLR}}(X)$-test.

Lemma 3. *If A is Demuth random and $B \in \text{Demuth}_{\text{BLR}}(A)$, then $A \oplus B$ is Demuth random.*

Proof. Again by contraposition. Suppose $A \oplus B$ is not Demuth random, and let $\langle V_n \rangle = \langle \lim_s V_n[s] \rangle$ be a test that it fails. Without loss of generality, assume there are infinitely many n such that $A \oplus B \in V_{2n}$ — if not, replace $2n$ with $2n+1$. For strings σ, τ, let $[\sigma \oplus \tau]^{\prec}$ denote the clopen set $\{X \oplus Y : \sigma \prec X, \tau \prec Y\}$. Let

$$\hat{V}_n = \{\sigma : \mu(V_{2n} \mid [\sigma \oplus \varnothing]^{\prec}) > 2^{-n}\}.$$

Then $\mu(\hat{V}_n) \leqslant 2^{-n}$, or else V_{2n} violates the measure condition. Furthermore, the approximation $V_{2n}[s]$ to V_{2n} induces an approximation $\hat{V}_n[s]$ to $\hat{V}_n$ which is c.e. (uniformly in n and s) and changes only when $V_{2n}[s]$ changes, and so witnesses that $\langle \hat{V}_n \rangle$ is a Demuth test.

If A fails this Demuth test, then it is not Demuth random and we are finished. Otherwise, there is a finite set F such that for all n, m with $n \notin F$,

$$\mu(V_{2n} \mid [A\!\upharpoonright\! m \oplus \varnothing]) \leqslant 2^{-n}.$$

Fix $n \notin F$, and let

$$U_n = \{\tau : [A\!\upharpoonright\! |\tau| \oplus \tau]^{\prec} \subseteq V_{2n}\}.$$

Let U_n^l be the set of strings of length l in U_n. Then

$$\mu(U_n^l) \leqslant \mu(V_{2n} \mid [A\!\upharpoonright\! l \oplus \varnothing]^{\prec}) \leqslant 2^{-n}.$$

Moreover, U_n is closed upwards under $\prec$, and hence $\mu(U_n) = \lim_l \mu(U_n^l) \leqslant 2^{-n}$. Using the approximation $U_n[t]$ to U_n induced by the approximation $V_{2n}[t]$ to V_{2n}, we see that U_n is the limit of a sequence of open sets which are c.e. in A, and change only at stages when the approximation to V_{2n} changes (so at most $f(2n)$ times). Hence $\langle U_n \rangle$ is a $\text{Demuth}_{\text{BLR}}(A)$ test, which B fails. □

Remark. As Demuth randomness is invariant under computable permutations, the results in this paper hold if the usual join is replaced by the *Z-join*

$$A \oplus_Z B \stackrel{\text{def}}{=} f_Z(A) \cup f_{\bar{Z}}(B),$$

where Z is some infinite, coinfinite computable set, and f_X is the function that enumerates X in increasing order.

3. Does a stronger version of van Lambalgen's theorem hold for Demuth randomness?

In the previous section, we showed that a version of van Lambalgen's theorem holds for the partial relativization of Demuth randomness. But what about the full relativization? Is it true that $A \oplus B$ is Demuth random if and only if A is Demuth random, and B is A-Demuth-random? Note that the "hard" direction is simply a weakening of Theorem 1, so the only question is the "easy" direction. This fails, because of the existence of a real in $\text{Demuth}_{\text{BLR}}(A)$ which is not Demuth random relative to A. Rod Downey and Keng Meng Ng [3] proved that lowness for Demuth randomness implies being computably dominated, by directly constructing, for each non-dominated set A, a Demuth random set B which is not Demuth random relative to B. Their construction can be partially relativized, to give the following theorem.

Theorem 4. *If A is not computably dominated, there is some $B \in \text{Demuth}_{\text{BLR}}(A)$ which is not Demuth random relative to A.*

Proof. The proof is a straightforward partial relativization of the Downey–Ng proof, so rather than rewriting their proof, we describe only the needed modifications to their proof.

Their proof is a construction, relative to a non-computably-dominated set A, of a real Z which is Demuth random, but not Demuth random relative to A. To make $Z \in \text{Demuth}_{\text{BLR}}(A)$, all that is needed is to replace the eth Demuth test $\mathcal{U}^e = \langle U^e_x \rangle$ with the eth $\text{Demuth}_{\text{BLR}}(A)$ test $\mathcal{U}^e(A) = \langle U^e_x(A) \rangle$. The rest of the construction is identical.

As this is an A-oracle construction, these sets can be uniformly enumerated just as easily as the sets U^e_x, so this does not affect the complexity of the construction. We still get the same property that $U^e_x(A)$ does not have a change of index until the eth partial order-function h_e converges on input x, and the number of changes of index is bounded by h_e. As these are $\text{Demuth}_{\text{BLR}}$ tests, the functions h_e are computable, not merely A-computable. The verification now proceeds exactly as in Downey–Ng, and shows that there is a $\Delta^0_2(A)$ set Z which is not Demuth random relative to A, but which is not captured by any of the tests $\mathcal{U}^e(A)$, and is therefore $\text{Demuth}_{\text{BLR}}(A)$ random. □

We now get the following corollary.

Corollary 5. *There is a Demuth random real $A \oplus B$ such that B is not Demuth random relative to A. Moreover, A can be chosen to be an arbitrary Demuth random.*

Proof. Let A be Demuth random. By a result of Miller and Nies [10], A is not computably dominated. By Theorem 4, there is some $B \in \text{Demuth}_{\text{BLR}}(A) \setminus \text{Demuth}(A)$. By Theorem 1, $A \oplus B$ is Demuth random. □

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