STRONG JUMP TRACEABILITY AND DEMUTH RANDOMNESS

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Abstract. We solve the covering problem for Demuth randomness, showing that a computably enumerable set is computable from a Demuth random set if and only if it is strongly jump-traceable. We show that on the other hand, the class of sets which form a base for Demuth randomness is a proper subclass of the class of strongly jump-traceable sets.

1. Introduction

The notion of relative information content for sets of natural numbers was first formalised by Turing [41]. A set B contains at least as much information as a set A if given answers to membership queries regarding B, as a "black box", one can compute A by following a finite algorithm. Colourful terminology envisions a computer with access to an "oracle" (the set B). Formally, we say that A is *Turing reducible* to B. The information content of a set A is captured by its *Turing degree*, the collection of all sets which are bi-reducible (or *Turing equivalent*) to A.

It is not surprising that much complexity is observed in the structure of the Turing degrees (partially ordered by the relation induced by Turing reducibility). After all, we can describe sets with much information, beginning with the halting problem (the set of halting Turing machines), through true arithmetic (the set of correct first-order statements about the semi-ring of natural numbers), and up to sets which code complicated set-theoretic information such as sharps. A number of results formalise the intuition that the structure of the Turing degrees itself is as complicated as possible, for example Simpson's [39] result showing that the theory of the structure is equivalent to second-order arithmetic.

If the Turing degrees are complicated because we can devise complex sets of numbers, a natural question to ask is whether this complexity can be contained by considering the degrees of only fairly simple sets. Studied more than any other collection of degrees is the sub-structure of degrees of *computably enumerable* sets – those sets which can be enumerated effectively, but membership thereof cannot necessarily be effectively decidable. In arithmetic, these are the sets which are definable by very simple formulas, allowing only a single existential quantifier, which corresponds to one operation of unbounded search.

The computable sets – those which contain the least amount of information – are c.e., as well as the halting problem, which is the most complicated c.e. set. Again, the main thrust of research since the 50's was to find whether the structure of the c.e. degrees is complicated or not. Beginning with Friedberg and Muchnik's [22, 33]

2010 *Mathematics Subject Classification.* 03D25,03D32.

Both authors were supported by the Marsden Fund of New Zealand, the second author as a postdoctoral fellow.

construction of an incomplete c.e. degree (showing the structure has more than two elements), via minimal pair results [32, 42] refuting Shoenfield's conjecture that the structure is saturated and so homogenous, up to coding results [26, 38], showed that again, the structure of c.e. degrees is as complicated as possible (with theory equivalent to first-order arithmetic). So even sets which are easily described and do not contain outlandish information can interact in very complicated ways. To date, seemingly simple problems, such as whether there are intermediate definable degrees, whether the structure is rigid, or identifying the finite lattices embeddable in the structure, remain open.

One path to further understanding of the structure of the c.e. degrees is to look not at particular degrees, but to instead look at *classes* of degrees. There we have a number of nice definability results (as in [2, 20, 16]), usually showing that there are several ways of understanding the degrees by separating them into "simple" and "complex" degrees, and sometimes even giving a hierarchy of such simplicity. In particular, attention is given to *lowness notions*, classes of c.e. degrees which resemble the computable ones in some ways. The striking aspect of this study is the relationship we sometimes get between the algebraic structure of the degrees (as a partial ordering) and the dynamic properties of the degrees in a particular class. Each nicely behaved class is characterised by the way sets of those degrees can be enumerated. Often this captures a class of constructions which can be performed by degrees in the class. For example, the promptly simple degrees are those which allow prompt permitting [2]; the array-non-recursive degrees [19] are those which allow multiple permitting; and the K-trivial degrees are those which have an enumeration which is amenable to the decanter method [36]. In the nice cases, these dynamic properties are matched up with the algebraic structure of the classes, sometimes even giving natural definitions of the class, as in the cases of the promptly simple degrees, which coincide with the non-cappable degrees [2]; the contiguous degrees [20, 1]; and the totally ω-c.a. degrees [14].

Among the notions of lowness, two which have been studied are K-triviality [40, 18] and strong jump-traceability [21]. These classes are collections of degrees of sets which contain very little information: they are either useless as oracles in performing some tasks, are easy to compute, or can be described as efficiently as computable sets. Again, these classes are characterised by constructions – the decanter method for K-trivials, and the box-promotion method for strong jump-traceable degrees – which exhibit fascinating dynamics not present in previously known constructions. And the close proximity of these sets to the computable ones allows these two collections to exhibit pleasing regularity in terms of their algebraic structure, for instance they form ideals [36, 6]. Thus, new light is shed on the c.e. degrees, and the notion of computation as a whole, by investigating these classes. The full picture, though, comes to light only when considering interactions with randomness.

Randomness is quite the opposite of computable enumerability. Random sets – those which escape a variety of statistical tests – are by nature hard to pin down individually, and cannot have concise definitions. In general, relative information content, i.e. Turing reducibility, has limited use when considering randomness. The intuition is that random sets contain much "white noise", and so are both far from being computable, but also cannot contain useful information. The quickly expanding study of algorithmic randomness (see [17, 37]) analyses the notion via a different approach – giving a hierarchy of randomness notions, which attempts to capture the notion of amount of randomness by specifying, for each notion of randomness, the collection of statistical tests which define it. Unlike the Turing degrees, this hierarchy is almost linear.

It would seem that random sets and computably enumerable sets are so different, that there could be no interaction between them. And mostly, this intuition is correct; for example, no incomplete c.e. degree can compute a random set, and a sufficiently random set cannot compute a non-computable c.e. set. However, surprisingly, some interaction using Turing reducibility is observed between random sets and very low c.e. sets. The first step was taken by Kučera [29], who showed that every random set which itself is computable from the halting problem, does in fact compute a non-computable c.e. set. He used this result to give an injury-free solution to Post's problem.

Recently it was found that this interaction between random sets and c.e. sets concentrates on the very low c.e. sets. Hirschfeldt, Nies and Stephan [27] showed that every c.e. set which is computable from an incomplete Martin-Löf random set is K-trivial. Greenberg, Hirschfeldt and Nies [25, 24] showed that a c.e. set is strongly jump-traceable if and only if it is computable from many random sets. Incidentally, together, these results show the containment of the strongly jump-traceable degrees in the K-trivial degrees, a result earlier obtain by a combinatorial argument in [6]. Since unlike K-triviality, strong jump-traceability has a purely discrete definition, avoiding any reference to Lebesgue measure or to prefix-free Kolmogorov complexity, the result from [24] shows that this notion has "dual nationality": it lives in both the discrete and the continuous worlds. In particular, this was the first instance of a definition of a class of c.e. degrees using their interaction with random sets. Consequently, attempts were made to go the other way and give a discrete, combinatorial characterization of K-triviality, using the notion of traceability, but the problem of doing so remains open.

After Greenberg [23] constructed a Δ^0_2 Martin-Löf random set which only computes strongly jump-traceable c.e. sets, Kučera and Nies [30] showed that any c.e. set computable from any *Demuth* random set is strongly jump-traceable. Demuth randomness was introduced by Demuth [8, 9] in order to study differentiability of constructive functions; he showed that every constructive function satisfies the Denjoy alternative at any Demuth random real (the converse fails, but it is known that some strengthening of Martin-Löf randomness is required; see for example [10]). Demuth randomness is a strengthening of Martin-Löf randomness which has some nice properties which resemble Cohen 1-genericity: it implies generalised lowness (and so in particular incompleteness), but unlike weak 2-randomness is compatible with being Δ^0_2.

Kučera's and Nies's result, much like the Hirschfeldt-Nies-Stephan result mentioned above, raises the question of whether the converse holds. In this paper, we provide a positive solution to this problem.

Theorem 1.1. *A c.e. set is strongly jump-traceable if and only if it is computable from some Demuth random set.*

Theorem 1.1 is the analogue of the recent covering result for difference randomness, which characterises K-triviality ([3, 4, 7]). We note that recently, the authors, together with D. Diamondstone, showed [12] that every strongly jump-traceable set is computable from a computably enumerable one. Thus we get a characterisation of the strongly jump-traceable sets: a set $A \in 2^\omega$ is strongly jump-traceable if and only if it is computable from a c.e. set which is computable from a Demuth random set.

The proof of Theorem 1.1 is involved, combining novel techniques with the *box-promotion* method used in the investigation of strongly jump-traceable sets. This is the first example using the full power of strong jump-traceability, rather than an approximation in the form of h-jump-traceability for some sufficiently slow growing order function h. A general argument in the style of [24] is impossible here, since no Δ^0_2 Demuth random set computes all strongly jump-traceable c.e. sets. The Demuth random set constructed computing a given c.e., strongly jump-traceable set can be made Δ^0_3; it remains open whether it can be made Δ^0_2. We also do not know yet if there is a single Demuth random set which computes all strongly jump-traceable sets.

Being a *base* for a notion of randomness is a lowness notion emanating from the interplay of randomness and Turing reducibility. If $\mathcal{R}$ is a relativisable class of randomness, then we say that a set A is a base for $\mathcal{R}$ if there is some $X \in \mathcal{R}^A$ which computes A. That the cone above A intersects $\mathcal{R}^A$ is taken as evidence that A is too weak to comprehend that this cone is null. The robustness of the class of K-trivial degrees is witnessed by its coincidence with the class of bases for Martin-Löf randomness (Hirchfeldt, Nies and Stephan [27]). Nies [35] showed that every base for Demuth randomness is strongly jump-traceable, and asked if the converse holds. That is, whether Theorem 1.1 can be improved to produce not merely a Demuth random set computing a given strongly jump-traceable set A, but indeed a Demuth^A random set computing A. We show that the converse fails.

Theorem 1.2. *There is a strongly jump-traceable c.e. set which is not a base for Demuth randomness.*

Nies showed that the class of c.e. bases for Demuth randomness properly contains a sub-ideal of the c.e. jump-traceable sets, namely those c.e. sets computable from every ω^2-computably approximable Martin-Löf random sets. Thus, the collection of bases for Demuth randomness forms a new class, about which we know close to nothing. For example, it is not clear if it induces an ideal in the Turing degrees.

It is easy to prove that every K-trivial set is a base for Martin-Löf randomness, once it is shown that K-triviality implies *lowness* for Martin-Löf randomness. That is, if A is K-trivial, then every Martin-Löf random set is Martin-Löf random relative to A. By the Kučera-Gács theorem, A is computable from a Martin-Löf random set Z (indeed every K-trivial set is Δ^0_2, so A is computable from Chaitin's Ω), and so Z witnesses that A is a base for Martin-Löf randomness. A naïve attempt to show that every c.e., strongly jump-traceable set is a base for Demuth randomness would start by utilising Theorem 1.1 as an analogue to the Kučera-Gács theorem, and then go on to show that every strongly jump-traceable set is low for Demuth randomness. Unfortunately, the latter fails. Indeed, Downey and Ng [15] showed that lowness for Demuth randomness implied hyperimmune-freeness, whereas Downey and Greenberg [13] showed that every strongly jump-traceable set is Δ^0_2, and so the only strongly jump-traceable sets that are low for Demuth are the computable ones.

The proof of Theorem 1.2 relies on the fact that the full relativisation of Demuth randomness to an oracle A allows for an A-computable bound on the number of mind-changes for the value of the function giving the index for components of a Demuth test. So does the Downey-Ng theorem. Further investigations into lowness for Demuth randomness [5] yield the observation that the partial relativisation of Demuth randomness, which allows only computable bounds on the number of changes of each component of a test, is much better behaved. As a relation between oracles and random sets, this partial relativisation, denoted by $\text{Demuth}_{\text{BLR}}$ can be seen as either a separate notion of randomness (which coincides with Demuth randomness on the hyperimmune-free oracles), or as in some way a "correct" relativisation of Demuth randomness. Evidence for this come from the exact characterisation of lowness for $\text{Demuth}_{\text{BLR}}$ by a tracing notion [5], and the fact that van-Lambalgen's theorem holds for $\text{Demuth}_{\text{BLR}}$ but not for the full relativisation of Demuth randomness [11].

Now we see another way in which $\text{Demuth}_{\text{BLR}}$ is better behaved than the full relativisation of Demuth randomness: we can fully characterise the sets which are bases for $\text{Demuth}_{\text{BLR}}$. We first note that Nies's result from [35] actually shows that every set which is a base for $\text{Demuth}_{\text{BLR}}$ is strongly jump-traceable. We also note that by [5], every superlow c.e. set, and so certainly every strongly jump-traceable c.e. set, is low for $\text{Demuth}_{\text{BLR}}$. We can now carry out the naïve plan discussed above. Armed with Theorem 1.1 and with the fact, mentioned above, that every strongly jump-traceable set is computable from a c.e. one, we get:

Corollary 1.3. *The sets which are bases for* $\text{Demuth}_{\text{BLR}}$ *are precisely the strongly jump-traceable sets.*

We prove Theorem 1.1 in Section 3, and Theorem 1.2 in Section 4.

2. Definitions of Demuth randomness and other notions

We begin with some basic notation which we will need later.

Definition 2.1. For $d \in \omega$, let $\omega^{[d]} = \{\langle n, d\rangle : n \in \omega\}$, where $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ is the standard pairing function.

For finite or infinite binary strings $\sigma, \tau \in 2^{<\omega} \cup 2^{\omega}$, we write $\sigma \preccurlyeq \tau$ to indicate that σ is a (not necessarily proper) initial segment of τ, and $\sigma \prec \tau$ to indicate that σ is a proper initial segment of τ.

We next recall the definition of strong jump-traceability.

Definition 2.2.

(1) An *order function* is a computable, nondecreasing and unbounded function $h : \omega \to \omega \backslash \{0\}$.
(2) A *c.e. trace* is a uniformly c.e. sequence of finite sets. A c.e. trace $T = \langle T(x)\rangle_{x<\omega}$ *traces* a partial function $\psi : \omega \to \omega$ if for all $x \in \operatorname{dom}\psi$, $\psi(x) \in T(x)$.
(3) If h is an order function, then an h-trace is a c.e. trace $T = \langle T(x)\rangle$ such that for all $x < \omega$, $|T(x)| \leqslant h(x)$.
(4) A set A is *strongly jump-traceable* if for every order function h, every A-partial computable function ψ is traced by an h-trace.

Next, we discuss notation for subsets of Cantor space 2^{ω}, and define Demuth randomness.

Definition 2.3. A sequence of functions $\langle f_s\rangle_{s<\omega}$ is an *approximation* of a function $f\colon \omega \to \omega$ if for all n, for all but finitely many s, $f_s(n) = f(n)$. We often write $f(n,s)$ for $f_s(n)$. A *computable approximation* is a uniformly computable sequence which is an approximation. Shoenfield's limit lemma says that a function has a computable approximation if and only if it is computable from $\mathbf{0}'$.

If $\langle f_s\rangle$ is an approximation, then the associated *mind-change* function $m_{\langle f_s\rangle}$ is defined by

$$m_{\langle f_s\rangle}(n) = \#\{s : f_{s+1}(n) \neq f_s(n)\}.$$

A computable approximation $\langle f_s\rangle$ is an *ω-computable approximation* if $m_{\langle f_s\rangle}$ is bounded by a computable function. A function is *ω-computably approximable* (or *ω-c.a.*) if it has an ω-computable approximation.

Definition 2.4. For a finite binary string $\sigma \in 2^{<\omega}$, we let $[\![\sigma]\!]$, the clopen subset defined by σ, be the collection of reals $X \in 2^\omega$ which extend σ. If W is a set of strings, then

$$[\![W]\!] = \bigcup_{\sigma\in W} [\![\sigma]\!]$$

is the open (or $\mathbf{\Sigma}^0_1$) subset of 2^ω defined by W. If W is c.e., then $[\![W]\!]$ is called *effectively open* (or Σ^0_1). By compactness, a subset $\mathcal{V}$ of Cantor space is clopen if and only if $\mathcal{V} = [\![D]\!]$ for some finite subset D of $2^{<\omega}$.

If $\mathcal{W} = [\![W]\!]$ and $\langle W_s\rangle$ is an effective enumeration of the c.e. set W, then we often write $\mathcal{W}_s$ for $[\![W_s]\!]$. We call $\langle \mathcal{W}_s\rangle$ an *effective enumeration* of $\mathcal{W}$.

We interrupt the stream of definitions to remark that we will be using Lachlan's notation [31] of appending the stage in square brackets to a complicated expression to indicate that every element of the expression is intended to be evaluated at that stage. For example, if $\langle f_s\rangle$ is a computable approximation of a function f, and $\mathcal{V}_s$ is an effective enumeration of $\mathcal{V}$, then we write $\mathcal{V} \cup [\![W_{f(n)}]\!]\,[s]$ rather than $\mathcal{V}_s \cup [\![W_{f_s(n),s}]\!]$.

Definition 2.5. A *test* is a sequence $\langle \mathcal{V}_n\rangle_{n<\omega}$ of open subsets of Cantor space 2^ω such that for all n, $\lambda(\mathcal{V}_n) \leqslant 2^{-n}$; here λ denotes the fair coin measure on Cantor space. We say that a set $X \in 2^\omega$ *passes* the test $\langle \mathcal{V}_n\rangle$ if $X \in \mathcal{V}_n$ for only finitely many n. Otherwise, the set X *fails* the test. The collection of sets which fail a test is a null class.

A test $\langle \mathcal{V}_n\rangle$ is *effectively open* if each $\mathcal{V}_n$ is an effectively open subset of Cantor space. If $\langle \mathcal{V}_n\rangle$ is effectively open, then an *index function* for $\langle \mathcal{V}_n\rangle$ is a function $f\colon \omega \to \omega$ such that for all n, $\mathcal{V}_n = [\![W_{f(n)}]\!]$; here $\langle W_e\rangle$ is an effective list of all c.e. sets. Thus, for example, an effectively open test is a Martin-Löf test if it has a computable index function. A *Demuth test* is an effectively open test which has an ω-c.a. index function. A set $X \in 2^\omega$ is *Demuth random* if it passes all Demuth tests.

Rather than working with Demuth tests, it will be convenient to work with a more restrictive (yet equally powerful) notion of tests.

Definition 2.6. A test $\langle \mathcal{V}_n\rangle$ is *clopen* if each $\mathcal{V}_n$ is a clopen subset of 2^ω. If $\langle \mathcal{V}_n\rangle$ is a clopen test, then a *clopen index function* for $\langle \mathcal{V}_n\rangle$ is a function $f\colon \omega \to \omega$ such that for all n, $\mathcal{V}_n = [\![D_{f(n)}]\!]$; here $\langle D_e\rangle$ is an effective list of all finite sets of strings. Thus, for example, a Kurtz test is a clopen test which has a computable clopen index function. A *Demuth clopen test* is a clopen test which has an ω-c.a. clopen index function.

The following lemma is implicit in [28]. We give a proof for completeness.

Lemma 2.7. *A set $X \in 2^\omega$ is Demuth random if and only if it passes every clopen Demuth test.*

Proof. Every clopen Demuth test is a Demuth test. Hence every Demuth random set passes every clopen Demuth test.

For the converse, we show that for any Demuth test $\langle \mathcal{V}_n \rangle$ there is a clopen Demuth test $\langle \mathcal{U}_n \rangle$ such that every set which fails the test $\langle \mathcal{V}_n \rangle$ also fails the test $\langle \mathcal{U}_n \rangle$. Let $\langle \mathcal{V}_n \rangle$ be a Demuth test.

The idea is to copy $\bigcup_n \mathcal{V}_n$ into various $\mathcal{U}_n$'s in discrete steps. For each $\mathcal{U}_n$, we set a threshold $\epsilon(n)$. We then copy $\mathcal{V}_n$ into $\mathcal{U}_n$ only at stages at which the measure of $\mathcal{V}_n$ passes some integer multiple of $\epsilon(n)$. At other stages, the part of $\mathcal{V}_n$ which hasn't yet been copied to $\mathcal{U}_n$ is split up and copied to $\mathcal{U}_m$ for various $m > n$, depending on the measure of that part and its relation to the thresholds $\epsilon(m)$. At a later stage, if the measure of $\mathcal{V}_n$ crosses another integer multiple of $\epsilon(n)$, we recall that part of $\mathcal{V}_n$ which has been passed to $\mathcal{U}_m$ for $m > n$, and copy it to $\mathcal{U}_n$. Because $\epsilon(n)$ is fixed, $\mathcal{U}_n$ is changed only a finite number of times, and so $\mathcal{U}_n$ is clopen.

Actually, this description is not quite correct, because we can set $\epsilon(n)$ to be greater than $\lambda(\mathcal{V}_n)$, so $\lambda(\mathcal{V}_{n,s})$ never crosses an integer multiple of $\epsilon(n)$. What we in fact track, when defining $\mathcal{U}_n$, is the total measure of the parts of $\mathcal{V}_{k,s}$ for $k < n$ which are passed down to $\mathcal{U}_n$.

To assist with the construction, we will define auxiliary clopen sets $\langle \mathcal{S}_{n,s} \rangle$. These consist of the measure passed on to $\mathcal{U}_n$ by $\mathcal{U}_{n-1}$, together with $\mathcal{V}_{n,s}$. Let f be an ω-c.a. index function for $\langle \mathcal{V}_n \rangle$, and let $\langle f_s \rangle$ be an ω-computable approximation for f. We let $\mathcal{V}_{n,s} = [\![W_{f_s(n),s}]\!]$. Since each set $W_{e,s}$ is finite, each set $\mathcal{V}_{n,s}$ is clopen (in fact, a canonical index d such that $W_{f_s(n),s} = D_d$ can be obtained effectively from n and s). We may assume that for all n and s, $\lambda(\mathcal{V}_{n,s}) \leqslant 2^{-n}$, and that for all s, for all $n \geqslant s$, $\mathcal{V}_{n,s} = \varnothing$.

For all n, we let $\epsilon(n) = 2^{-n}$.

Construction. At stage 0, we let $\mathcal{U}_{n,0} = \mathcal{S}_{n,0} = \varnothing$ for all n. At stage $s > 0$ we define $\mathcal{S}_{n,s}$ and $\mathcal{U}_{n,s}$ for all n by recursion on n. We first let $\mathcal{S}_{0,s} = \varnothing$. Let $n < \omega$, and suppose that $\mathcal{S}_{n,s}$ is already defined.

If

$$\lambda\left(\mathcal{S}_{n,s} \backslash \mathcal{U}_{n,s-1}\right) > \epsilon(n),$$

then $\mathcal{U}_n$ needs to change; we let $\mathcal{U}_{n,s} = \mathcal{S}_{n,s}$, and for all $m > n$ we let $\mathcal{U}_{m,s} = \mathcal{S}_{m,s} = \varnothing$. Otherwise, we let $\mathcal{U}_{n,s} = \mathcal{U}_{n,s-1}$, let

$$\mathcal{S}_{n+1,s} = \mathcal{V}_{n,s} \cup \left(\mathcal{S}_{n,s} \backslash \mathcal{U}_{n,s}\right),$$

and proceed to define $\mathcal{U}_{n+1,s}$.

Verification. For all n and s,

$$\lambda\left(\mathcal{S}_{n+1,s}\right) \leqslant \epsilon(n) + \lambda\left(\mathcal{V}_{n,s}\right) \leqslant 2^{-n+1}.$$

Hence, for all n and s, $\lambda\left(\mathcal{U}_{n,s}\right) \leqslant 2^{-n+1}$.

We also see that even though this was not required for the construction to be computable, every stage of the construction is in fact finite. We show, by induction on s, that for almost all n, $\mathcal{S}_{n,s} = \mathcal{U}_{n,s} = \varnothing$. Suppose this holds at stage $s-1$. Suppose, for contradiction, that for infinitely many n we have $\mathcal{S}_{n,s} \neq \varnothing$; so no $\mathcal{U}_n$ "acts" at stage s, and for all n we have $\mathcal{U}_{n,s} = \mathcal{U}_{n,s-1}$. Since for all $n \geqslant s$, $\mathcal{V}_{n,s} = \varnothing$, and for almost all n, $\mathcal{U}_{n,s-1}$ is empty, for almost all n, we have $\mathcal{S}_{n+1,s} = \mathcal{S}_{n,s}$; so we are assuming that this stable set is nonempty, and hence has positive measure. Since $\epsilon(n) \to 0$, there is some n such that $\epsilon(n) < \lambda(\mathcal{S}_{n,s})$, and this n would act at stage s and set $\mathcal{S}_{m,s} = \varnothing$ for all $m > n$, yielding a contradiction. Hence, for almost all n, $\mathcal{S}_{n,s} = \varnothing$; this implies that for almost all n, $\mathcal{U}_{n,s} = \mathcal{U}_{n,s-1} = \varnothing$.

There is a uniformly computable sequence $\langle h_s \rangle$ of functions such that for all n and s, $\mathcal{U}_{n,s} = [\![D_{h_s(n)}]\!]$.

Claim 2.7.1. The sequence $\langle h_s \rangle$ is an ω-computable approximation.

Proof. Fix $n < \omega$. Let $s_0 > 0$ be a stage s such that $\mathcal{U}_{n,s} \neq \mathcal{U}_{n,s-1}$. Suppose further that for all $m < n$, $\mathcal{U}_{m,s_0} = \mathcal{U}_{m,s_0-1}$. Hence at stage s_0 we define $\mathcal{U}_{n,s_0} = \mathcal{S}_{n,s_0}$, but for all $m < n$, we have $\mathcal{S}_{m+1} = \mathcal{V}_m \cup (\mathcal{S}_m \backslash \mathcal{U}_m)\ [s_0]$.

Suppose that there is some stage $s > s_0$ such that $\mathcal{U}_{n,s} \neq \mathcal{U}_{n,s-1}$; let s_1 be the least such stage. We claim that there is some $m < n$ for which one of the following holds:

(1) $\mathcal{U}_{m,s_1} \neq \mathcal{U}_{m,s_1-1}$.
(2) There is some $s \in [s_0, s_1)$ such that $f_s(m) \neq f_{s-1}(m)$.
(3) $\lambda\left(\mathcal{V}_{m,s_1} \backslash \mathcal{V}_{m,s_0}\right) > \epsilon(n)/n$.

Suppose that (1) and (2) do not hold. To show that (3) holds, we show that in this case,

$$\mathcal{S}_{n,s_1} \backslash \mathcal{U}_{n,s_0} \subseteq \bigcup_{m<n} \left(\mathcal{V}_{m,s_1} \backslash \mathcal{V}_{m,s_0}\right);$$

(3) then follows from the fact that the minimality of s_1 ensures that $\mathcal{U}_{n,s_1-1} = \mathcal{U}_{n,s_0}$, and from the fact that $\lambda\left(\mathcal{S}_{n,s_1} \backslash \mathcal{U}_{n,s_1-1}\right) > \epsilon(n)$. To verify the containment, let $X \in \mathcal{S}_{n,s_1} \backslash \mathcal{U}_{n,s_0}$. Since (1) does not hold, for all $m < n$, $\mathcal{S}_{m+1} = \mathcal{V}_m \cup (\mathcal{S}_m \backslash \mathcal{U}_m)\ [s_1]$. By minimality of s_1, and since (2) does not hold, for all $m < n$, $\mathcal{U}_{m,s_1} = \mathcal{U}_{m,s_0}$. Since $\mathcal{S}_n \subseteq \bigcup_{m<n} \mathcal{V}_m\ [s_1]$, there is some $m < n$ such that $X \in \mathcal{V}_{m,s_1}$; pick m^* to be the greatest such m. Then $X \in \mathcal{S}_{n,s_1}$ implies that for all $m \in (m^*, n)$, $X \notin \mathcal{U}_{m,s_1}$. Now if $X \in \mathcal{V}_{m^*,s_0}$, then the fact that $X \notin \mathcal{V}_{m,s_0}$ for all $m \in (m^*, n)$ would imply that $X \in \mathcal{S}_{n,s_0}$ and so $X \in \mathcal{U}_{n,s_0}$. Hence $X \in \mathcal{V}_{m^*,s_1} \backslash \mathcal{V}_{m^*,s_0}$ as required.

This analysis allows us to recursively define a bound $k(n)$ for $m_{\langle h_s \rangle}$. Let g be a bound for $m_{\langle f_s \rangle}$. We can let $k(0) = 0$, as $\mathcal{U}_{0,s} = \varnothing$ for all s. If $k(m)$ is defined for all $m < n$, then we can let

$$k(n) = \left(\sum_{m<n} k(m)\right) \cdot \left(\sum_{m<n} g(m)\right) \cdot \frac{n^2}{\epsilon(n)}.$$

Note that k depends only on g and not on f. □

Let $h = \lim_s h(s)$; for $n < \omega$, let $\mathcal{U}_n = [\![D_{h(n)}]\!] = \lim_s \mathcal{U}_{n,s}$. Hence $\langle \mathcal{U}_{n+1} \rangle_{n<\omega}$ is a clopen Demuth test. It remains to see that every set $X \in 2^\omega$ which fails the test $\langle \mathcal{V}_n \rangle$ also fails the test $\langle \mathcal{U}_{n+1} \rangle$. This follows from the following claim.

Claim 2.7.2. For all $n < \omega$,

$$\mathcal{V}_n \subseteq \bigcup_{m>n} \mathcal{U}_m.$$

Proof. Let $n < \omega$ and let $X \in \mathcal{V}_n$. Let s_0 be a stage sufficiently late so that for all $s \geqslant s_0$, $X \in \mathcal{V}_{n,s}$, and so that for all $s \geqslant s_0$, $\mathcal{U}_{n,s} = \mathcal{U}_{n,s-1}$. Hence for all $s \geqslant s_0$, we let $\mathcal{S}_{n+1} = \mathcal{V}_n \cup (\mathcal{S}_n \backslash \mathcal{U}_n)\ [s]$, and so for all $s \geqslant s_0$, $X \in \mathcal{S}_{n+1,s}$.

First, we see that for all $s \geqslant s_0$ there is some $m > n$ such that $X \in \mathcal{U}_{m,s}$. We saw above that there is some $m > n$ such that $\mathcal{S}_{m,s} = \varnothing$; so there is some $m > n$ such that $X \in \mathcal{S}_{m,s} \backslash \mathcal{S}_{m+1,s}$. Either $\mathcal{U}_{m,s} = \mathcal{S}_{m,s}$, or $\mathcal{S}_{m+1,s} \supseteq \mathcal{S}_{m,s} \backslash \mathcal{U}_{m,s}$; in either case, $X \in \mathcal{U}_{m,s}$. For $s \geqslant s_0$, let $m(s)$ be the least $m > n$ such that $X \in \mathcal{U}_{m,s}$.

The function $m(s)$ is nonincreasing. To see this, let $s > s_0$, and suppose that $m(s) \neq m(s-1)$. Suppose, for contradiction, that $m(s) > m(s-1)$. Then for all $m \in (n, k]$, we have $X \notin \mathcal{U}_{m,s}$. Since $X \in \mathcal{U}_{k,s-1}$, we have $\mathcal{U}_{k,s} \neq \mathcal{U}_{k,s-1}$. This implies that for all $m > k$, $\mathcal{U}_{m,s} = \varnothing$, contradicting $m(s) > m$ and $X \in \mathcal{U}_{m(s),s}$.

Hence $m = \lim_s m(s)$ exists, and for almost all s, $X \in \mathcal{U}_{m,s}$. Hence $X \in \mathcal{U}_m$, as required. □

This completes the proof of Lemma 2.7. □

We would like to draw the reader's attention to certain terminology that was used in the last proof, and will be used throughout the paper. In a couple of instances, the word "measure" meant "a nonempty clopen subset of Cantor space", as in "the measure passed on to $\mathcal{U}_n$ by $\mathcal{U}_{n-1}$". This incorrect usage of the word "measure" makes for smoother sentences, but also emphasises that we often don't quite care which particular nonempty clopen sets we are dealing with, but rather care about their measure.

To keep future calculations smoother, we employ quick tests.

Definition 2.8. A test $\langle \mathcal{V}_n \rangle$ is *quick* if for all n, $\lambda(\mathcal{V}_n) \leqslant 2^{-2n}$.

Lemma 2.9. *A set X is Demuth random if and only if it passes every quick clopen Demuth test.*

Proof. Let $\langle \mathcal{V}_n \rangle$ be a clopen Demuth test. For $n < \omega$, let $\mathcal{U}_n = \mathcal{V}_{2n+1} \cup \mathcal{V}_{2n+2}$. Then

$$\lambda(\mathcal{U}_n) \leqslant 2^{-2n+1} + 2^{-2n+2} < 2^{-2n},$$

so $\langle \mathcal{U}_n \rangle$ is a quick test, and it is easy to see that $\langle \mathcal{U}_n \rangle$ is a clopen Demuth test. If X fails $\langle \mathcal{V}_n \rangle$ then it fails $\langle \mathcal{U}_n \rangle$. □

In general, it can be shown that if $\langle q_n \rangle$ is a computable, nonincreasing sequence of rational numbers, and $\sum_n q_n$ converges to a computable real number, then a set is Demuth random if and only if it passes all clopen Demuth tests $\langle \mathcal{V}_n \rangle$ satisfying $\lambda(\mathcal{V}_n) \leqslant q_n$ for all n. In this paper we do not make use of this more general fact.

We fix an enumeration of quick clopen Demuth tests. Using a uniform enumeration of all ω-c.a. functions, we fix an effective list $\langle \mathcal{V}^e_{n,s} \rangle$ of clopen sets (that is, canonical indices are given effectively), and an effective list $\langle g^e \rangle$ of partial computable functions, such that:

- For all n, e and s, $\lambda(\mathcal{V}^e_{n,s}) \leqslant 2^{-2n}$;
- For all n and e, if $n \in \operatorname{dom} g^e$, then $\#\{s : \mathcal{V}^e_{n,s} \neq \mathcal{V}^e_{n,s+1}\} \leqslant g^e(n)$;
- For all n and e, $\mathcal{V}^e_{n,0} = \varnothing$;
- For all n and e, if $n \notin \operatorname{dom} g^e$, then for all s, $\mathcal{V}^e_{n,s} = \varnothing$;
- For all e, the domain of g^e is an initial segment of ω;

- For every quick clopen Demuth test $\langle \mathcal{V}_n \rangle$, there is an e such that g^e is total and $\mathcal{V}_n = \mathcal{V}^e_n$, where $\mathcal{V}^e_n = \lim_s \mathcal{V}^e_{n,s}$; and
- g^0 is total.

3. Proof of Theorem 1.1

By the Kučera-Nies result from [30], it is sufficient to show that every strongly jump-traceable c.e. set is computable from some Demuth random set. Let A be a strongly jump-traceable c.e. set. Let $\langle A_s \rangle$ be an effective enumeration of A.

We want to construct a Demuth random set that computes A. To do so, we enumerate a Turing functional Γ. A typical axiom, enumerated into Γ at a stage s of the construction, will map a clopen subset $\mathcal{C}$ of Cantor space to some initial segment of A_s. At the end we let, for $X \in 2^\omega$,

$$\Gamma^X = \bigcup \alpha \ \llbracket \text{for some clopen } \mathcal{C} \text{ we have } X \in \mathcal{C} \ \& \ (\mathcal{C}, \alpha) \in \Gamma \rrbracket.$$

Because A is c.e., to keep Γ consistent it is sufficient (and necessary) to ensure that if $\mathcal{C}$ is added to the domain of Γ at stage s, then $\mathcal{C}$ is disjoint from the *error set*:

$$\mathcal{E}_s = \left\{ Y \in 2^\omega \ : \ \Gamma^Y_s \not\preccurlyeq A_s \right\}.$$

Our aim is to construct Γ so that there is some X such that $\Gamma^X = A$, and X passes every Demuth test. There are therefore three tasks at hand:

- Ensure that $X \notin \mathcal{E}$, where
$$\mathcal{E} = \bigcup_s \mathcal{E}_s = \left\{ Y \in 2^\omega \ : \ \Gamma^Y \not\preccurlyeq A \right\};$$
- Ensure that for all k, $|\Gamma^X| \geqslant k$;
- Ensure that for all e such that g^e is total, there is some n_e such that for all $n \geqslant n_e$, $X \notin \mathcal{V}^e_n$.

3.1. **Towards a full strategy.** We begin by illustrating simplified approaches to the construction, explain what goes wrong, and discuss the added complexity needed to address these issues.

For every k, we would like to have some clopen set $\mathcal{U}^k$ with $\Gamma(\mathcal{U}^k) = A \upharpoonright_k$. We would also like these to be nested, so that $\mathcal{U}^{k+1} \subseteq \mathcal{U}^k$. Then by compactness there is some $X \in \bigcap_k \mathcal{U}^k$. For such an X we would have $\Gamma^X = A$.

The simplest approach to constructing these is to simply select some clopen set $\mathcal{U}^k$ and define $\Gamma(\mathcal{U}^k) = A_k \upharpoonright_k$. Of course, assuming A is non-computable, there will be k such that $A_k \upharpoonright_k \neq A \upharpoonright_k$. When we see $A \upharpoonright_k$ change, all the measure in $\mathcal{U}^k$ becomes "bad" (it enters $\mathcal{E}$), so we need to select new measure from $\mathcal{U}^{k-1}$ and use that to redefine $\mathcal{U}^k$ (defining $\Gamma(\mathcal{U}^k) = A_s \upharpoonright_k$ for this new $\mathcal{U}^k$). Now given $A \upharpoonright_{k-1}$, $A \upharpoonright_k$ can change at most once, and so we can prepare for this eventuality by splitting every version of $\mathcal{U}^{k-1}$ into two potential versions for $\mathcal{U}^k$. For understanding the current construction, it is best to view this dynamically. We imagine that when $\mathcal{U}^{k-1}$ is established, it carves out a piece of itself and passes it down as $\mathcal{U}^k$ to an agent who is responsible for defining Γ on $\mathcal{U}^k$. When that agent observes that it defined Γ incorrectly, it passes that measure back to $\mathcal{U}^{k-1}$, and asks for replacement. $\mathcal{U}^{k-1}$ has in reserve some measure, free of extra Γ definitions, and so not yet in $\mathcal{E}$, which it gives as replacement measure, to define a new version of $\mathcal{U}^k$. All we need to do is make sure that $\mathcal{U}^{k-1}$ always has sufficiently much free measure to provide when asked to. This is done by counting how many times it may need to replace measure (in this simplified scenario, only once), and then making the measure of $\mathcal{U}^k$ sufficiently small relative to the measure of $\mathcal{U}^{k-1}$ (in this case, at most half).

Of course, the above approach makes no effort to ensure that X is Demuth random. So suppose $\langle \mathcal{V}_n \rangle$ were some Demuth test; we wish to ensure that X is not captured by (i.e. passes) $\langle \mathcal{V}_n \rangle$. The easiest approach would be to assign to some k the task of avoiding the test. The Solovay passing condition for Demuth tests implies that $\mathcal{U}^k$ has to consider all $\mathcal{V}_n$ or at least all but finitely many. The natural instructions would be, whenever any $\mathcal{V}_{n,s}$ covers part of $\mathcal{U}^k$, remove $\mathcal{V}_{n,s} \cap \mathcal{U}^k$ from $\mathcal{U}^k$ and take replacement measure from $\mathcal{U}^{k-1}$. Of course, if n is small compared to k, it might be that $\mathcal{U}^{k-1} \subseteq \mathcal{V}_{n,s}$, so there would be no good replacement measure to take. This can be solved by choosing an appropriately large value n_k and only considering $\mathcal{V}_{n,s}$ with $n \geqslant n_k$. Making $\mathcal{U}^k$ disjoint from all $\mathcal{V}_n$ for $n \geqslant n_k$ suffices to ensure every element of $\mathcal{U}^k$ passes $\langle \mathcal{V}_n \rangle$.[1]

There are still problems with this approach, however. First, because $\mathcal{U}^k$ is trying to avoid infinitely many $\mathcal{V}_n$, it will never settle; there will always be an n for which $\mathcal{V}_n$ is still "moving" — the ω-computable approximation function is still changing. This $\mathcal{V}_n$ will cause measure to move out of $\mathcal{U}^k$, and so $\mathcal{U}^k$ will always be changing. Thus there is no good limit for $\mathcal{U}^k$: if we use the inner limit of the $\mathcal{U}^k_s$, the result could be empty, since each $\mathcal{V}_n$ could potentially move "through" all of $\mathcal{U}^{k-1}$ before settling; if we use the outer limit, the result will not in general be a closed set, and the compactness argument above will fail; if we take the closure of the outer limit, we may include reals that are captured by $\langle \mathcal{V}_n \rangle$.

Our solution here is the same as in Lemma 2.7: we only change $\mathcal{U}^k$ when a critical amount of "badness" has built up. In particular, we only remove measure from $\mathcal{U}^k$ when the amount that needs to be replaced is at least $1/4$ the total measure of $\mathcal{U}^k$. Because the $\mathcal{V}_n$ shrink quickly, there will be some m such that only those $\mathcal{V}_n$ with $n_k \leqslant n \leqslant m$ need be considered; the total combined measure of the $\mathcal{V}_n$ for $n > m$ cannot possibly be enough to trigger a change in $\mathcal{U}^k$. Once the approximation function has settled for $n \leqslant m$, $\mathcal{U}^k$ will have settled. So $\mathcal{U}^k$ will be closed (actually clopen) as desired.

However, letting $\mathcal{W}_k = \bigcup_{n \geqslant n_k} \mathcal{V}_n$, the set $\mathcal{U}^k \cap \mathcal{W}_k$ of reals in $\mathcal{U}^k$ which may be captured by the test is nonempty. This will be an open set (not necessarily effectively so) of measure at most $1/4$ the measure of $\mathcal{U}^k$, so $\mathcal{U}^k - \mathcal{W}^k$ will be a closed, nonempty set. So we use $\mathcal{U}^k - \mathcal{W}^k$ in place of $\mathcal{U}^k$ in the compactness argument above.

Now we need to worry about the sequence being nested. $\mathcal{U}^k \cap \mathcal{W}^k$ may only be a fraction of the size of $\mathcal{U}^k$, but it could be that $\mathcal{U}^{k+1} \subseteq \mathcal{W}^k$. So $\mathcal{U}^{k+1}$ will need to avoid $\mathcal{W}^k$ in addition to avoiding whatever test $\langle \mathcal{V}'_n \rangle$ it is assigned. Again, $\mathcal{U}^{k+1}$ only replaces measure when the amount needing replacement is at least $1/4$ its total measure. Now, however, it concerns itself not only with replacing measure covered by its "own" test $\langle \mathcal{V}'_n \rangle$, but also with replacing measure covered by $\mathcal{W}^k$. Since $\mathcal{W}^k$ is small compared to $\mathcal{U}^k$, and because $\mathcal{U}^{k+1}$ chooses a large n_{k+1} for its own test, the amount covered at any one time is small, so $\mathcal{U}^{k+1}$ should always be able to find good replacement measure in $\mathcal{U}^k$ when it needs it.

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[1] As we shall later see, measure considerations cause us to consider all $\mathcal{V}_n$ for $n \geqslant n_k$, rather than letting subsequent $\mathcal{U}^{k'}$ consider more and more $\mathcal{V}_n$'s. Once we decided n_k, we have to avoid $\mathcal{V}_n$ for all $n \geqslant n_k$. The measure of $\mathcal{U}^{k+1}$ will be much smaller than the measure of $\mathcal{U}^k$, so will not be able to supply, for example, $\mathcal{U}^{k+2}$ with replacement measure for areas covered by (say) $\mathcal{V}_{n_k+1}$.

Now we revisit the first part of our construction, ensuring that $\Gamma(\mathcal{U}^k) = A\restriction_k$, and consider how it interacts with this new process. Suppose $\mathcal{U}^k$ is assigned the task of avoiding $\langle \mathcal{V}_n \rangle$, and $\mathcal{V}_n$ is some component of the test such that $\lambda(\mathcal{V}_n)$ is smaller than $1/4$ the measure of $\mathcal{U}^k$, but larger than $1/4$ the measure of $\mathcal{U}^{k+1}$. So, on its own, $\mathcal{V}_n$ is enough to cause $\mathcal{U}^{k+1}$ to change, but not enough to cause $\mathcal{U}^k$ to change. For the moment we ignore the actions of any other components of the test.

At stage s, $\Gamma(\mathcal{U}^k) = A_s\restriction_k$ and $\Gamma(\mathcal{U}^{k+1}) = A_s\restriction_{k+1}$. Suppose that at this stage, $\mathcal{V}_n$ changes to cover measure in $\mathcal{U}^{k+1}$. Then $\mathcal{U}^{k+1}$ will remove that measure and seek replacement measure from $\mathcal{U}^k$. However, the Γ-computation for $A_s\restriction_{k+1}$ still exists on the removed measure. If $\mathcal{V}_n$ changes again to cover new measure in $\mathcal{U}^{k+1}$, then $\mathcal{U}^{k+1}$ will again seek new measure. There is a bound on the number of times $\mathcal{V}_n$ can change, but it could potentially be large with respect to $\lambda(\mathcal{U}^k)/\lambda(\mathcal{U}^{k+1})$. So in this fashion, $\mathcal{U}^k$ could be filled with Γ-computations for $A_s\restriction_{k+1}$. All of this measure is being "risked": suppose that after this happens, k enters A. Then all of the measure in $\mathcal{U}^k$ is bad, since it miscomputes A, while all the Γ-computations defined by the agent monitoring $\mathcal{U}^k$ are correct. This process magnifies itself when we consider the action for various $\mathcal{U}^{k+\ell}$ and possibly more tests. Cumulatively, the measure in $\mathcal{U}^k$ could all go bad because of numbers much larger than k entering A, numbers which are far beyond the scope of comprehension of $\mathcal{U}^k$. The end result is that $\mathcal{E}$ covers the entire space and a set X as required cannot be found.

Partly, this happened because $\mathcal{U}^k$ was wasteful when it gave replacement measure to $\mathcal{U}^{k+1}$ when $\mathcal{V}_n$ moved. Say that a piece of measure $\mathcal{C}_1$, covered by $\mathcal{V}_n$, is rejected by $\mathcal{U}^{k+1}$ and is replaced by another piece, $\mathcal{C}_2$. Then $\mathcal{V}_n$ moves and covers $\mathcal{C}_2$. This movement may have uncovered $\mathcal{C}_1$, which is now free to return to $\mathcal{U}^{k+1}$. The measure in $\mathcal{C}_1$ was already risked, so it would be foolish to risk new measure when we could use $\mathcal{C}_1$.

But this is not the whole story. Even if $\mathcal{U}^k$ is economical with its replacement measure, it is still the case that the cumulative effect of the work of agents $\mathcal{U}^l$ for l far greater than k may make all the measure in $\mathcal{U}^k$ go bad. Suppose n is such that $\lambda(\mathcal{V}_n)$ is smaller than $1/4$ the measure of $\mathcal{U}^k$, but larger than $1/4$ the measure of $\mathcal{U}^{k+1}$, and δ is a positive rational number with $\delta < \lambda(\mathcal{V}_n) - 1/4 \cdot \lambda(\mathcal{U}^{k+1})$. Let l be far greater than n and k. At stage s, we have $\Gamma(\mathcal{U}^{l+1}) = A_s\restriction_{l+1}$. Suppose $\mathcal{V}_n$ then changes to cover measure in $\mathcal{U}^{k+1}$, and in particular covers all of $\mathcal{U}^{l+1}$. Then $\mathcal{U}^{k+1}$ will remove that measure and seek replacement measure from $\mathcal{U}^k$, some of this replacement measure will eventually be provided to $\mathcal{U}^{l+1}$, and $\mathcal{U}^{l+1}$ will put Γ-computations on it.

Suppose the measure with Γ-computations for $A_s\restriction_{l+1}$ is now smaller than δ (this measure consists of the measure which is currently in $\mathcal{U}^{l+1}$ and the measure which just left $\mathcal{U}^{l+1}$). Then $\mathcal{V}_n$ is large enough that it can cover all the measure with Γ-computations for $A_s\restriction_{l+1}$, and also cover at least $1/4$ the measure of $\mathcal{U}^{k+1}$. So it can cover all the measure with Γ-computations for $A_s\restriction_{l+1}$ and also force $\mathcal{U}^{k+1}$ to seek replacement measure. Suppose $\mathcal{V}_n$ changes to do this. Then again $\mathcal{U}^{k+1}$ will get replacement measure; some of this measure will have been newly uncovered by the $\mathcal{V}_n$ change, since $\mathcal{U}^k$ is being economical, but none of it will have Γ-computations for $A_s\restriction_{l+1}$ on it, since $\mathcal{V}_n$ covers all of that measure. So $\mathcal{U}^{l+1}$ will put down new Γ-computations on $\lambda(\mathcal{U}^{l+1})$ much measure. We can continue in this fashion until $\mathcal{U}^{l+1}$ has put Γ-computations on at least δ measure. If l then enters A, δ much of $\mathcal{U}^k$ has gone bad because of the action of l. This process can then be repeated multiple times with greater l, and so all of $\mathcal{U}^k$ might go bad because of numbers much larger than k entering A.

This is where the strong jump-traceability of A is used. The idea is that every piece of measure $\mathcal{C}$ which is risked needs to be *claimed* by some actor O. When $\mathcal{C}$ is moved to O, then from that moment on, actor O *claims responsibility* for all $Y \in \mathcal{C}$. The actor keeps being responsible for such Y until Y is moved to some other actor, at which time O is no longer responsible for Y. It will be possible for this to happen without Y leaving O: for example, if Y is part of the measure that $\mathcal{U}^k$ provides to $\mathcal{U}^{k+1}$, then $\mathcal{U}^{k+1}$ has claimed responsibility for Y, and $\mathcal{U}^k$ has discharged its responsibility, without Y leaving $\mathcal{U}^k$.

Before such measure is claimed (and so before the actor puts Γ computations on $\mathcal{C}$), the actor O can ask for some evidence that this piece of measure $\mathcal{C}$ will not go bad. There may be already some Γ computations defined on subsets of $\mathcal{C}$; and the actor O may plan to put some new computation on $\mathcal{C}$. There is some initial segment α of A_s so that if $\alpha \prec A$ then none of the measure in $\mathcal{C}$ will go bad. The actor O will want some evidence that $\alpha \prec A$. Strong jump-traceability allows us to devise a process of *testing* of such initial segments α.

O will *perform a test* by selecting some input x for some partial functional Ψ and enumerating the axiom $\Psi^\alpha(x) = \alpha$. O then waits. Eventually, either we will observe that α is no longer an initial segment of A_t (and so cannot be an initial segment of A), or the test will be *successful*, meaning we will see $\alpha \in T(x)$, where T is a trace for Ψ^A. We delay for the moment any discussion of where Ψ, x or T come from. In the first case, the measure $\mathcal{C}$ is rejected by O, and any measure in $\mathcal{C}$ which went bad because of the change in A is charged to other actors, whoever claimed it before it was offered to O. In the second case, the actor O accepts $\mathcal{C}$ and may put a new computation on $\mathcal{C}$. Certainly, the fact that the test returned positively does not guarantee that α is in fact an initial segment of A, so some measure under O's watch may yet go bad. But the testing procedure will be designed so that this can happen only a small number of times, which we determine in advance. We do this by controlling the h for which T is an h-trace.

The bound on the total amount of measure in $\mathcal{E}$ which will be *charged* to the actor O will be the product of two numbers: the number of times measure under its watch can go bad; and a bound on the amount of measure which can be claimed by O at any given time. We then bound the total size of $\mathcal{E}$ by distributing "garbage quotas" to the actors, whose sum is bounded away from 1.

What are the actors? First, each $\mathcal{U}^k$ is an actor, responsible for the measure in the set $\mathcal{U}^k$ itself. Thus, the bound on the amount of measure claimed by $\mathcal{U}^k$ at each stage will be given by a fixed bound on the size of $\mathcal{U}^k$. This means that measure $\mathcal{C}$ returned by $\mathcal{U}^{k+1}$ to $\mathcal{U}^k$, say at some stage s, must no longer be claimed by $\mathcal{U}^{k+1}$: it only claims the measure in its new version $\mathcal{U}^{k+1}_{s+1}$, which does not include $\mathcal{C}$. Who claims this rejected measure? If $\mathcal{C}$ is part of $\mathcal{E}_s$, then no-one needs to claim it: it was already charged to the last actor which claimed it. Otherwise, it would seem that we could return $\mathcal{C}$ to $\mathcal{U}^k$'s guardianship. After all, before $\mathcal{C}$ was ever given to $\mathcal{U}^{k+1}$ by $\mathcal{U}^k$, it was part of $\mathcal{U}^k$, and so $\mathcal{U}^k$ took responsibility for it then, say at stage $t < s$; and the fact that $\mathcal{C}$ is not bad at stage s indicates that the tests which prompted $\mathcal{U}^k$ to claim $\mathcal{C}$ at stage t are still valid at stage s. But not all claims are alike. Between stages t and s, $\mathcal{U}^{k+1}$ and actors $\mathcal{U}^l$ for $l > k+1$ could have taken some parts of $\mathcal{C}$ and defined long Γ computations on them. Thus, to take responsibility for $\mathcal{C}$ at stage s, we need to test a longer initial segment of A than was necessary for claiming $\mathcal{C}$ at stage t. In other words, the measure in $\mathcal{C}$ got riskier with time.

We cannot require $\mathcal{U}^k$ to re-test on behalf of $\mathcal{C}$. The proper use of strong jump-traceability requires that we find out in advance how many possible tests would be run in parallel by each actor. For $\mathcal{U}^k$, this will correspond to the number of possible versions of $\mathcal{U}^k$. The return of $\mathcal{C}$ to $\mathcal{U}^k$ has prompted a change in $\mathcal{U}^{k+1}$, but $\mathcal{U}^k$ may stay the same, and so a new test for $\mathcal{U}^k$ is not budgeted for. (The reader may ask why this restriction on the number of tests for $\mathcal{U}^k$. Can't $\mathcal{U}^k$ figure out how many times $\mathcal{U}^{k+1}$ will move and budget for that? The answer is no, but the reason may not yet be apparent, because we did not introduce the tree of strategies yet. In reality, we will have infinitely many versions of $\mathcal{U}^{k+1}$, which will in total move infinitely many times, and $\mathcal{U}^k$ is only allowed to have finitely many tests.)

For this reason we introduce another kind of actors, the *bins*. Their task is to take responsibility for measure which is rejected by $\mathcal{U}^{k+1}$ and has not yet gone bad. Recall that such measure is rejected because it is covered by some $\mathcal{V}_n$. A bin associated with $\mathcal{U}^k$ and $\mathcal{V}_n$ takes control of such measure. The frugality, discussed above, of $\mathcal{U}^k$ when choosing measure to give to $\mathcal{U}^{k+1}$, is used to get the bound on the amount of measure claimed by a bin at any given time. The size of the bin associated with $\mathcal{V}_n$ will be bounded by the size of $\mathcal{V}_n$, which is 2^{-2n}; thus, whenever new measure is given to the bin, we need to extract from the bin (and return to $\mathcal{U}^{k+1}$) measure which $\mathcal{V}_n$ no longer covers. The number of tests for the bin that we need to budget for will be tied to the number of times $\mathcal{V}_n$ can move. Because $\mathcal{V}_n$ is a component of a Demuth test, we can find out this number in advance as is required for setting up the testing mechanism for the bin.

These are the ideas which make the construction work. What is left is to provide the details. We start with setting up the overall mechanism of the tree of strategies. We will then give the construction, so that the reader has some global picture of what is going on; only afterwards will we flesh out the details. Much of the verification is in showing that the construction makes sense – that it can actually be performed – and we will provide the arguments for this as we go along. Along the way, we will specify rules which we promise the construction abides by, and use them in our verifications. We also make claims about the construction which we will only be able to prove after we have explained all or most of the construction. Actually, we have already made one such claim, which we restate:

Claim 3.1. If a new axiom, defining a computation on a clopen set $\mathcal{C}$, is added to Γ at stage s, then $\mathcal{C}$ is disjoint from $\mathcal{E}_s$.

We rely on the various claims while we describe the construction. While this may seem circular, what really is happening is a grand induction on the stages; we assume all the claims and rules hold at stage $s-1$, use that information to describe what we do at stage s, and then show that the claims and rules hold at the end of stage s as well.

3.2. **Strategies.** Continuing the discussion from above, Suppose for simplicity that $\mathcal{U}^k$ needs to avoid the tests among $\langle \mathcal{V}_n^0 \rangle, \langle \mathcal{V}_n^1 \rangle, \dots, \langle \mathcal{V}_n^k \rangle$. To calculate a bound on the number of versions of $\mathcal{U}^k$, we need to know a bound on how many times various components $\mathcal{V}_n^e$ which may trouble $\mathcal{U}^k$ will move; the bound for $\langle \mathcal{V}_n^e \rangle_{n<\omega}$ is given by g^e. Some of the functions g^e may be partial; we cannot wait forever, so $\mathcal{U}^k$ will guess which of the bounding functions g^e is total. Hence the need for a tree of strategies, and so for a number of versions for each $\mathcal{U}^k$, which will be indexed by strategies on the tree.

We need even more guessing. This has to do with the testing procedure. The testing is done by defining A-partial computable functions Ψ^A, and observing the results which show up in a c.e. trace for the function. The bound on the number of possible errors by some actor is eventually determined by an order function which bounds the trace. As we go down the tree[2], we need more and more tests and so set up more functions Ψ^A and require the traces to be bounded by slower-growing order functions.

There is no reason to believe that there is a uniform procedure which produces, given a partial function Ψ^A and an order function h, an h-trace for Ψ^A. All we can do is uniformly list all h-traces, and then guess which one is the correct trace for Ψ^A.

Now the natural thing to do would be to have three kinds of nodes – for introducing Demuth tests (and guessing whether the next one on the list is total); for providing "boxes" for testing, i.e., introducing A-partial computable functions and bounds, and guessing traces; and for enumerating Γ-computations. It turns out this is not a wise choice. To see why, suppose that σ is a node (a strategy) which introduces a new test $\langle \mathcal{V}^e_n \rangle$. The node σ would have two children, corresponding to the guesses whether g^e is total or not, say $\sigma\hat{}\texttt{fin}$ and $\sigma\hat{}\texttt{tot}$. Suppose that the strategy $\sigma\hat{}\texttt{tot}$ is used for testing, so it defines some Ψ^A and has infinitely many children, each guessing a trace for Ψ^A. Because $\sigma\hat{}\texttt{tot}$ instructs its children to avoid a tail of the test $\langle \mathcal{V}^e_n \rangle$, the number of movements of the sets $\mathcal{U}^\tau$ for children τ of $\sigma\hat{}\texttt{tot}$ is influenced by the number of movements of the test components, i.e., by the function g^e. In turn, the number of movements of the children determines the amount of space which is required for testing on behalf of these children. So a node ρ which defines a function Φ^A which is used for testing for $\sigma\hat{}\texttt{tot}$'s children must have access to the values of g^e, hence must extend σ. On the other hand, in order to find the results of the tests, $\sigma\hat{}\texttt{tot}$ needs to guess a trace for Φ^A, which means that $\sigma\hat{}\texttt{tot}$ must properly extend ρ. The only possibility then is $\rho = \sigma$, but we already have given σ a different task. The reasonable way to deal with this is to give both tasks to the children of σ: to guess whether g^e is total or not, and to guess traces for the function defined by σ.

Hence, we have only one kind of node. A node σ, of length e, will define a p.c. functional Ψ_σ and an order function h_σ. The node will have a child $\sigma\hat{}\texttt{fin}$ which guesses that g^e is partial. If instead g^e is total, then so will be h_σ, and so Ψ^A_σ will have some h_σ-trace. We let $\left\langle T^{\sigma\hat{}d} \right\rangle_{d<\omega}$ be a list of all h_σ-traces[3], and let $\sigma\hat{}d$ be the child of σ which guesses that $T^{\sigma\hat{}d}$ is a trace for Ψ^A_σ. We order these outcomes as $0 <_L 1 <_L 2 <_L \cdots <_L \texttt{fin}$. Finally, we will give σ the task of enumerating axioms into Γ which map clopen sets to strings of length $e-1$ (we later explain why not e).

[2] We like to imagine our tree growing downward, so going down the tree means moving further from the root. As k grows larger, the strategy responsible for building $\mathcal{U}^k$ will be located lower on the tree.

[3] The function h_σ may be partial. This listing means: enumerate nothing into $T^{\sigma\hat{}d}(z)$ until we see that $h_\sigma(z)\downarrow$. Then copy (say) $W^{[z]}_d$, but stop after $h_\sigma(z)$ many elements.

As mentioned, at each stage, each node σ is equipped with a clopen set $\mathcal{U}^\sigma_s$. These sets mirror the tree structure:

Rule 1. If τ extends σ then $\mathcal{U}^\tau_s \subset \mathcal{U}^\sigma_s$, but if σ and τ are incomparable then $\mathcal{U}^\tau_s$ and $\mathcal{U}^\sigma_s$ are disjoint.

The aim is that if σ lies on the true path, then $\mathcal{U}^\sigma_s$ eventually stabilises to a final value $\mathcal{U}^\sigma$. The node σ aims to let Γ map (most of) $\mathcal{U}^\sigma$ to $A\restriction_{|\sigma|-1}$.

Let F be the set of nonempty nodes that do not end with `fin`, that is, nodes of the form $\pi\hat{\ }d$. This is the set of nodes ρ which have a guess for a trace for the A-partial computable function introduced before them, and guess that $g^{|\rho|-1}$ is total; and so, these are the nodes which must avoid another test. Throughout, for a nonzero node τ, we let τ^- be τ's immediate predecessor, technically the string $\tau\restriction_{|\tau|-1}$. For each $\rho \in F$, we let $\langle \mathcal{V}^\rho_n \rangle = \left\langle \mathcal{V}^{|\rho^-|}_n \right\rangle$ be the test which ρ instructs its descendants to avoid, and $g^\rho = g^{|\rho^-|}$ be the bound on the changes of this test.

For every $\rho \in F$, if ever accessible, we will choose some $n_\rho < \omega$. This will be the point from which ρ's descendants have to avoid the test $\langle \mathcal{V}^\rho_n \rangle$. For every node σ, we let

$$F^{\subset\sigma} = \{\rho \in F \,:\, \rho \subset \sigma\}$$

and

$$F^{\subseteq\sigma} = \{\rho \in F \,:\, \rho \subseteq \sigma\}\,;$$

we let

$$\mathcal{W}^\sigma_s = \bigcup \mathcal{V}^\rho_{n,s} \;\; [\![\rho \in F^{\subset\sigma} \;\;\&\;\; n_\rho \leqslant n \leqslant s]\!].$$

This is a clopen set which consists of the measure which is currently covered by those test components which σ has been instructed to avoid by its predecessors. Its "limit" is

$$\mathcal{W}^\sigma = \bigcup \mathcal{V}^\rho_n \;\; [\![\rho \in F^{\subset\sigma} \;\;\&\;\; n_\rho \leqslant n]\!],$$

which is open, but not effectively so. If σ lies on the true path, we need to ensure that $X \notin \mathcal{W}^\sigma$, and of course to make sure that $X \notin \mathcal{E}$, so at the end, we let X be the unique element in the intersection of the sets $\mathcal{U}^\sigma \backslash (\mathcal{W}^\sigma \cup \mathcal{E})$ where σ ranges over nodes on the true path.

Of course, to do this, we need to ensure that for no σ do we get $\mathcal{U}^\sigma \subseteq \mathcal{W}^\sigma \cup \mathcal{E}$. We do this by ensuring that the measure of $(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E}))\,[s]$ is smaller than the measure of $\mathcal{U}^\sigma_s$. We will set a rational number δ_σ (in fact an integer power of 2), and stipulate that:

Rule 2. If σ is accessible at stage s, then the measure of $\mathcal{U}^\sigma_s$ is $4\delta_\sigma$.

and

Rule 3. If σ is accessible at stage s, then the measure of $\mathcal{U}^\sigma_s \cap (\mathcal{W}^\sigma_s \cup \mathcal{E}_s)$ is bounded by δ_σ.

The aim is to ensure that at most a quarter of $\mathcal{U}^\sigma$ is covered by $\mathcal{W}^\sigma \cup \mathcal{E}$. To ensure that Rule 2 holds, whenever σ^- is accessible and thinks that σ may be the next accessible node, if the measure of $\mathcal{U}^\sigma_s$ is smaller than $4\delta_\sigma$ then rather than letting σ be accessible, the parent will try to give $\mathcal{U}^\sigma$ more measure. To ensure that Rule 3 holds, whenever σ^- is accessible and thinks that σ may be the next accessible node, if the measure of $\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})$ $[s]$ has grown beyond δ_σ, σ^- will try to extract most of it from $\mathcal{U}^\sigma_s$ and give replacement measure.

As discussed above, each strategy σ sets up *bins* which are used to accommodate measure rejected by the children of σ. Each bin corresponds to some test component $\mathcal{V}^\rho_n$ which these children are instructed to avoid. A child τ of σ is instructed to avoid the components $\mathcal{V}^\rho_n$ for $\rho \in F^{\subset\tau}$ and $n \geqslant n_\rho$. Noting that $F^{\subset\tau} = F^{\subseteq\sigma}$, the node σ defines bins $\mathcal{B}^\sigma_s(n,\rho) \subset \mathcal{U}^\sigma$ for $\rho \in F^{\subseteq\sigma}$ and $n \geqslant n_\rho$. Again, we keep all "playing grounds" disjoint:

Rule 4. The bins $\mathcal{B}^\sigma_s(n,\rho)$ are pairwise disjoint, and are each disjoint from $\mathcal{U}^\tau_s$ for all children τ of σ, and from $\mathcal{E}_{s-1}$.

Note that in contrast with the description in the previous subsection, measure in the bin comes from a variety of children of σ and so can be exchanged between them. It is not possible for us to limit the amount of measure which was rejected by a single child τ of σ, but for each component $\mathcal{V}^\rho_n$ we can limit the total amount of measure which was rejected, by any child of σ, because of $\mathcal{V}^\rho_n$.

Finally, we recall that the construction involves *actors*. Each $\mathcal{U}^\tau$ is an actor, and there will be actors associated with bins (in fact, actors $\mathcal{B}^\sigma(n,\rho,l)$ for $\rho \in F^{\subseteq\sigma}$, $n \geqslant n_\sigma$ and infinitely many numbers l). Each actor O has a dual existence. At each stage, temporal instantiation of the actor O is a clopen set O_s (so $\mathcal{U}^\tau_s$, and $\mathcal{B}^\sigma_s(n,\rho,l) \subseteq \mathcal{B}^\sigma_s(n,\rho)$). But the ideal form of the actor is an agent in the construction, which has personality just as nodes on the tree of strategies do. It takes responsibility for measure, rejects measure, and so on. While the sets O_s may change from stage to stage, each actor O is constant and indivisible.[4]

While we delay until Section 3.4 the full description of the actors (and the explanation why we don't take a full bin to be an actor, rather than a collection of "sub-bins"), we remark now that each actor O will be associated with a unique node $\sigma = \sigma(O)$ which "owns" the actor. The actors associated with σ are the sub-bins $\mathcal{B}^\sigma(n,\rho,l)$ and the sets $\mathcal{U}^\tau$ for τ a child of σ. Note that $\mathcal{U}^\sigma$ is not associated with σ but with its parent σ^-.

3.3. The construction, skeleton.

We give the construction. This is the correct construction; we are no longer describing failed constructions as we did in Section 3.1. The specifics of some of the steps, however, will be left for the following sections. Thus this section gives the basic skeleton of the construction, and it will have specified spots into which later sections will insert the appropriate details.

For us to be able to carry out stage s of the construction, we need to make an inductive assumption.

Claim 3.2. $\lambda(\mathcal{E}_s) \leqslant 1/4$.

Given the claim holds at stage s, we describe our actions at stage s by recursively defining a path of nodes which are accessible at that stage, and for each accessible node, indicate the node's actions at that stage. The actions are:

- Extending the definition of h_σ;

[4] Of course, we could have formally defined the actor to be some other stationary "object", such as the node τ, or a triple (σ, n, ρ, l), or some such. But this would actually mask the idea behind the actors. Their role is to take responsibility, at each stage, for the measure in their corporeal body O_s.

- Defining $\mathcal{U}^\tau_{s+1}$ for children τ of σ;
- Defining the bins $\mathcal{B}^\sigma_{s+1}(n,\rho)$ for $\rho \in F^{\subseteq\sigma}$ and $n \geqslant n_\rho$;
- Enumerating axioms into Γ; and
- Choosing a child τ to attend to next.

The root $\langle\rangle$ is accessible at every stage, and we always let $\mathcal{U}^{\langle\rangle}_s = 2^\omega$. At stage 0, for all other actors O we let $O_0 = \varnothing$.

Let σ be a node which is accessible at stage s. If $|\sigma| = s$ we halt the stage. Otherwise, the action of σ depends on circumstances, and is done in a number of steps.

First step: Executing a swap. Let $t < s$ be the last stage at which σ was accessible (with $t = 0$ if there was no such previous stage). It is possible that at stage t, the node σ planned to *swap* some measure with one of its children τ. Before making the swap, the pieces involved in the swap needed to be *tested* for the actors involved (see Sections 3.8 and 3.11 for details). If these tests were not *cancelled* between stage t and stage s, then these tests have now returned and they either all *failed* or all *succeeded.*

If the tests were cancelled or if they failed, then the planned swap of measure is abandoned, and we proceed to the next step below. Of course if no tests for σ were started at stage t, we also proceed to the next step.

Otherwise, tests were started, not cancelled, and have now succeeded. We perform the swap, as described in Section 3.8. This gives a new value for $\mathcal{U}^\tau_{s+1}$, as well as for other actors associated with σ. This also influences τ and its descendants, as follows.

Let $\mathcal{D} = \mathcal{U}^\tau_s \backslash \mathcal{U}^\tau_{s+1}$ be the clopen set which is extracted from $\mathcal{U}^\tau$ at stage s. For every actor O which is associated with a node $\pi \supseteq \tau$, we let $O_{s+1} = O_s \backslash \mathcal{D}$. If a test is currently performed for such an actor O, then we cancel that test (the test is cancelled because even if successful, the planned swap the test is running for can not take place anymore.)

Let $\mathcal{C} = \mathcal{U}^\tau_{s+1} \backslash \mathcal{U}^\tau_s$ be the clopen set which was added to $\mathcal{U}^\tau$ at stage s. We enumerate the axiom $\mathcal{C} \mapsto A_s \upharpoonright_{|\sigma|}$ into Γ_{s+1}. We then end the stage.

Second step: extending h_σ. Let $t < s$ be the last σ-*expansionary* stage before stage s (with $t = 0$ if there was no such previous stage). The stage s is σ-expansionary if and only if $\operatorname{dom} g^{|\sigma|}_s > t$, and there is some $d < s$ such that for all $z \in \omega^{[d]} \cap \operatorname{dom} \Psi^A_\sigma\,[s]$ we have $\Psi^A(z) \in T^{\sigma\hat{}d}(z)\ [s]$.[5]

If s is a σ-expansionary stage, then we extend the definition of h_σ, as is described in Section 3.10. In either case, we then move to the next step.

Third step: planning a swap. We choose a child τ of σ to attend to. If s is not σ-expansionary then we choose $\tau = \sigma\hat{}\mathtt{fin}$. If s is σ-expansionary, we find the least d witnessing this fact, and choose $\tau = \sigma\hat{}d$.

If $\sigma = \langle\rangle$ and $\tau = \langle\mathtt{fin}\rangle$, then we end the stage. This is because we required g_0 to be total, so $\langle\mathtt{fin}\rangle$ will not be on the true path. We discuss in Section 3.9 why this matters.

[5]The inputs in $\omega^{[d]}$ will be the ones used by $\sigma\hat{}d$ and some of its extensions for testing. Definitions of $\Psi^A_\sigma(z)$ for $z \in \omega^{[d]}$ will only be made at stages when $\sigma\hat{}d$ is accessible; this will be important for ensuring the existence of the true path (see Section 3.13.)

We consider whether $\mathcal{U}^\tau$ needs to change. If either $\lambda(\mathcal{U}^\tau_s) < 4\delta_\tau$,[6] (recall Rule 2) or $\lambda\left(\mathcal{U}^\tau \cap (\mathcal{W}^\tau \cup \mathcal{E})\right)[s] \geqslant \delta_\tau$ (recall Rule 3), then we select measure from $\mathcal{U}^\tau$ to extract, as specified in Section 3.6, and τ requests to perform a swap with σ, as explained in Section 3.8.

It is important to note that we do not perform the swap immediately (this was alluded to in the first step). Rather, now that the desired measure swap has been determined, each actor which is supposed to take control of a piece of measure tests it. The details of how the testing is performed are provided in Section 3.11. We start the tests and then end the stage.

It is possible, though, that the desired tests cannot be performed, because the required "testing grounds" have not yet been prepared (this is explained in Section 3.6). In that case, we give up: no swap is planned, and we end the stage. (Of course, we will argue that if τ is truly the outcome we want, and the bad measure in $\mathcal{U}^\tau$ persists, then eventually the required testing grounds will be available and we will be able to plan and later execute a required swap.)

On the other hand, if $\sigma = \langle\rangle$ (and $\tau = \langle d\rangle$), then we do not need to perform tests (see Section 3.9). We prepare a swap and execute it immediately (again, the details are found in Section 3.8), and then end the stage.

If $\lambda(\mathcal{U}^\tau_s) = 4\delta_\tau$ and $\lambda\left(\mathcal{U}^\tau \cap (\mathcal{W}^\tau \cup \mathcal{E})\right)[s] < \delta_\tau$ then no change to $\mathcal{U}^\tau_s$ is required. We set $\mathcal{U}^\tau_{s+1} = \mathcal{U}^\tau_s$, and let τ be the next accessible node.

End of Stage. If the value of any set $\mathcal{U}^\tau$ at stage $s+1$ has not been explicitly defined during stage s, then we let $\mathcal{U}^\tau_{s+1} = \mathcal{U}^\tau_s$. If the contents of a sub-bin $\mathcal{B}^\sigma(n,\rho,l)$ at stage $s+1$ have not been explicitly defined at stage s, then we set $\mathcal{B}^\sigma_{s+1}(n,\rho,l) = \mathcal{B}^\sigma_s(n,\rho,l)\backslash\mathcal{E}_s$.

This is the skeleton of the construction. In what follows we flesh out the details. We note a claim which we stated during the construction:

Claim 3.3. If a node σ plans a swap with $\mathcal{U}^\tau$ (for a child τ of σ) at some stage t, and s is the next stage at which σ is accessible, then the tests which we started at stage t for actors owned by σ have either all been cancelled by stage s, or are all successful at stage s, or have all failed by stage s.

It is not that important that the tests all return by stage s; we could have simply instructed σ to wait – but we can arrange for the tests to return at stage s and this makes for a smoother presentation of the construction. However, it is very important that it is not the case that some tests succeed while others fail. Claim 3.3 is proved on page 33.

3.4. **Actors and certainty.** In giving the details, we start by working toward defining the numbers δ_σ and n_ρ. The main driver behind these definitions is the attempt to limit the size of $\mathcal{E}$. We first explain how the bound on $\lambda(\mathcal{E})$ will be achieved. Recall that we decided to employ *actors*, which *claim responsibility* for pieces of measure on which Γ-computations have been defined. Eventually, limiting the size of $\mathcal{E}$ boils down to:

Claim 3.4. Let s be a stage. Every real in $\mathcal{E}_s\backslash\mathcal{E}_{s-1}$ is the responsibility of some actor O at stage s, with $\sigma(O) \neq \langle\rangle$.

[6]This happens if s is the first stage at which we consider τ, and so $\mathcal{U}^\tau_s = \varnothing$; or if at a previous stage an ancestor of τ was involved in a swap and this led to measure being extracted from $\mathcal{U}^\tau$.

Thus, each addition to $\mathcal{E}$ can be *charged* to some actor.

The main concept is the *level of certainty* of a claim. Each actor O (with $\sigma(O) \neq \langle\rangle$) will be assigned a level of certainty $k = k(O)$.

Claim 3.5. For every actor O, there is a set of stages $R(O)$, such that $R(O)$ has size at most $k(O)$, and if O is responsible for a real Y at the stage it enters $\mathcal{E}$, then $Y \in O_s$ for some $s \in R(O)$.

Rule 5. For all s, $\lambda(O_s) \leqslant 2 \cdot 2^{-k(O)}$.

Corollary 3.6. The total amount charged to an actor O is bounded by $2k(O)2^{-k(O)}$.

We indicated earlier what the actors are, and now give all the details and explain. An important point to note is that in order to set up the testing for an actor O, we will need to calculate how many parallel tests we will need to possibly run for O. Roughly, this number of tests is tied to the number of possible changes to the contents of the actor. For $\mathcal{U}^\tau$ we will be able to do this directly, and so we declare each $\mathcal{U}^\tau$ to be an actor, and set:

- $k(\mathcal{U}^\tau) = -\log_2 \delta_\tau - 1$,

motivated by Rules 2 and 5; this will give $\lambda(\mathcal{U}^\tau_s) \leqslant 4\delta_\tau = 2 \cdot 2^{-k(\mathcal{U}^\tau)}$, as desired.

However, even though we will be provided with a number $g^e(n)$ which will bound the number of changes to $\mathcal{V}^\rho_n$, we cannot in advance tell how many times the bin $\mathcal{B}^\sigma(n,\rho)$ will change. The reason is that if a new version of $\mathcal{V}^\rho_n$ appears at some stage s, it may interfere with $\mathcal{U}^{\sigma\hat{\ }d}_s$ for many d, much larger than n (but smaller than s). Each such $\mathcal{U}^{\sigma\hat{\ }d}$ will in turn cause a change in the bin $\mathcal{B}^\sigma(n,\rho)$, when it rejects measure covered by $\mathcal{V}^\rho_n$, and so the number of changes to $\mathcal{B}^\sigma(n,\rho)$ will be related to the number of changes to $\mathcal{U}^{\sigma\hat{\ }d}$, which may be much larger than $g^e(n)$.

To overcome this problem, we split the bin $\mathcal{B}^\sigma(n,\rho)$ into infinitely many pieces. We will set a bound $2^{-k^\sigma(n,\rho)}$ for the possible size of $\mathcal{B}^\sigma(n,\rho)$, and partition the bin $\mathcal{B}^\sigma_s(n,\rho)$ into clopen sets $\mathcal{B}^\sigma_s(n,\rho,l)$, for $l > k^\sigma(n,\rho)$. Each "sub-bin" $\mathcal{B}^\sigma(n,\rho,l)$ is an actor, and we set

- $k(\mathcal{B}^\sigma(n,\rho,l)) = l$.

We note that we will not only allow pieces of measure to be swapped between bins and sets $\mathcal{U}^\tau$, but also pieces of measure will be transferred inside the bin, between the various "sub-bins".

3.5. Defining δ_σ and n_σ.

To limit $\lambda(\mathcal{E})$, we distribute "waste targets" among all actors, ensure that no actor exceeds the bound, and then ensure that the sum of the waste targets is bounded by 1/4. We then find conditions on the numbers δ_σ and n_σ which will ensure that the waste targets are respected.

For every node σ we computably define a rational number ϵ_σ. We aim to ensure that the total amount of bad measure charged to the actor $\mathcal{U}^\tau$ is at most ϵ_τ, and for $\rho \in F$, the total amount of bad measure charged to all actors $\mathcal{B}^\sigma(n,\rho,l)$ (for σ extending ρ, $n \geqslant n_\rho$, and $l > k^\sigma(n,\rho)$) is bounded by ϵ_ρ. Hence, to bound the size of $\mathcal{E}$, we just need to ensure that

$$\sum_\sigma 2\epsilon_\sigma \leqslant \frac{1}{4}.$$

The details of how ϵ_σ is defined are less important. For example, we can let $\epsilon_\sigma = 2^{-(n+5)}$, where σ is the n^{th} node in some effective ω-listing of all nodes.

We first consider an actor $\mathcal{U}^\tau$. By Rule 2, the total amount of measure ever claimed by $\mathcal{U}^\tau$ at any given stage is $4\delta_\tau$. From Corollary 3.6 we conclude that the total contribution of $\mathcal{U}^\tau$ to $\mathcal{E}$ will be bounded by $2\cdot k(\mathcal{U}^\tau)\cdot 2^{-k(\mathcal{U}^\tau)} = 4(-1-\log_2 \delta_\tau)\cdot \delta_\tau$. Some differentiation tells us that $\lim_{x\to 0^+} 4(-1-\log x)x = 0$, and further it approaches 0 monotonically for $x < (2e)^{-1}$. Hence we can (by search if necessary) find a positive rational number $c_\tau < (2e)^{-1}$ such that $4(-1-\log c_\tau)c_\tau < \epsilon_\tau$. If we ensure that **(a):** $\delta_\tau \leqslant c_\tau$, then we will have ensured that the total amount from $\mathcal{E}$ charged to $\mathcal{U}^\tau$ is bounded by ϵ_τ as required.

Next, we consider the bins. For $\rho \in F$, we need to ensure that the total contribution to $\mathcal{E}$ of all actors $\mathcal{B}^\sigma(n,\rho,l)$ for $\sigma \supseteq \rho$, $n \geqslant n_\rho$, and $l > k^\sigma(n,\rho)$ is bounded by ϵ_ρ. From Corollary 3.6 we conclude that the total amount charged to each sub-bin $\mathcal{B}^\sigma(n,\rho,l)$ is bounded by $2l\cdot 2^{-l}$, and so the contribution of a single bin $\mathcal{B}^\sigma(n,\rho)$ is bounded by

$$\sum_{l>k^\sigma(n,\rho)} 2l2^{-l} = 2(k^\sigma(n,\rho)+2)2^{-k^\sigma(n,\rho)}. \tag{3.1}$$

We wish to choose the number $k^\sigma(n,\rho)$, so that $2^{-k^\sigma(n,\rho)}$ bounds the possible size of the bin $\mathcal{B}^\sigma_s(n,\rho)$ at any stage s. That is, for $l = k^\sigma(n,\rho)+1$, the entire contents of $\mathcal{B}^\sigma_s(n,\rho)$ could fit in $\mathcal{B}^\sigma_s(n,\rho,l)$ without violating Rule 5. The following rule gives the necessary bounds on this size:

Rule 6. For $\rho \in F$, $\sigma \supseteq \rho$, $n \geqslant n_\rho$, and $s < \omega$, we have

$$\lambda(\mathcal{B}^\sigma_s(n,\rho)) \leqslant \min\{2^{-2n}, \delta_\sigma\}.$$

The bound $2^{-2n} = \lambda(\mathcal{V}^\rho_n)$ will be obtained by extracting from the bin, whenever new measure is transferred to the bin, measure which $\mathcal{V}^\rho_n$ no longer covers. The second bound, if $\sigma \neq \rho$, follows from the fact that $\mathcal{V}^\rho_n \subseteq \mathcal{W}^\sigma$, and from Rule 3. If $\sigma = \rho$ then the second bound will follow from the choice of n_ρ (item **f** below).

Hence we will set:

- $2^{-k^\sigma(n,\rho)} = \min\left\{2^{-2n}, \delta_\sigma\right\}$.

For $m \geqslant 9$ we have

$$2(m+2)2^{-m} \leqslant 2^{-m/2}.$$

Fixing a bin $\mathcal{B}^\sigma(n,\rho)$, we have $n \geqslant n_\rho$, and so $k^\sigma(n,\rho) \geqslant 2n_\rho$. So if we require that **(b):** $n_\rho \geqslant 5$ for all $\rho \in F$, then $k^\sigma(n,\rho) \geqslant 10$ and by Equation (3.1) we will have ensured that the total amount charged to the sub-bins of $\mathcal{B}^\sigma(n,\rho)$ is bounded by $2^{-k^\sigma(n,\rho)/2}$. The definition of $k^\sigma(n,\rho)$ shows that this charge is bounded by both 2^{-n} and by $\sqrt{\delta_\sigma}$.

We sum the bounds of charges to all bins (as σ and n vary) in two parts: the bins $\mathcal{B}^\sigma(n,\rho)$ for $\sigma = \rho$, and for all proper extensions of ρ. We require that the total charge to each part is bounded by $\epsilon_\rho/2$.

First we consider $\sigma = \rho$. The charge to the bin $\mathcal{B}^\sigma(n,\rho)$ is bounded by 2^{-n}, and the total charge to all such bins (for $n \geqslant n_\rho$) is bounded by $\sum_{n\geqslant n_\rho} 2^{-n} = 2^{-n_\rho+1}$. Then $2^{-n_\rho+1} \leqslant \epsilon_\rho/2$ is guaranteed once we require that **(c):** $n_\rho \geqslant 2-\log\epsilon_\rho$.

Next, we consider the proper extensions of ρ. Fix some $\sigma \supsetneq \rho$. We saw that the total charge to $\mathcal{B}^\sigma(n,\rho)$ is bounded by both 2^{-n} and by $\sqrt{\delta_\sigma}$. For any number $p < \omega$ we can bound the total charge to all the bins $\mathcal{B}^\sigma(n,\rho)$ (for $n \geqslant n_\rho$) by using the bound 2^{-n} for $n > p$ and the bound $\sqrt{\delta_\sigma}$ for $n \in [n_\rho, p]$ and obtain the total $p\sqrt{\delta_\sigma} + 2^{-p}$ (as $n_\rho > 0$). For each $\sigma \supsetneq \rho$ we choose some $p_\sigma = p_\sigma(\rho) < \omega$

so that $\sum_{\sigma \supsetneq \rho} 2^{-p_\sigma} \leqslant \epsilon_\rho/4$. How this is done is unimportant; for example, we can enumerate all the extensions σ as $\langle \sigma_i \rangle$, and let $2^{-p_{\sigma_i}} = 2^{-i-3}\epsilon_\rho$. The total bound will be obtain by arranging that $\sum_{\sigma \supsetneq \rho} p_\sigma \sqrt{\delta_\sigma} \leqslant \epsilon_\rho/4$. This is done by requiring δ_σ to be small. For example, again using the enumeration $\langle \sigma_i \rangle$, we arrange that $p_{\sigma_i}\sqrt{\delta_{\sigma_i}} \leqslant 2^{-i-3}\epsilon_\rho$ by requiring that $\delta_{\sigma_i} \leqslant 2^{-2i-6}\epsilon_\rho^2/(p_{\sigma_i})^2$. So that we can forget the enumeration, we let $\delta_{\sigma_i}(\rho) = 2^{-2i-6}\epsilon_\rho^2/(p_{\sigma_i})^2$. Focussing on σ, the requirement translates to requiring that **(d):** for all $\rho \in F^{\subset \sigma}$, $\delta_\sigma \leqslant \delta_\sigma(\rho)$.

This concludes the calculations giving the bound on the measure of $\mathcal{E}$, but in order for the construction to work, we have some other concerns when choosing our constants δ_σ and n_σ.

First, recall that a node σ may be required to give fresh measure to a child τ. Some replacement measure will come from the bins, but all the measure from the bins may not suffice to make the measure of the new $\mathcal{U}^\tau$ the required $4\delta_\tau$ (Rule 2), and so some extra measure will be provided by σ. To keep Rule 1, this fresh measure will need to be disjoint from $\mathcal{U}^{\tau'}$ for all children τ' of σ. We will also want this extra piece of measure to be disjoint from $\mathcal{E}$ and from $\mathcal{W}^\tau$.[7] By Rule 3, the measure of $\mathcal{W}^\sigma \cup \mathcal{E}$ in $\mathcal{U}^\sigma$ is bounded by δ_σ. If $\sigma \in F$, though, then $\mathcal{W}^\tau = \mathcal{W}^\sigma \cup \bigcup_{n \geqslant n_\sigma} \mathcal{V}_n^\sigma$. To ensure that σ has enough free measure to give $\mathcal{U}^\tau$ we require:

Rule 7. For any node σ and any stage $s < \omega$, the total measure of $\mathcal{U}_s^\tau$, as τ ranges over the children of σ, is bounded by δ_σ.

and

Rule 8. For any node $\sigma \in F$ and any stage $s < \omega$,

$$\lambda\left(\bigcup_{n \geqslant n_\sigma} \mathcal{V}_n^\sigma\right) \leqslant \delta_\sigma.$$

Then, at least δ_σ from the total $4\delta_\sigma$ consisting of $\mathcal{U}^\sigma$ is free to give $\mathcal{U}^\tau$ (some of it may need to be extracted from bins). By Rule 2, Rule 7 is guaranteed by requiring that **(e):** the sum of $4\delta_\tau$, as τ ranges over the children of σ, is bounded by δ_σ. As $\lambda\left(\bigcup_{n \geqslant n_\sigma} \mathcal{V}_n^\sigma\right) \leqslant 2^{-n_\sigma}$, Rule 8 is guaranteed by requiring that **(f):** $2^{-n_\sigma} \leqslant \delta_\sigma$.

Further restrictions on δ_σ and n_σ are required to make the testing possible. As will be described in Section 3.9 and later, a node η will provide space for performing tests on behalf of a variety of actors. This will drive the definition of the function h_η. This function needs to be unbounded, and this requirement translates into a restriction on the number of actors O which depend on η for testing and whose level of certainty $k(O)$ is bounded by a number k.

Consider an actor O which belongs to some node σ extending $\eta\hat{\ }d$ for some $d < \omega$. If O is a sub-bin of some bin $\mathcal{B}^\sigma(n, \rho)$, then the strongest possible certainty required by O is $k^\sigma(n, \rho) + 1$, which is bounded below by both $2n_\rho$ and $1 - \log \delta_\sigma$. If $O = \mathcal{U}^\tau$ for some child τ of σ, then the level of certainty required is $-1 - \log \delta_\tau$, which is greater than $1 - \log \delta_\sigma$. Hence, putting lower bounds on $-\log \delta_\sigma$ and on n_σ (in case $\sigma = \rho$) related to d will bound the number of children $\eta\hat{\ }d$ of η which may make h_η small. The way to do this is to require that **(g):** if $\rho = \rho^-\hat{\ }d$ then $n_\rho \geqslant d$; and that **(h):** if $\sigma = \sigma^-\hat{\ }d$ then $\delta_\sigma \leqslant 2^{-d-3}$.

[7]Although the difference in time between the stage at which the swap is planned and the stage at which the swap is executed may mean that changes in $\mathcal{W}^\tau$ will result in this piece being covered by $\mathcal{W}^\tau$. See Section 3.7 for more details.

Choosing δ_σ and n_σ obeying the restraints **(a)**–**(h)** is now straightforward. We start by letting $\delta_{\langle\rangle} = 1/4$ (here again $\langle\rangle$ denotes the empty string, that is, the strategy at the root of the tree). We then proceed recursively. Given δ_σ, for $\tau = \sigma\hat{}\texttt{fin}$ we let

$$\delta_\tau = \min\left\{c_\tau, 2^{-3}\delta_\sigma, \delta_\tau(\rho) \,:\, \rho \in F^{\subseteq\sigma}\right\},$$

and for $\tau = \sigma\hat{}d$

$$\delta_\tau = \min\left\{c_\tau, 2^{-4-d}\delta_\sigma, \delta_\tau(\rho) \,:\, \rho \in F^{\subseteq\sigma}\right\}.$$

Then, for $\tau = \sigma\hat{}d$, we let

$$n_\tau = \max\left\{5, d, 2 - \log\epsilon_\tau, -\log\delta_\tau\right\}.$$

By choosing c_τ, ϵ_σ and $p_\sigma(\rho)$ (and hence $\delta_\sigma(\rho)$) judiciously, or alternatively shrinking δ_σ, we may assume that each δ_σ is indeed an integer power of 2. Also, if $\pi\hat{}d \subseteq \sigma$ then $\delta_\sigma \leqslant 2^{-d}$.

We note a restriction on the size of $\mathcal{V}^\rho_n$ inside $\mathcal{U}^\sigma$, related to Rule 6.

Claim 3.7. Suppose that σ is accessible at stage s. Let $\rho \in F^{\subseteq\sigma}$ and $n \geqslant n_\rho$. Then

$$\lambda\left(\left(\mathcal{V}^\rho_n \cap \mathcal{U}^\sigma\right)[s]\right) \leqslant 2^{-k^\sigma(n,\rho)}.$$

Proof. The bound $\lambda(\mathcal{V}^\rho_{n,s} \cap \mathcal{U}^\sigma_s) \leqslant 2^{-2n}$ follows, of course, from the fact that $\lambda(\mathcal{V}^\rho_{n,s}) \leqslant 2^{-2n}$. We need to show that $\lambda(\mathcal{V}^\rho_{n,s} \cap \mathcal{U}^\sigma_n) \leqslant \delta_\sigma$.

If $\sigma = \rho$, then this bound follows from the bound $2^{-n_\sigma} \leqslant \delta_\sigma$ which we just arranged (**(f)** above), and again noting that $\lambda(\mathcal{V}^\rho_{n,s} \cap \mathcal{U}^\sigma_n) \leqslant 2^{-2n} \leqslant 2^{-2n_\sigma}$.

If σ is a proper extension of ρ, then $\mathcal{V}^\rho_{n,s} \subseteq \mathcal{W}^\sigma_s$, and the required bound follows from Rule 3. ☐

3.6. **What we actually extract.** Suppose that at some stage t, a node σ is accessible. It has decided that its child τ should be next accessible, but notices that $\lambda\left(\mathcal{U}^\tau \cap (\mathcal{W}^\tau \cup \mathcal{E})[t]\right) \geqslant \delta_\tau$, and so, instead of letting τ be accessible, it plans to exchange measure with $\mathcal{U}^\tau$ so that Rule 3 will hold. We will soon (Section 3.8) give the complete details of how this exchange is planned and executed. But we start by examining the part of $\mathcal{U}^\tau_t$ which $\mathcal{U}^\sigma$ tries to extract from $\mathcal{U}^\tau$ (and then replace).

It would seem reasonable to try to extract all of $(\mathcal{U}^\tau \cap (\mathcal{W}^\tau \cup \mathcal{E}))[t]$. However, trying to do so will run into problems. As we described above, measure extracted from $\mathcal{U}^\tau \cap \mathcal{W}^\tau$ will need to be claimed by various bins, and before they accept the measure, they test it. The "testing ground" is a collection of *boxes* which will be provided by some ancestor η of σ, which we identify later (Section 3.9). Boxes are simply inputs z for Ψ^A_η, and we call $h_\eta(z)$ the *size* of the box (really, this is a bound on the size of the trace $T^{\eta\hat{}d}(z)$). What is important now is that a test for an actor O is carried out on boxes of size $k(O)$; the size of the box (roughly) gives a bound on the number of incorrect successful tests. Because h_η needs to be both computable and monotone, before the test can begin, the node η needs to know *how many* boxes of size $k(O)$ will ever be required by *anyone* appealing to η for boxes of this size, not only O. This information will be calculated by (finitely many) values of functions g^ρ for $\rho \in F^{\subseteq\eta}$, and $g^{|\eta|}$, and so, as indicated during the construction, we may need to give up on tests if the required convergences of g^ρ have not yet occurred.

Now the point is that when we plan an exchange at some stage t, if we want to extract $(\mathcal{U}^\tau \cap \mathcal{V}^\rho_n)[t]$ from $\mathcal{U}^\tau_t$ for *all* $n < t$, the certainty required by the bin would be at least $k^\sigma(n,\rho) \geqslant 2n$, and possibly l much larger than $k^\sigma(n,\rho)$, which will roughly correspond to the amount of measure we plan to move to the bin. That is $l \approx -\log\lambda(\mathcal{V}^\rho_{n,t} \cap \mathcal{U}^\tau_t)$, which can be as large as t. The functions g^ρ might converge slowly, and so at no stage t would we have sufficiently many boxes to test a planned swap.

However, to obey Rule 3, it is sufficient to extract only most of $\mathcal{W}^\tau_t$ from $\mathcal{U}^\tau$, and this can be done while bounding the n's involved. For each node τ we compute a number m_τ such that $|F^{\subset\tau}|\sum_{n>m_\tau} 2^{-2n} < \delta_\tau/4$. The following will suffice:

Definition 3.8. Let $m_\tau = -\log(\delta_\tau/4 \cdot |F^{\subset\tau}|)$.

What we get is that for all s,

$$\lambda\left(\bigcup \mathcal{V}^\rho_{n,s}\,[\![\rho \in F^{\subset\tau}\ \ \&\ \ n > m_\tau]\!]\right) \leqslant \delta_\tau/4.$$

We let

$$\hat{\mathcal{W}}^\tau_s = \bigcup \mathcal{V}^\rho_{n,s}\ [\![\rho \in F^{\subset\tau}\ \ \&\ \ n \in [n_\rho, m_\tau]]\!].$$

It is then sufficient to extract $\hat{\mathcal{W}}^\tau \cup \mathcal{E}$ from $\mathcal{U}^\tau$, rather than $\mathcal{W}^\tau \cup \mathcal{E}$, and this is what we will do.

3.7. **Bounding the number of tests.** Before we can perform tests for an actor O, we need to find a bound on the total number of tests for O which will either succeed or be cancelled. The reason is the following. Say that we test a string $\alpha < A_t$ for some actor O. If $\alpha < A$, then the test will be successful, but this also means that the boxes which were used for the test will not be available for other tests. Similarly, cancelled tests could potentially also test correct initial segments of A, and so the boxes used are also lost. And so, we need to set up in advance sufficiently many boxes, so that we never run out.

We introduce some notation which will be useful now and later.

Suppose that a node σ is accessible at some stage s. We let $s^-(\sigma)$ be the previous stage at which σ was accessible (if there was one; otherwise we let $s^-(\sigma) = 0$). We let $s^+(\sigma)$ be the next stage at which σ will be accessible (if there is one; otherwise we let $s^+(\sigma) = \omega$).

Let O be an actor, associated with a node $\sigma \neq \langle\rangle$. We let $S(O)$ be the set of stages t at which a test is started on behalf of O, and that test returns successfully at stage $t^+(\sigma)$ (in particular, $t^+(\sigma) < \omega$). We let $C(O)$ be the set of stages t at which a test is started on behalf of O, and that test is later cancelled (this cancellation will happen before stage $t^+(\sigma)$).

We wish to effectively find a bound $b(O)$ on $|S(O) \cup C(O)|$. However, calculating $b(O)$ will depend on a variety of (finitely many) values of functions g^ρ. In order to run a test for O, we will have to wait for convergences of functions g^ρ, so that we can find $b(O)$. This is why sometimes we will not be able to test and will immediately abandon an intended swap.

We start with $O = \mathcal{U}^\tau$.

For any node τ with $|\tau| > 1$, let $S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ be the set of stages $t \in S(\mathcal{U}^\tau)$ at which $\lambda(\mathcal{U}^\tau \cap (\mathcal{W}^\tau \cup \mathcal{E})[t]) \geqslant \delta_\tau$. At stages $t \in S_{\mathsf{Rej}}(\mathcal{U}^\tau)$, a test is started because $\mathcal{U}^\tau$ wants to reject some of its measure. For $\tau = \langle d\rangle$, let $S(\mathcal{U}^{\langle d\rangle})$ be the set of stages

t at which $\mathcal{U}^{\langle d\rangle}$ exchanges measure with its parent, the root $\langle\rangle$; let $S_{\mathsf{Rej}}(\mathcal{U}^{\langle d\rangle}) = S(\mathcal{U}^{\langle d\rangle})\backslash\{\min S(\mathcal{U}^{\langle d\rangle})\}$; note that for all $t \in S_{\mathsf{Rej}}(\mathcal{U}^{\langle d\rangle})$, $\lambda(\mathcal{U}^{\langle d\rangle}_t \cap \mathcal{E}_t) \geqslant \delta_{\langle d\rangle}$.

We first bound the size of $S_{\mathsf{Rej}}(\mathcal{U}^\tau)$. In order to do that, we need two more rules.

Rule 9. Replacement measure which at some stage t, a parent τ^- plans to give τ, is disjoint from $\mathcal{E}_t$.

and:

Rule 10. Suppose that at stage t, as part of a planned swap, we intend to add a clopen set $\mathcal{C}$ to a bin $\mathcal{B}^\sigma(n,\rho)$ and to extract a clopen set $\mathcal{D}$ from $\mathcal{B}^\sigma_t(n,\rho)$. Then $\mathcal{C} \subseteq \mathcal{V}^\rho_{n,t}\backslash\mathcal{E}_t$ and $\mathcal{D}$ is contained in $\mathcal{E}_t \cup (\mathcal{U}^\sigma_t\backslash\mathcal{V}^\rho_{n,t})$.

Let τ be any nonzero node.

Let $t \in S_{\mathsf{Rej}}(\mathcal{U}^\tau)$. At stage $s = t^+(\tau^-)$, $\mathcal{U}^\tau_t \cap \mathcal{E}_t$ is extracted from $\mathcal{U}^\tau_s$, and by Rule 9, the new measure given to $\mathcal{U}^\tau_{s+1}$ is disjoint from $\mathcal{E}_t$.[8] By induction on stages $v \geqslant s+1$, using Rule 9, and the fact that $\langle\mathcal{E}_v\rangle$ is increasing, we see that $\mathcal{U}^\tau_v$ is disjoint from $\mathcal{E}_t$. This shows that there can be at most $4/\delta_\tau$ many stages $t \in S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ with $\lambda(\mathcal{U}^\tau_t \cap \mathcal{E}_t) \geqslant \delta_\tau/4$.

If $t \in S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ and $\lambda(\mathcal{U}^\tau_t \cap \mathcal{E}_t) < \delta_\tau/4$, then $|\tau| > 1$ and $\lambda(\mathcal{U}^\tau_t \cap \mathcal{W}^\tau_t) > 3\delta_\tau/4$. By the definition of m_τ, we see that $\lambda(\mathcal{U}^\tau_t \cap \hat{\mathcal{W}}^\tau_t) > \delta_\tau/2$.

Claim 3.9. Suppose that I is an interval of stages during which $\mathcal{V}^\rho_{n,-}$ is constant for all $\rho \in F^{\subset\tau}$ and $n \in [n_\rho, m_\tau]$, and that no new measure is added to $\mathcal{U}^{\tau^-}$ during stages in I. Then there are at most $2/\delta_\tau$ many stages $t \in I \cap S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ at which $\lambda(\mathcal{U}^\tau_t \cap \hat{\mathcal{W}}^\tau_t) > \delta_\tau/2$.

Proof. Let $\mathcal{V}^\rho_{n,I} = \mathcal{V}^\rho_{n,t}$ for $t \in I$. For brevity, for $t \in I$ let, in this proof, $\mathcal{D}_t = \mathcal{U}^\tau_t \cap \hat{\mathcal{W}}^\tau_t$.

Suppose that $t < t'$ are stages in $I \cap S_{\mathsf{Rej}}(\mathcal{U}^\tau)$. Let $s = t^+(\tau^-)$. By Claim 3.3, we have $s < t'$. At stage s, $\mathcal{D}_t$ is extracted from $\mathcal{U}^\tau$, and $\mathcal{D}_t\backslash\mathcal{E}_s$ is distributed to bins $\mathcal{B}^\rho$ (again, $\rho \in F^{\subset\tau}$ and $n \in [n_\rho, m_\tau]$). Each piece added to a bin $\mathcal{B}^\sigma(n,\rho)$ is contained in $\mathcal{V}^\rho_{n,t} = \mathcal{V}^\rho_{n,I}$, and so by Rule 10, parts of it are only extracted from the bin if they enter $\mathcal{E}$ or are extracted from $\mathcal{U}^{\tau^-}$. In either case, such measure cannot be returned to $\mathcal{U}^\tau$ during I. Hence, $\mathcal{D}_t$ and $\mathcal{D}_{t'}$ are disjoint. The Claim follows. ☐

It follows that the number of possible stages t in $S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ with $\lambda(\mathcal{U}^\tau_t \cap \hat{\mathcal{W}}^\tau_t) > \delta_\tau/2$ is bounded by $2/\delta_\tau$ multiplied by the number of possible versions of $\langle\mathcal{V}^\rho_n\rangle$ (for the appropriate pairs (n,ρ)), multiplied by the number of times measure is added to $\mathcal{U}^{\tau^-}$. The number of versions of $\langle\mathcal{V}^\rho_n\rangle$ is bounded by the sum of the numbers $g^\rho(n)$ for the appropriate pairs (n,ρ), for which $g^\rho(n)\downarrow$. It will actually turn out that:

Rule 11. Suppose that at some stage t, a node τ^- starts a test for $\mathcal{U}^\tau$. Then for all $\rho \in F^{\subset\tau}$, for all $n \in [n_\rho, m_\tau]$, we have $g^\rho_t(n)\downarrow$.

(Otherwise, we will not have enough information to set up boxes for the testing to begin.) Note that the rule holds vacuously for $|\tau| = 1$.

Overall, we see that the number of stages $t \in S_{\mathsf{Rej}}(\mathcal{U}^\tau)$ with $\lambda(\mathcal{U}^\tau_t \cap \mathcal{E}_t) \leqslant \delta/4$ is bounded by

$$|S(\mathcal{U}^{\tau^-})| \cdot 2/\delta_\tau \cdot \sum g^\rho(n)\, [\![\rho \in F^{\subset\tau} \ \ \&\ \ n \in [n_\rho, m_\tau]]\!].$$

[8] In fact, it will also be disjoint from $\mathcal{E}_s$, but this will require proof (and ensure that Claim 3.1 holds).

Next, we consider other stages in $S(\mathcal{U}^\tau) \cup C(\mathcal{U}^\tau)$. Let $t \in C(\mathcal{U}^\tau)$. Then between stage t and $t^+(\tau^-)$, some node $\pi \precneqq \tau$ exchanges measure with its parent, and by considering accessibility, we see that this exchange was planned at some stage between t and $t^+(\tau^-)$ as well; that stage is in $S(\mathcal{U}^\pi)$. Hence, $|C^\tau| \leqslant \sum_{\pi \precneqq \tau} |S(\mathcal{U}^\pi)|$.

Suppose that $t \in S(\mathcal{U}^\tau) \backslash S_{\mathsf{Rej}}(\mathcal{U}^\tau)$. For example, $\min S(\mathcal{U}^\tau)$ is such a stage. If $t \neq \min S(\mathcal{U}^\tau)$, let t' be the last stage in $S(\mathcal{U}^\tau)$ before stage t. At stage $v = (t')^+(\tau^-)+1$ we have $\lambda(\mathcal{U}^\tau_v) = 4\delta_\tau$, and $v \leqslant t$. In fact, $v < t$ because $\lambda(\mathcal{U}^\tau_t) < 4\delta_\tau$. Between stages v and t, measure was extracted from $\mathcal{U}^\tau$, and this happened because of a swap of measure out of $\mathcal{U}^\pi$ for some $\pi \precneqq \tau$. That swap was also planned at some stage between v and t. So the interval (t', t) contains a stage in $S(\mathcal{U}^\pi)$, which shows that $|S(\mathcal{U}^\tau) \backslash S_{\mathsf{Rej}}(\mathcal{U}^\tau)| \leqslant 1 + \sum_{\pi \precneqq \tau} |S(\mathcal{U}^\pi)|$.

We thus define $\tau \mapsto b(\mathcal{U}^\tau)$ by recursion on τ. For $\tau = \langle d \rangle$ we let $b(\mathcal{U}^{\langle d \rangle}) = 1 + 4/\delta_{\langle d \rangle}$. Suppose that $b(\mathcal{U}^\tau)$ has been defined for every nonzero $\pi \precneqq \tau$, and that $g^\rho(n)\downarrow$ for all $\rho \in F^{\subset \tau}$ and $n \in [n_\rho, m_\tau]$. Then we let

$$b(\mathcal{U}^\tau) = 4/\delta_\tau + b(\mathcal{U}^{\tau^-}) \cdot 2/\delta_\tau \cdot \sum g^\rho(n) \, [\![\rho \in F^{\subset \tau} \;\; \& \;\; n_\rho \leqslant n \leqslant m_\tau]\!] \; + \; 1 + 2 \sum_{\pi \precneqq \tau} b(\mathcal{U}^\pi).$$

As mentioned above, this function is partial computable. The calculations and Rule 11 ensure:

Claim 3.10. If a test is started for $\mathcal{U}^\tau$ at some stage t, then $b_t(\mathcal{U}^\tau)\downarrow$. For such τ we have $|S(\mathcal{U}^\tau) \cup C(\mathcal{U}^\tau)| < b(\mathcal{U}^\tau)$.

The bound $b(\mathcal{U}^\tau)$ is not the sharpest (we just picked the shortest formula to define it). However, it turns out that with the definition that we gave, the number $b(\mathcal{U}^\tau)$ bounds the number of stages s such that $\mathcal{U}^\tau_{s+1} \neq \mathcal{U}^\tau_s$. We will not really make use of this fact.

Before we go on to the sub-bins, we stop to discuss an aspect of the construction which is somewhat counter-intuitive. The reason for swapping measure is to "clean the system": extract bad measure from $\mathcal{U}^\tau$, and, it would seem reasonable to assume, *replace it by good measure*. The construction does not quite work that way. We already mentioned (footnote in page 22) that the time elapsed between the stage t at which a swap is planned and the stage s at which it is executed could make the swap useless; $\hat{\mathcal{W}}^\tau$ would have shifted between stages t and s, possibly causing extraction of measure which has become good and giving $\mathcal{U}^\tau$ measure which has become bad. We could try to review the situation before making the swap and cancelling the swap if necessary, but it turns out this is unnecessary. We can make a lousy swap, and then at the next stage, $\mathcal{U}^\tau$ will complain again, and require a new swap. The fact that $\hat{\mathcal{W}}^\tau$ eventually stabilises means that eventually, good swaps do not become bad when executed; and this is good enough for us.

However, even this is not the whole picture. It may be that sometimes we *plan a bad swap*. Consider, for simplicity, two test components $\mathcal{V}^\rho_n$ and $\mathcal{V}^{\rho'}_{n'}$. At some point $\mathcal{V}^\rho_n$ covers much of $\mathcal{U}^\tau$, and we plan to give $\mathcal{U}^\tau_t \cap \mathcal{V}^\rho_{n,t}$ to the bin $\mathcal{B}^{\tau^-}(n, \rho)$. In return, the bin $\mathcal{B}^{\tau^-}(n, \rho)$ plans to give some measure $\mathcal{C}$ to $\mathcal{U}^\tau$. Rule 10 says that $\mathcal{C}$ must be disjoint from $\mathcal{V}^\rho_{n,t}$, but it could be that $\mathcal{C}$ is covered by $\mathcal{V}^{\rho'}_{n',t}$. The reason for $\mathcal{C}$ being in the bin $\mathcal{B}^{\tau^-}(n, \rho)$ in the first place was that in some prehistoric time, it was covered by $\mathcal{V}^\rho_n$; in the meantime, $\mathcal{V}^\rho_n$ moved away from $\mathcal{C}$, but $\mathcal{C}$ got covered by $\mathcal{V}^{\rho'}_{n'}$.

This seems silly, and indeed we could do it otherwise: instead of giving $\mathcal{C}$ to $\mathcal{U}^\tau$, the bin $\mathcal{B}(n,\rho)$ would give $\mathcal{C}$ directly to the bin $\mathcal{B}(n',\rho')$, and as part of a triple exchange, the bin $\mathcal{B}(n',\rho')$ will give some other measure to $\mathcal{U}^\tau$. Of course in turn this new measure could be covered by some other test, so we might need to plan some grand swap involving many bins. This is possible, but is unnecessarily complicated. We simply give the bad measure $\mathcal{C}$ to $\mathcal{U}^\tau$, and $\mathcal{U}^\tau$ can later complain and reject $\mathcal{C}$ (and it would pass to $\mathcal{B}(n',\rho')$). Again, the fact that $\hat{\mathcal{W}}^\tau$ stabilises will show that this does not go on forever, and that eventually, $\mathcal{U}^\tau$ will get good measure.

We now turn to defining $b(O)$ for $O = \mathcal{B}^\sigma(n,\rho,l)$. To bound the number of tests required by $\mathcal{B}^\sigma(n,\rho,l)$, we need new information on its action. The only purpose of the following rule is, in fact, to bound the number of tests required by the sub-bin.

Rule 12. Suppose that at some stage we plan to add a clopen set $\mathcal{C}$ to $\mathcal{B}^\sigma(n,\rho,l)$ (and test $\mathcal{C}$ for O). Then $\lambda(\mathcal{C}) \geqslant 2^{-l}$.

The following claim will be proved on page 30.

Claim 3.11. Let I be an interval of stages such that $\mathcal{V}^\rho_{n,s}$ is constant as s ranges over I, and during which no new measure is added to $\mathcal{U}^\sigma$. Suppose that $t_0 < t_1$ are stages in I at which we start testing pieces of measure ($\mathcal{C}_0$ and $\mathcal{C}_1$, respectively) for $\mathcal{B}^\sigma(n,\rho,l)$. Suppose further that $t_0 \in S(\mathcal{B}^\sigma(n,\rho,l))$, i.e. the test for $\mathcal{C}_0$ which began at stage t_0 is successful. Then $\mathcal{C}_1$ is disjoint from $\mathcal{C}_0$.

So we let $b(\mathcal{B}^\sigma(n,\rho,l)) = b(\mathcal{U}^\sigma)\cdot 2^l g^\rho(n) + 1$. Similarly to Rule 11, we have:

Rule 13. Suppose that at some stage t, we start a test for $\mathcal{B}^\sigma(n,\rho,l)$. Then $g^\rho_t(n)\downarrow$.

Therefore:

Claim 3.12. If a test is started for $\mathcal{B}^\sigma(n,\rho,l)$ at some stage t, then $b_t(\mathcal{B}^\sigma(n,\rho,l))\downarrow$. For such a sub-bin we have $|S(\mathcal{B}^\sigma(n,\rho,l)) \cup C(\mathcal{B}^\sigma(n,\rho,l))| < b(\mathcal{B}^\sigma(n,\rho,l))$.

Note that in all of the calculations in this section, we used the fact that if a test for an actor O returns successfully, then the planned swap is executed. This relies on the important part of Claim 3.3.

3.8. **Planning and executing a swap.** We now describe how to do a swap. Recall that at some stage t, an accessible node σ decides to pay attention to a child τ, and τ requests new measure. Rules 1 and 9 say that any new measure which we plan to give $\mathcal{U}^\tau$ must be disjoint from any $\mathcal{U}^{\tau'}_t$ for any child τ' of σ and from $\mathcal{E}_t$. We also want any extra measure given to $\mathcal{U}^\tau$ directly from $\mathcal{U}^\sigma$ (not as part of an exchange with a bin) to be disjoint from $\mathcal{W}^\tau$ and from bins $\mathcal{B}^\sigma_t(n,\rho)$ for $n > m_\tau$. We first argue that this is possible.

Claim 3.13. Suppose that σ is accessible at stage t. For any child τ of σ,

$$\lambda\left(\mathcal{U}^\sigma \cap \left(\mathcal{W}^\tau \cup \mathcal{E} \cup \bigcup_{\tau' \text{ a child of } \sigma} \mathcal{U}^{\tau'} \cup \bigcup_{\rho\in F^{\subseteq\sigma} \;\&\; n>m_\tau} \mathcal{B}^\sigma(n,\rho)\right)[t]\right) \leqslant 3.5\cdot\delta_\sigma.$$

Thus, at least $\delta_\sigma/2$ of $\mathcal{U}^\sigma_t$ is good measure, and we recall that $4\delta_\tau \leqslant \delta_\sigma/2$.

Proof. By Rule 7, $\lambda\left(\bigcup_{\tau'}\mathcal{U}^{\tau'}_t\right) \leqslant \delta_\sigma$. If $\sigma \neq \langle\rangle$, then because σ is accessible at stage t, we know that $\lambda\left(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})[t]\right) \leqslant \delta_\sigma$ (otherwise it would ask its parent for more measure and not be accessible). For $\sigma = \langle\rangle$ we apply Claim 3.2 and note that $\delta_{\langle\rangle} = 1/4$ and that $\mathcal{W}^{\langle\rangle} = \varnothing$. If $\sigma \notin F$ then $\mathcal{W}^\sigma_t = \mathcal{W}^\tau_t$. Otherwise, $\mathcal{W}^\tau_t = \mathcal{W}^\sigma_t \cup \bigcup_{n\in[n_\sigma,t)} \mathcal{V}^\sigma_{n,t}$. By Rule 8, $\lambda(\bigcup_{n\in[n_\sigma,t)} \mathcal{V}^\sigma_{n,t}) \leqslant \delta_\sigma$. Finally, by Rule 6 we know that $\lambda\left(\bigcup_{\rho\in F^{\subseteq\sigma}\ \&\ n>m_\tau} \mathcal{B}^\sigma_t(n,\rho)\right) < \delta_\tau/4 < \delta_\sigma/2$. □

If nonempty, the node τ wants to return $\mathcal{U}^\tau \cap (\hat{\mathcal{W}}^\tau \cup \mathcal{E})[t]$ to $\mathcal{U}^\sigma$. It then needs new measure, so that in the end it has $4\delta_\tau$ in total. Rejected measure in $\mathcal{E}_t$ does not need to be tested, but measure in $\mathcal{U}^\tau \cap (\hat{\mathcal{W}}^\tau \backslash \mathcal{E})[t]$ needs to be distributed among bins $\mathcal{B}^\sigma(n,\rho)$. In exchanging with the bins, we need to follow Rules 10 and 6.

We plan to exchange measure between the bins and $\mathcal{U}^\tau$, and from $\mathcal{U}^\sigma$ and $\mathcal{U}^\tau$. Thus, we are seeking clopen sets:

- $\mathcal{C}(n,\rho) = \mathcal{C}_t(n,\rho)$ – measure passed from $\mathcal{U}^\tau$ to $\mathcal{B}^\sigma(n,\rho)$;
- $\mathcal{Y}(n,\rho) = \mathcal{Y}_t(n,\rho)$ – measure passed from $\mathcal{B}^\sigma(n,\rho)$ to $\mathcal{U}^\tau$;
- $\mathcal{Y}^* = \mathcal{Y}^*_t$ – measure passed directly from $\mathcal{U}^\sigma$ to $\mathcal{U}^\tau$.

Here and below, (n,ρ) and (n',ρ') range over $\rho \in F^{\subseteq\sigma}$ and $n \in [n_\rho, m_\tau]$. We make the following requirements of these sets:

- $\{\mathcal{C}(n,\rho)\}$ is a partition of $\mathcal{C} = (\mathcal{U}^\tau \cap (\hat{\mathcal{W}}^\tau \backslash \mathcal{E}))[t]$, and for all pairs (n,ρ), $\mathcal{C}(n,\rho) \subseteq \mathcal{V}^\rho_{n,t}$.
- $\mathcal{Y}(n,\rho) \subseteq \mathcal{B}^\sigma_t(n,\rho)$ and is disjoint from $\mathcal{V}^\rho_{n,t}$.
- $\mathcal{Y}^*$ is disjoint from each $\mathcal{Y}(n,\rho)$, from each $\mathcal{U}^{\tau'}_t$, from $\mathcal{W}^\tau_t$, and from bins $\mathcal{B}^\sigma(n,\rho)$ for $n > m_\tau$.
- All pieces mentioned are disjoint from $\mathcal{E}_t$.

We also need to make sure that the bins do not get too full and that the correct amount of measure is given to $\mathcal{U}^\tau$. Recall that $2^{-k^\sigma(n,\rho)} = \min\{2^{-2n}, \delta_\sigma\}$. Let

$$\mathcal{Y} = \mathcal{Y}^* \cup \bigcup_{n,\rho} \mathcal{Y}(n,\rho)$$

be the set we intend to give to $\mathcal{U}^\tau$; let

$$\mathcal{D}(n,\rho) = \mathcal{Y}(n,\rho) \cup (\mathcal{B}^\sigma_t(n,\rho) \cap (\mathcal{Y}^* \cup \mathcal{E}_t))$$

be the set which we plan to extract from the bin $\mathcal{B}^\sigma_t(n,\rho)$, and note that $\mathcal{C}(n,\rho)$ is the set which we intend to give the bin $\mathcal{B}^\sigma(n,\rho)$. We require:

- $\lambda(\mathcal{U}^\tau_t) + \lambda(\mathcal{Y}) - \lambda(\mathcal{C}) - \lambda(\mathcal{U}^\tau_t \cap \mathcal{E}_t) = 4\delta_\tau$; and
- for each pair (n,ρ), $\lambda(\mathcal{B}^\sigma_t(n,\rho)) + \lambda(\mathcal{C}(n,\rho)) - \lambda(\mathcal{D}(n,\rho))$ is bounded by $2^{-k^\sigma(n,\rho)}$.

We order all the pairs (n,ρ) (with $n \geqslant n_\rho$) by an ordering $<_\sigma$ of order-type ω; we arrange $(\rho,n) <_\sigma (\rho', n+1)$ and so the collection of pairs (n,ρ) with $n \leqslant m_\tau$ is an initial segment of $<_\sigma$. We carve out maximal pieces from $\mathcal{C}$ in order. That is, if $\mathcal{C}(n',\rho')$ has been defined for all $(n',\rho') <_\sigma (n,\rho)$, then we let

$$\mathcal{C}(n,\rho) = (\mathcal{C} \cap \mathcal{V}^\rho_{n,t}) \backslash \bigcup_{(n',\rho') <_\tau (n,\rho)} \mathcal{C}(n',\rho').$$

Now, for each pair (n,ρ) (again $n \leqslant m_\tau$), we find some clopen

$$\mathcal{Y}(n,\rho) \subseteq (\mathcal{B}^\sigma_t(n,\rho) \backslash (\mathcal{E}_t \cup \mathcal{V}^\rho_{n,t}))$$

with $\lambda(\mathcal{Y}(n,\rho)) \leqslant \lambda(\mathcal{C}(n,\rho))$ and

$$\lambda(\mathcal{B}^\sigma_t(n,\rho)) + \lambda(\mathcal{C}(n,\rho)) - \lambda(\mathcal{Y}(n,\rho)) - \lambda(\mathcal{B}^\sigma_t(n,\rho) \cap \mathcal{E}_t) \leqslant 2^{-k^\sigma(n,\rho)}.$$

This is possible because $\mathcal{C}(n,\rho) \subseteq \mathcal{V}^{\rho}_{n,t}$, $\lambda(\mathcal{B}^{\sigma}_t(n,\rho)) \leqslant 2^{-k^{\sigma}(n,\rho)}$ (Rule 6) and $\lambda(\mathcal{U}^{\sigma}_t \cap \mathcal{V}^{\rho}_{n,t}) \leqslant 2^{-k^{\sigma}(n,\rho)}$ (Claim 3.7). Since $\mathcal{Y}(n,\rho) \cup (\mathcal{B}^{\sigma}_t(n,\rho) \cap \mathcal{E}_t) \subseteq \mathcal{D}(n,\rho)$, it will follow that

$$\lambda(\mathcal{B}^{\sigma}_t(n,\rho)) + \lambda(\mathcal{C}(n,\rho)) - \lambda(\mathcal{D}(n,\rho)) \leqslant 2^{-k^{\sigma}(n,\rho)}.$$

So far, we plan to give $\bigcup_{(n,\rho)} \mathcal{Y}(n,\rho)$ to $\mathcal{U}^{\tau}$ and extract $\mathcal{C}$ and $\mathcal{U}^{\tau}_t \cap \mathcal{E}_t$ from it. Let

$$q = \lambda(\mathcal{U}^{\tau}_t) - \lambda(\mathcal{C}) - \lambda(\mathcal{U}^{\tau}_t \cap \mathcal{E}_t) + \sum_{(n,\rho)} \lambda(\mathcal{Y}(n,\rho)).$$

Because we picked $\lambda(\mathcal{Y}(n,\rho)) \leqslant \lambda(\mathcal{C}(n,\rho))$, we see that $q \leqslant 4\delta_{\tau}$. So we need to find $\mathcal{Y}^*$ as above, disjoint from each $\mathcal{Y}(n,\rho)$, of measure $4\delta_{\tau} - q$. This is possible by Claim 3.13; note that $\sum \lambda(\mathcal{Y}(n,\rho)) \leqslant q$. Note that $\mathcal{B}^{\sigma}_t(n,\rho) \cap \mathcal{Y}^*$ is disjoint from $\hat{\mathcal{W}}^{\tau}_t \supseteq \mathcal{V}^{\rho}_{n,t}$ and so $\mathcal{D}(n,\rho)$ satisfies Rule 10.

We have now decided which sets are intended for which actors. We start a test on $\mathcal{Y}$ for $\mathcal{U}^{\tau}$. If $\sigma = \langle\rangle$ (and $\tau = \langle d \rangle$) then there will be no bins ($\mathcal{C} = \varnothing$) and actually no need to test $\mathcal{Y}$; we then immediately let $\mathcal{U}^{\tau}_{t+1} = (\mathcal{U}^{\tau}_t \backslash \mathcal{E}_t) \cup \mathcal{Y}$.

For each pair (n,ρ), we need to start a test on some clopen set (including $\mathcal{C}(n,\rho)$) for some actor $\mathcal{B}^{\sigma}(n,\rho,l)$. Recall that we have to obey Rules 5 and 12: the set we add to the actor has to have measure at least 2^{-l}, but the total resulting measure claimed by the actor has to be bounded by twice that amount. Also, in light of the planned Claim 3.11, the new piece needs to be new to the actor (since the last move of $\mathcal{V}^{\rho}_n$). For $l \geqslant k^{\sigma}(n,\rho)$, let $\mathcal{B}'(n,\rho,l) = \mathcal{B}^{\sigma}_t(n,\rho,l) \backslash \mathcal{D}(n,\rho)$. We find some $k > k^{\sigma}(n,\rho)$ such that:

- $\lambda\left(\mathcal{C}(n,\rho) \cup \bigcup_{l \geqslant k} \mathcal{B}'(n,\rho,l)\right) \leqslant 2 \cdot 2^{-k}$; and
- $\lambda\left(\mathcal{C}(n,\rho) \cup \bigcup_{l > k} \mathcal{B}'(n,\rho,l)\right) \geqslant 2^{-k}$.

Claim 3.14. Such k exists.

Proof. Note that the sets $\mathcal{B}'(n,\rho,l)$ are pairwise disjoint, and they are all disjoint from $\mathcal{C}(n,\rho)$. Let k be the least $k > k^{\sigma}(n,\rho)$ such that

$$(*) \qquad \lambda(\mathcal{C}(n,\rho)) + \sum_{l > k} \lambda(\mathcal{B}'(n,\rho,l)) \geqslant 2^{-k}.$$

Such k exists; for example, any k with $2^{-k} \leqslant \lambda(\mathcal{C}(n,\rho))$. If $k > k^{\sigma}(n,\rho) + 1$, then the failure of $(*)$ for $k-1$ shows that k is as required. If $k = k^{\sigma}(n,\rho) + 1$, then the suitability of k follows from

$$\lambda(\mathcal{C}(n,\rho)) + \sum_{l > k^{\sigma}(n,\rho)} \lambda(\mathcal{B}'(n,\rho,l)) = \lambda\left((\mathcal{C}(n,\rho) \cup \mathcal{B}^{\sigma}_t(n,\rho)) \backslash \mathcal{D}(n,\rho)\right) \leqslant 2^{-k^{\sigma}(n,\rho)}$$

which was our design. □

For later application, we note that the k we chose is bounded by m such that $2^{-m} \leqslant \lambda(C) < 2 \cdot 2^{-m}$. I.e.:

Porism 3.15. $k \leqslant 1 - \log \lambda(\mathcal{C})$.

Having chosen a suitable k, we test $\mathcal{C}(n,\rho) \cup \bigcup_{l > k} \mathcal{B}'(n,\rho,l)$ for $\mathcal{B}^{\sigma}(n,\rho,k)$. We halt the narrative to note that all clopen sets which are tested are disjoint from $\mathcal{E}_t$.

If the tests are successful then when we perform the swap, at some stage s, we will set:

- $\mathcal{U}^{\tau}_{s+1} = (\mathcal{U}^{\tau}_t \backslash (\hat{\mathcal{W}}^{\tau}_t \cup \mathcal{E}_t)) \cup \mathcal{Y}$; and

- for each n, ρ, with k chosen as above, we let, for $l > k^\sigma(n,\rho)$,

$$\mathcal{B}^\sigma_{s+1}(n,\rho,l) = \begin{cases} \mathcal{B}^\sigma_t(n,\rho,l)\backslash\mathcal{D}(n,\rho), & \text{for } l < k; \\ \left(\bigcup_{l \geqslant k} \mathcal{B}^\sigma_t(n,\rho,l)\backslash\mathcal{D}(n,\rho)\right) \cup \mathcal{C}(n,\rho), & \text{for } l = k; \\ \varnothing, & \text{for } l > k. \end{cases}$$

Thus, the sub-bins for $l > k$ have been moved to the k^{th} sub-bin. Note that this is one-directional: from larger to smaller l. This allows us to prove Claim 3.11.

Proof of Claim 3.11. Let I be an interval of stages such that $\mathcal{V}^\rho_{n,s}$ is constant as s ranges over I, and during which $\mathcal{U}^\sigma$ receives no new measure. Let $t_0 \in I \cap S(\mathcal{B}^\sigma(n,\rho,l))$, and let $\mathcal{C}_0$ be the set tested for the sub-bin at stage t_0; let $s_0 = t_0^+(\sigma)$ be the stage at which $\mathcal{C}_0$ is added to the sub-bin.

We follow the fate of pieces of $\mathcal{C}_0$ along I after stage s_0. It is possible that parts of $\mathcal{C}_0$ are passed to sub-bins $\mathcal{B}^\sigma(n,\rho,l')$ for $l' < l$. From these sub-bins they will never be directly returned to $\mathcal{B}^\sigma(n,\rho,l)$. It is also possible that pieces of $\mathcal{C}_0$ are removed from $\mathcal{U}^\sigma$; but during I, they are not returned to $\mathcal{U}^\sigma$. Otherwise, if a part $\mathcal{C}'$ of $\mathcal{C}_0$ leaves the bin at any stage $s > s_0$ in I, then it must be that $\mathcal{C}'\backslash\mathcal{E}_s$ is disjoint from $\mathcal{V}^\rho_{n,s}$. No part of $\mathcal{E}_s$ is ever returned to the bin, and as long as $\mathcal{V}^\rho_n$ does not change, the rest of $\mathcal{C}_0$ remains disjoint from $\mathcal{V}^\rho_n$ and so none of it is moved back to the bin. Thus, no part of $\mathcal{C}_0$ can be offered to $\mathcal{B}^\sigma(n,\rho,l)$ at any stage $t > s_0$ in I. □

We observe that Rule 6 is observed in the construction. When we planned the swap, we were careful to extract unnecessary measure so that the measure of the resulting bin does not exceed $2^{-k^\sigma(n,\rho)}$. At other stages, if a swap is not performed, the only change to the bin could be the removal of measure due to a swap performed by an ancestor of σ.

Later, we will need:

Claim 3.16. Let σ be any node. For every $\rho \in F^{\subseteq\sigma}$ and $n \geqslant n_\rho$, there is some stage after which no new measure is ever added to the bin $\mathcal{B}^\sigma(n,\rho)$.

Proof. This is proved by induction on $<_\sigma$. Fix some pair (n,ρ).

Let s_0 be a stage such that the component $\mathcal{V}^\rho_{n,s}$ is constant for $s \geqslant s_0$, and no bin $\mathcal{B}^\sigma(n',\rho')$ for $(n',\rho') <_\sigma (n,\rho)$ ever receives new measure. We may assume that bins $\mathcal{B}^\sigma(n,\rho)$ with $n \geqslant s_0$ are empty at stage s_0; and that for all (n,ρ), the sub-bins $\mathcal{B}^\sigma(n,\rho,l)$ for $l \geqslant s_0$ are empty at stage s_0.

Now we note that if any $\mathcal{U}^\tau$ plans to extract any piece of $\mathcal{V}^\rho_{n,t}\backslash\mathcal{E}_t$, then our partition algorithm shows that this piece will be distributed among bins $\mathcal{B}^\sigma(n',\rho')$ with $(n',\rho') \leqslant_\sigma (n,\rho)$. So if this happens after stage s_0, the measure will be given to $\mathcal{B}^\sigma(n,\rho)$. By induction, at every stage $t \geqslant s_0$, every bin $\mathcal{B}^\sigma_t(n',\rho')$ with $n' \geqslant s_0$ is disjoint from $\mathcal{V}^\rho_n$, and every sub-bin $\mathcal{B}^\sigma(n',\rho',l)$ with $l \geqslant s_0$ and $(n',\rho') \neq (n,\rho)$ is also disjoint from $\mathcal{V}^\rho_n$.

Let $s_1 > s_0$ be a stage after which no sub-bin $\mathcal{B}^\sigma(n',\rho',l)$ for $l < s_0$ and $n' < s_0$ receives any measure (recall that $S(\mathcal{B}^\sigma(n',\rho',l))$ is finite). So no $\mathcal{U}^\tau$ receives measure from such sub-bins after stage s_1. All measure given to $\mathcal{U}^\tau$'s from $\mathcal{B}^\sigma(n,\rho)$ after stage s_0 is disjoint from $\mathcal{V}^\rho_n$. Any set $\mathcal{Y}^*$ given by $\mathcal{U}^\sigma$ to such $\mathcal{U}^\tau$ at $t \geqslant s_0$ is disjoint from $\mathcal{W}^\tau_t$ and so from $\mathcal{V}^\rho_n$. Overall, we see that no measure given to any $\mathcal{U}^\tau$ after stage s_1 intersects $\mathcal{V}^\rho_n$. So by induction, we see that for all $d \geqslant s_1$, for all $t \geqslant s_1$, $\mathcal{U}^{\sigma\hat{\ }d}_t$ is disjoint from $\mathcal{V}^\rho_n$.

Let $s_2 > s_1$ be a stage after which $\mathcal{U}^{\sigma\hat{}\mathtt{fin}}$ and $\mathcal{U}^{\sigma\hat{}d}$ for $d < s_1$ never receive any new measure. Then after stage s_2, no $\mathcal{U}^\tau$ ever extract any measure which intersects $\mathcal{V}^\rho_n \backslash \mathcal{E}_t$. Hence after stage s_2, no new measure is added to $\mathcal{B}^\sigma(n, \rho)$. □

3.9. **The identity of the examiners.** Our last task is to describe precisely how the testing works. We now identify which node sets up boxes for tests for a given actor O. It would seem natural that an actor O associated with a node σ would use inputs for Ψ^A_σ to run these tests, but this will not work. Though σ is constructing h_σ and can see $\Psi^A_\sigma[s]$, it does not know the trace for Ψ^A_σ; only the child $\sigma\hat{}d$ which correctly guesses a trace can know what the trace is. Thus, we need to choose some predecessor of σ to perform the test. As we just mentioned, σ needs access to a trace for the function defined by that predecessor. So we choose some *tester* π for σ which is in F; we use the inputs for $\Psi^A_{\pi^-}$ to run tests for actors owned by σ, and observe the trace T^π in order to find if the test succeeded or not. The fact that σ needs access to the trace gives one restriction on the choice of π:

- We need to choose $\pi \in F^{\subseteq\sigma}$.

On the other hand, the node π^- needs to know how many boxes to allocate to O, and so needs to be able to calculate $b(O)$. For $O = \mathcal{U}^\tau$ (where τ is a child of σ), it needs access to values of g^ρ for $\rho \in F^{\subset\tau} = F^{\subseteq\sigma}$. For $O = \mathcal{B}^\sigma(n, \rho, l)$, we also need $g^\rho(n)$, where again $\rho \in F^{\subseteq\sigma}$. For $\rho \in F^{\subseteq\sigma}$, the nodes which guess that g^ρ is total and so have access to the values of g^ρ when they are accessible are the siblings of ρ in F (as $g^\rho = g^{|\rho^-|}$). In particular, $\rho \in F^{\subseteq\sigma}$ is the shortest predecessor of σ which "knows" g^ρ. Hence:

- We need to choose π extending every string in $F^{\subseteq\sigma}$.

The only possible choice is to take π to be the longest string in $F^{\subseteq\sigma}$.

From the point of view of π^-, we see that the nodes σ which require boxes of π^- are the extensions of $\pi^-\hat{}d$ for some $d < \omega$ for which there is no element of F properly extending $\pi^-\hat{}d$ and extended by σ. That is, nodes of the form $\pi^-\hat{}d\hat{}\mathtt{fin}^m$ for some $d, m < \omega$.

We note, however, that this assignment of tester π does not make sense in special cases, namely, nodes σ for which $F^{\subseteq\sigma}$ is empty. There are two kinds of such nodes: the root $\langle\rangle$, and certain nodes extending $\langle\mathtt{fin}\rangle$.

As mentioned during the construction, nodes $\sigma \supseteq \langle\mathtt{fin}\rangle$ do not play a role in the construction: we required g^0 to be total and so we know that fin is not the correct outcome of the root. Indeed, the reason that we made this requirement is precisely our inability to find a suitable tester for such nodes σ.

Of course, this solution does not work for the root, which definitely has a role in the construction. The root needs to apportion measure to its children $\langle d\rangle$ and replace it when necessary. We note, however, two facts:

(1) A node $\langle d\rangle$ does not put any Γ computations on $\mathcal{U}^{\langle d\rangle}$, as those computations would map to strings of length 0 (this is why we mapped $\mathcal{U}^\tau$ to $A_s\restriction_{|\tau|-1}$ rather than $A_s\restriction_{|\tau|}$).
(2) $\mathcal{W}^{\langle d\rangle} = \varnothing$ because $F^{\subset\langle d\rangle} = \varnothing$.

The second fact shows that there is no need for bins held by the root. And both facts together show that measure can be given to $\mathcal{U}^{\langle d\rangle}$ without testing it. All measure which the root ever gives to $\mathcal{U}^{\langle d\rangle}$ is measure which has never been given to any child of the root: any measure rejected by any $\mathcal{U}^{\langle d\rangle}$ is rejected because it is in $\mathcal{E}$, and so it is never given back to anyone. Thus, measure given to $\mathcal{U}^{\langle d\rangle}$ is free of any Γ-computations, and $\mathcal{U}^{\langle d\rangle}$ does not put any new Γ-computation on it. Hence no testing is required, as was indicated in the construction.

3.10. **Definition of h_η.** For any node η, in order to define h_η, we need to find, for any given $k<\omega$, the collection $\mathfrak{O}_\eta(k)$ of all actors O with $k(O)=k$ which ask η for boxes to perform their tests. Each such actor O will require $2^{b(O)}$ many k-boxes from the appropriate column of ω. As we shall see, for each O, the testing mechanism will utilise a hypercube of boxes devoted to O, with one axis for every test; hence the required number of boxes.

Fix $k<\omega$. The node η first needs to find the actors in $\mathfrak{O}_\eta(k)$. This is easily done.

Claim 3.17. The map

$$q\mapsto\{\sigma : \delta_\sigma\geqslant q\}$$

defined on positive rational numbers q, is computable.

Proof. Say $q=2^{-k}$. We know that if τ is a child of σ then $\delta_\tau<\delta_\sigma/2$ (in fact $\delta_\sigma/8$). Hence if $\delta_\sigma\geqslant q$ then $|\sigma|<k$. We also know that if σ extends some node $\tau\hat{\ }d$ then $\delta_\sigma<2^{-d}$. Hence we can bound the search to the finitely many nodes of length at most k which do not mention numbers greater than k. □

Given $k<\omega$, the actors O with $k(O)\leqslant k$ are either $\mathcal{U}^\tau$ with $\delta_\tau\geqslant 2^{-k+1}$, or $\mathcal{B}^\sigma(n,\rho,l)$ with $l\leqslant k$, which implies $k^\sigma(n,\rho)<k$, which implies $2n<k$ and $\delta_\sigma\geqslant 2^{-k}$. From Claim 3.17 we see that we can effectively find all such actors. We then check which ones have $k(O)$ precisely equal to k and which ones appeal to η for boxes, and obtain $\mathfrak{O}_\eta(k)$. Note that this is done effectively, without waiting for any functions g^ρ to show us any values.

However, to find $b(O)$ for $O\in\mathfrak{O}_\eta(k)$, we need such values. For every $O\in\bigcup_k\mathfrak{O}_\eta(k)$ we can find a number $u=u(O)$ such that $b(O)$ can be calculated given $g^\rho\restriction_{u(O)}$ for $\rho\in F^{\subseteq\eta}$, and $g^{|\eta|}\restriction_{u(O)}$. At each stage s, for some $K=K_{\eta,s}$, we will have already set up the k-boxes $k<K$; the collection of k-boxes is an interval I_k and these intervals are consecutive. If s is η-expansionary, and $\operatorname{dom} g_s^{|\eta|}, \operatorname{dom} g_s^\rho>u(O)$ for $\rho\in F^{\subseteq\eta}$ and all $O\in\mathfrak{O}_\eta(K)$, then we can define I_K to be long enough so that there are disjoint sets $I(O)\subseteq I_K$ for $O\in\mathfrak{O}_K(\eta)$ with $|I(O)|=2^{b(O)}$ and $I(O)\subset\omega^{[d]}$, where $\sigma(O)\supseteq\eta\hat{\ }d$. Then h_η is extended to I_K (by letting $h_\eta(z)=K$ for $z\in I_K$), and we let $K_{\eta,s+1}=K_{\eta,s}+1$.

If the node σ associated with O is accessible at some stage t and at that stage plans a swap with a child, and asks to test for O, then carrying out this test is only possible if $K_{\eta,t}>k(O)$, i.e., if I_k has been defined by stage t. If this fails for *some* actor O for which σ asks for a test, then as indicated above, the planned swap is abandoned and no test is started.

We note that now we can verify that Rules 11 and 13 are followed. If τ is accessible at some stage s, then at some stage $t<s$, τ^- tested measure that would go into $\mathcal{U}^\tau$, and so the required boxes were set up by stage t. In order to set up the boxes for $\mathcal{U}^\tau$, we need to calculate $b(\mathcal{U}^\tau)$. The same argument holds for the sub-bins.

3.11. **Testing.** The set $I(O)$ is the collection of inputs (boxes) z such that tests performed for O use $\Psi_\eta^A(z)$. We want to think of $I(O)$ as a hypercube of dimension $b(O)$ (and axis length 2). We write $D(O) = \{1, 2, \dots, b(O)\}$ for the set of axes of this hypercube, and so write

$$I(O) = \{z_\nu(O) \,:\, \nu\colon D(O) \to \{0,1\}\}\,.$$

When O is clear from context, we write z_ν for $z_\nu(O)$, D for $D(O)$, etc. What is important is to notice that $h_\eta(z) = k(O)$ for all $z \in I(O)$.

When at some stage t, we start a test for the actor O, we assign some axis $a_t = a_t(O) \in D(O)$ to this test. We then use the boxes in some flat subset $L_t(O)$ of $I(O)$.

We will test some string $\alpha_t \prec A_t$. So that Claim 3.3 holds, we need to test the same string for all actors testing at stage t. Let $\mathcal{C}_t(O)$ be the clopen set which we plan to give O at stage t. Also, let $T_t(O)$ be the set of stages $r < t$ at which a test for O was started, such that $\alpha_r \prec A_t$. We let

$$\alpha_t(O) = \bigcup_{r \in T_t(O)} \alpha_r \;\cup \bigcup_{X \in \mathcal{C}_t(O)} \Gamma_t^X.$$

We noted (in Section 3.8) that $\mathcal{C}_t(O)$ is disjoint from $\mathcal{E}_t$, and so $\alpha_t(O)$ is an initial segment of A_t. We then consider all actors O for which a test is started at stage t. Let σ be the node which owns all of these actors. We let

$$\alpha_t = A_t \restriction_{|\sigma|} \;\cup \bigcup \alpha_t(O) \;\; [\![\text{a test for } O \text{ is started at stage } t]\!].$$

To perform the test for O, we let, for all $z \in L_t(O)$, $\Psi_{\eta,t+1}^{A_t}(z) = \alpha_t$ with use $|\alpha_t|$. So what remains is to describe how to pick a_t and L_t, and then to show that Ψ_η is consistent.

Let π be σ's tester; so $\eta = \pi^-$. The test on $L_t(O)$ is *successful* at stage $s > t$ if $\alpha_t \prec A_s$, and for all $z \in L_t(O)$ we have $\alpha_t \in T_s^\pi(z)$. The test on $L_t(O)$ has *failed* by stage $s > t$ if $\alpha_t \nprec A_s$. Note that it is possible for a test to be successful at stage s_1 and then for the same test to fail by stage $s_2 > s_1$.

Proof of Claim 3.3. Suppose that a node $\sigma \neq \langle\rangle$ plans a swap with a child τ at stage t, and is next accessible at stage s. Let π be the tester for σ, and $\eta = \pi^-$. At stage t we begin tests for actors O associated with σ; we set $\Psi_\eta^{\alpha_t}(z) = \alpha_t$ for all $z \in L_t(O)$ for every actor O involved. If $\alpha_t \nprec A_s$ then all of these tests have failed by stage s. Otherwise, $\Psi_\eta^A(z)\,[s] = \alpha_t$ for all such z. Since π is accessible at stage s, we have $\alpha_t \in T_s^\pi(z)$ for all such z, and so all the tests have succeeded by stage s. ☐

What we need to note is that if a test which began at stage t has failed by stage s, then the computations $\Psi_\eta^{A_t}(z)$, for $z \in L_t$, no longer apply to A_s, and so these boxes have been "released" by that test and are available for tests which begin after stage s.

Still using the same scenario, we note that at a stage t at which σ wants to perform tests, we can calculate both $C_t(O) = C(O) \cap \{0, \dots, t-1\}$ and $S_t(O) = S(O) \cap \{0, \dots, t-1\}$. The set $C_t(O)$ is the set of stages $r < t$ at which a test, started for O at r, was cancelled before stage $r^+(\sigma) \leqslant t$. The set $S_t(O)$ is the set of stages $r < t$ at which a test, started for O at r, returned successfully at stage $r^+(\sigma) \leqslant t$.

The calculations of Section 3.7 show the following important claim:

Claim 3.18. If O requests a test at some stage t, then

$$\{a_r : r \in S_t \cup C_t\}$$

is a proper subset of $D(O)$.

Proof. This is essentially a restatement of Claims 3.10 and 3.12. □

Hence, at such a stage t, we can choose some $a_t \in D(O)$, distinct from a_r for $r \in S_t \cup C_t$. It does not matter which one. Then Claim 3.3, which was used implicitly above, ensures that:

Corollary 3.19. The map $t \mapsto a_t$ is injective on $C(O) \cup S(O)$.

To find L_t, we let

- $P_s = P_s(O)$ be the set of stages $r \in S_s$ such that the test started at stage r has failed by stage s; and
- $Q_s = C_s \cup (S_s \backslash P_s)$.

The set P_s increases with time. As mentioned above, the boxes in L_r, for $r \in P_s$, have been released to be used by other tests starting after stage s. Moreover, the traces $T^\pi_s(z)$ for $z \in L_r$ contain the string α_r, which lies strictly to the left of A_s. These boxes have been "promoted" – their trace $T^\pi(z)$ has "used up" one of its $k = h_\eta(z)$ many "slots". Of course, the price for this is that at some stage $s' < s$ we erroneously believed that $\alpha_r \prec A$, and the actor O took responsibility for $\mathcal{C}_r$. The fact that the test then failed shows that some of this measure entered $\mathcal{E}$, and this was charged to O (unless in the mean time, O passed the responsibility to some other actor). The grand plan is to keep reusing such boxes for future tests, so that each time this happens, some traces $T^\pi(z)$ will lose more and more of their slots. Each such $T^\pi(z)$ has at most k slots, and we will use this to bound the number of errors by k.

What guides us in the choice of L_t is that we must have L_t disjoint from L_r for all $r \in Q_t$, because the tests started at such r have not yet released their boxes; using these boxes could make Ψ_η inconsistent. The flat L_r is contained in the hyperplane $\{z_\nu : \nu(a_r) = 1\}$, and so we let

$$L_t = \{z_\nu \in I(O) : \nu(a_t) = 1 \ \& \ \nu(a_r) = 0 \text{ for all } r \in Q_t\}.$$

Corollary 3.19 implies that L_t is nonempty.

Claim 3.20. Ψ_η is consistent.

Proof. First note that if $z \in I(O)$, then $\Psi^{A_t}_\eta(z)$ is defined only at stages t at which a test is started for O; the cubes for distinct actors are disjoint. Suppose that such a test is started at stage t, and that $\Psi^{A_t}_{\eta,t}(z)\downarrow$ for some $z = z_\nu \in I(O)$, so already converges by a computation defined at some stage $r < t$. The use of that computation was $|\alpha_r|$, and so $\alpha_r \prec A_t$. Thus, the test started at stage r has not yet failed, and since $\sigma(O)$ is accessible at stage t, we see that $r \in C_t \cup S_t$, and that in fact $r \in Q_t$. Hence $\nu(a_r) = 1$, and so $z \notin L_t$, whence no new computation is defined on z at stage t. □

3.12. **Bounding the size of $\mathcal{E}$.** We have finished giving the details of the construction. Our first task is to make sure that the construction makes sense, that is, that the instructions we gave can actually be carried out. To do this, we need to observe that all the rules were obeyed, and that all the claims we stated along the way hold. What is left is to verify Claims 3.1 and 3.2, using Claims 3.4 and 3.5.

Recall from Section 3.1 how the claiming of responsibility works: if a clopen set $\mathcal{C}$ is moved to some actor O at some stage s, then from stage $s+1$, the actor O claims responsibility for all $Y \in \mathcal{C}$. The actor keeps being responsible for such Y until a stage at which Y is moved to some other actor. Note that this can happen without Y leaving O: if Y enters $\mathcal{U}^\tau$, and then at some later stage s it enters $\mathcal{U}^{\tau\hat{\ } d}$, then from stage $s+1$, $\mathcal{U}^{\tau\hat{\ } d}$ is responsible for Y and $\mathcal{U}^\tau$ is not.

Claim 3.21. If $\Gamma^Y_s \neq \langle\rangle$ and $Y \notin \mathcal{E}_{s-1}$ then at stage s, Y is the responsibility of some actor O with $\sigma(O) \neq \langle\rangle$.

Proof. By induction on s. At the first stage s at which a Γ-computation pertaining to Y is enumerated (into Γ_s), Y is claimed by some $\mathcal{U}^\tau$ with $|\tau| \geqslant 2$. We then observe that once a real Y is claimed by some O with $\sigma(O) \neq \langle\rangle$, it can only move to another actor O' with $\sigma(O') \neq \langle\rangle$, or into $\mathcal{E}$; as we mentioned earlier, measure returned to $\mathcal{U}^{\langle\rangle}$ is in $\mathcal{E}$, and measure given to any $\mathcal{U}^{\langle d\rangle}$ is free from Γ-computations. □

Claim 3.4 follows. Next, we check that Γ actually works.

Claim 3.22. Suppose that a swap is planned at stage t and is executed at stage s. Then $\alpha_t < A_s$.

Proof. The tests which were started at stage t are all successful at stage s. □

Claim 3.23. Suppose that at some stage u, an actor O is responsible for a real Y. Let $s < u$ be the stage at which the actor took responsibility of Y, and let $t = s^-(\sigma(O))$ be the stage at which the swap, which was executed at stage s, was planned (we assume that $\sigma(O) \neq \langle\rangle$). Then $\Gamma^Y_u \preccurlyeq \alpha_t$.

Proof. By the definition of α_t, we have $\Gamma^Y_t \preccurlyeq \alpha_t$. We have $Y \in \mathcal{U}^\sigma_t = \mathcal{U}^\sigma_s$, and σ is not accessible between stages t and s. It follows (really by Rule 1) that no new Γ-computation is defined on Y between stages t and s, so $\Gamma^Y_s = \Gamma^Y_t$.

At stage s we may add a Γ-definition which pertains to Y. (This may happen if $O = \mathcal{U}^\tau$ for some τ, a child of σ.) The axiom, added to Γ_{s+1}, maps the new clopen subset of $\mathcal{U}^\tau$ to $A_s \upharpoonright_{|\sigma|}$. By Claim 3.22, $\alpha_t < A_s$, and by design, $|\alpha_t| \geqslant |\sigma|$. Hence $\Gamma^Y_{s+1} \preccurlyeq \alpha_t$. Between stages $s+1$ and u, no new Γ-computation applies to Y, as new computations always involve a shift in responsibility. □

Proof of Claim 3.1. In fact, if a clopen set $\mathcal{C}$ is moved to any actor O (not only $\mathcal{U}^\tau$) at stage s, then $\mathcal{C}$ is disjoint from $\mathcal{E}_s$. For let $Y \in \mathcal{C}$ and let $t = s^-(\sigma(O))$. Claim 3.23 implies that $\Gamma^Y_s \preccurlyeq \alpha_t$ (as $\Gamma^Y_s \preccurlyeq \Gamma^Y_{s+1}$). By Claim 3.22, $\alpha_t < A_s$. Hence $Y \notin \mathcal{E}_s$. □

So Γ is a consistent functional.

Proof of Claim 3.5. Let $P(O) = \bigcup_s P_s(O)$, i.e., it is $P_s(O)$ for sufficiently late s. For every $t \in P(O)$, let $v(t)$ be the stage at which t enters $P(O)$, i.e. the least stage v such that $\alpha_t \not< A_v$. Let P' be the set of $t \in P(O)$ such that the interval $(t, v(t))$ contains no element of $P(O)$.

First, we observe that $|P'| \leqslant k(O)$. Define $\nu\colon D(O) \to \{0,1\}$ by letting $\nu(a_t) = 1$ for all $t \in P'$ and $\nu(b) = 0$ for all other $b \in D(O)$. Let $t \in P'$, and consider $t' < t$ in P'. As $v(t') \leqslant t$, we see that $t' \in P_t(O)$. It follows that $z_\nu \in L_t$. Since $P(O) \subseteq S(O)$, we know that $\alpha_t \in T^\rho(z_\nu)$ for all $t \in P'$. However, if $t' < t$ are in P', then $v(t') \leqslant t$ and $\alpha_t \prec A_t$ implies that $\alpha_{t'}$ lies strictly to the left of α_t. In particular, the strings α_t for $t \in P'$ are distinct. Hence $|P'| \leqslant |T^\rho(z_\nu)| \leqslant k(O)$.

The proof of the claim will be finished once we show that if at some stage u, a real Y for which O is responsible at stage u enters $\mathcal{E}_u$, then $Y \in O_{(t')^+(\sigma(O))+1}$ for some $t' \in P'$. Let Y be such a real and u be such a stage. Let s be the last stage before u at which Y was moved to O, and let $t = s^-(\sigma(O))$; note that $\sigma(O) \neq \langle\rangle$.

By Claim 3.23, we have $\Gamma^Y_u \preccurlyeq \alpha_t$, and since $Y \in \mathcal{E}_u$ we conclude that $\alpha_t \nprec A_s$. So $t \in P(O)$ and $v(t) \leqslant u$.

We note: if $t < t'$, a test for O is started at stage t', and $v(t) > t'$ then $\alpha_t \preccurlyeq \alpha_{t'}$. This follows from the definition of $\alpha_t(O)$, as $\alpha_t \prec A_{t'}$ (we have $t \in T_{t'}(O)$). It follows that if in addition $t' \in P(O)$ then $v(t') \leqslant v(t)$.

Let t' be the greatest element of $P(O)$ in the interval of stages $[t, v(t))$. The argument in the previous paragraph shows that $t' \in P'$. Let $s' = (t')^+(\sigma(O))$; because $t' \in S(O)$, $\alpha'_t \prec A_{s'}$ and so $s' < v(t') \leqslant v(t) \leqslant u$. As $s' \geqslant s$ we have $Y \in O_{s'+1}$, as required. □

Finally, the calculations in Section 3.5 show that Claim 3.2 holds, and with that, we conclude that the construction can be carried out as described.

3.13. **The true path.** The true path is defined recursively, starting with the root. If σ is on the true path, then the child of σ which is on the true path is the leftmost one which is accessible infinitely often. Here the ordering on outcomes is $0 <_L 1 <_L 2 <_L \cdots <_L \texttt{fin}$. We need to show that the true path is infinite. Suppose that σ lies on the true path.

We first show that there is a leftmost child of σ which is *considered* by σ infinitely often. Then we will show that this child is accessible infinitely often.

We need:

Claim 3.24. Suppose that σ lies on the true path, and that $g^{|\sigma|}$ is total. Then h_σ is total.

Proof. Let $\rho \in F^{\subseteq\sigma}$. Because $\rho \in F$ is on the true path, we see that g^ρ is total (otherwise there are only finitely many ρ^--expansionary stages). The calculations in Sections 3.7 and 3.10 show that for any k, we eventually set up k-boxes (define I_k). □

We identify the "correct child" τ.

- If $g^{|\sigma|}$ is partial, then we let $\tau = \sigma\hat{\ }\texttt{fin}$.
- Otherwise, we let $\tau = \sigma\hat{\ }d$ for d the least such that for all $z \in \omega^{[d]} \cap \operatorname{dom} \Psi^A_\sigma$ we have $\Psi^A_\sigma(z) \in T^{\sigma\hat{\ }d}$. Such d exists: as h_σ is total, there is some $d < \omega$ such that $T^{\sigma\hat{\ }d}$ is a trace for Ψ^A_σ.

Claim 3.25. Every child τ' of σ which lies to the left of τ is considered by σ only finitely many times.

Proof. If $\tau = \sigma\hat{\ }\texttt{fin}$ then there are only finitely many σ-expansionary stages, and no $\sigma\hat{\ }d$ is considered at stages which are not σ-expansionary.

Suppose that $\tau = \sigma\hat{\ }d$, and let $\tau' <_L \tau$; $\tau' = \sigma\hat{\ }d'$ for some $d' < d$. Then $T^{\sigma\hat{\ }d'}\restriction_{\omega^{[d']}}$ is not a trace for $\Psi^A_\sigma\restriction_{\omega^{[d']}}$, which means that there is some $z \in \omega^{[d']} \cap \operatorname{dom}\Psi^A_\sigma$ such that $\Psi^A_\sigma(z) \notin T^{\sigma\hat{\ }d'}(z)$. There is some stage s such that for all $t \geqslant s$, $\Psi^A_\sigma(z)[t] = \Psi^A_\sigma(z)$. After stage s, $\sigma\hat{\ }d'$ will not be considered by σ. □

Let r_0 be a stage after which no $\tau' <_L \tau$ is considered by σ.

Claim 3.26. τ is considered infinitely often by σ.

Proof. There are infinitely many stages s at which σ does not perform a previously planned swap with one of its children.

If $\tau = \sigma\hat{\ }\texttt{fin}$, then at every stage $s > r_0$ at which σ is accessible and is not performing a swap, σ will consider τ.

Suppose that $\tau = \sigma\hat{\ }d$. Definitions of $\Psi^A_\sigma(z)$ for $z \in \omega^{[d]}$ are only made when τ is accessible. If τ is not considered infinitely often, then eventually $\Psi^A_\sigma[t]\restriction_{\omega^{[d]}}$ stablises and equals $\Psi^A_\sigma\restriction_{\omega^{[d]}}$. As $T^\tau\restriction_{\omega^{[d]}}$ traces $\Psi^A_\sigma\restriction_{\omega^{[d]}}$, there is some stage $s_0 > r_0$ such that for every $t \geqslant s_0$, for every $z \in \omega^{[d]} \cap \operatorname{dom}\Psi^A_\sigma[t]$, we have $\Psi^A_\sigma(z) \in T^\tau\ [t]$. Thus, τ would be considered at every σ-expansionary stage after stage s_0. Hence, there are no σ-expansionary stage after stage s_0. Let $t > s_0$ be a stage at which $\operatorname{dom} g^{|\sigma|}_t > s_0$, and σ is accessible at stage t and is not performing a swap at stage t. Then t is a σ-expansionary stage; contradiction. □

We recall that $S(\mathcal{U}^\pi)$ is finite for all $\pi \preccurlyeq \tau$ (Section 3.7). It follows that $\mathcal{U}^\pi_-$ eventually stabilises to a clopen set $\mathcal{U}^\pi$; we increase r_0 so that $\mathcal{U}^\pi_t = \mathcal{U}^\pi$ for all $t \geqslant r_0$ and $\pi \preccurlyeq \tau$. Note that no test which is started for $\mathcal{U}^\tau$ after stage r_0 can be cancelled. We can also make r_0 sufficiently large so that $\mathcal{V}^\rho_{n,-}$ is stable for all $\rho \in F^{\subset\tau}$ and $n \in [n_\rho, m_\tau]$.

Claim 3.27. τ is accessible infinitely often.

Proof. We suppose that it is not, for a contradiction. There is a stage $r_1 > r_0$ after which τ is never accessible. We note that since no new measure is added to $\mathcal{U}^\tau$ after stage r_1, and τ is never accessible after stage r_1, no new Γ-computations are put on any subset of $\mathcal{U}^\tau$ after stage r_1. So only finitely many Γ-axioms apply to reals in $\mathcal{U}^\tau$. This means that eventually, $\mathcal{E}_t \cap \mathcal{U}^\tau$ is stable. So by choosing r_1 sufficiently large, we may assume that $\mathcal{U}^\tau_t \cap \mathcal{E}_t = \mathcal{U}^\tau \cap \mathcal{E}$ for all $t \geqslant r_1$.

At every stage $t > r_1$ at which τ is considered by σ, $\mathcal{U}^\tau$ requests new measure from $\mathcal{U}^\sigma$ (otherwise τ would be accessible). A swap is planned. Either we cannot perform the test because of a lack of boxes, or we do test. The test cannot be cancelled, and cannot succeed, so it must fail by stage $t^+(\sigma)$.

Let $t > r_1$ be a stage at which σ considers τ. Consider some $\rho \in F^{\subseteq\sigma}$ and $n \in [n_\rho, m_\tau]$; at stage t we may plan to give the bin $\mathcal{B}^\sigma(n,\rho)$ a clopen set $\mathcal{C}_t(n,\rho)$. Porism 3.15 states that we plan to test $\mathcal{C}_t(n,\rho)$ for $\mathcal{B}(n,\rho,l)$ for some l bounded by $1 - \log\lambda(\mathcal{C}_t(n,\rho))$. Because $\mathcal{U}^\tau_-$, $\mathcal{V}^\rho_{n,-}$ and $\mathcal{U}^\tau_- \cap \mathcal{E}_-$ are all constant after r_1, the set $\mathcal{C}_- = \mathcal{U}^\tau_- \cap \hat{\mathcal{W}}^\tau_-$ is constant after r_1. Since the process of carving $\mathcal{C}_t$ up into $\mathcal{C}_t(n,\rho)$ is fixed, we see that after stage r_1, $\mathcal{C}_t(n,\rho)$ does not depend on t. This implies that there is a fixed bound N on the certainty $k(O)$ for any actor O which is involved in any plan to swap measure for $\mathcal{U}^\tau$, after stage r_1. As we noted above, because σ lies on the true path, g^ρ is total for all $\rho \in F^{\subseteq\sigma}$. The analysis in Section 3.10 shows that eventually, $\eta = \pi^-$ will set up k-boxes for all $k \leqslant N$, where π is σ's tester. Hence, after some stage $r_2 > r_1$, we can perform the test for $\mathcal{U}^\tau$ whenever we choose to.

So it remains to show, for the contradiction, that at some stage $t > r_2$ we will start a test for $\mathcal{U}^\tau$ which does not fail. This will immediately follow from a bound M we find for $|\alpha_t|$ for all t at which a test for $\mathcal{U}^\tau$ is started.

We just saw that the set $\mathcal{C}_t(n,\rho)$ offered to the bin $\mathcal{B}^\sigma(n,\rho)$ at stages $t > r_2$ is constant. By assumption, it never leaves $\mathcal{U}^\tau$; as $\mathcal{U}^\tau$ is never accessible, after stage r_2, no new Γ-computations are added that pertain to any reals in $\mathcal{C}_t(O)$.

There is a finite set $\mathfrak{B}$ of bins $\mathcal{B}^\sigma(n,\rho)$ which are involved in tests for $\mathcal{U}^\tau$ after stage r_2. By Claim 3.16, there is a stage $r_3 > r_2$ after which none of these bins receive any measure.

Consider a set $\mathcal{C}_t(O)$ offered to some sub-bin $O = \mathcal{B}^\sigma(n,\rho,l)$ when testing for $\mathcal{U}^\tau$ after stage r_2. This set comprises of measure from $\mathcal{U}^\tau$ and of measure from other sub-bins of $\mathcal{B}^\sigma(n,\rho)$. No new Γ-computations were added to either since stage r_3. Hence $\alpha_t(O)$ is bounded after stage r_3.

It remains to consider $O = \mathcal{U}^\tau$. Measure offered to $\mathcal{U}^\tau$ is either some $\mathcal{Y}(n,\rho)$ or $\mathcal{Y}^*$. The sets $\mathcal{Y}(n,\rho)$ come from bins in $\mathfrak{B}$, and so no new Γ-computations apply to reals in these sets after stage r_3. The set $\mathcal{Y}^*$ is always chosen to be disjoint from bins outside $\mathfrak{B}$, so it is composed of measure which was never in $\mathcal{U}^{\tau'}$ for any child τ' of σ, and of measure from bins in $\mathfrak{B}$. In either case, no Γ-computations enumerated after stage r_3 apply. So $\alpha_t(\mathcal{U}^\tau)$ is bounded after stage r_3. This completes the proof of the claim. □

3.14. **Endgame.** So the true path is infinite. For any node σ, we let $\mathcal{U}^\sigma$ be the final version of $\mathcal{U}^\sigma_t$.

Claim 3.28. If σ lies on the true path, then $\lambda(\mathcal{U}^\sigma) = 4\delta_\tau$, and $\lambda(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})) \leqslant \delta_\sigma$.

Proof. There is a stage s at which σ is accessible, and such that $\mathcal{U}^\sigma_s = \mathcal{U}^\sigma$. Then $\lambda(\mathcal{U}^\sigma) = 4\delta_\sigma$ follows from Rule 2.

Suppose that $\lambda(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})) > \delta_\sigma$; let $q < \lambda(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})) - \delta_\sigma$ be a positive rational number. The significant components of $\mathcal{W}^\sigma$ eventually stabilise, and so there is some stage s such that $\lambda(\mathcal{E}\backslash\mathcal{E}_s) < q/4$ and for all $t > s$, $\lambda(\mathcal{W}^\sigma \triangle \mathcal{W}^\sigma_t) < q/4$. Then for all sufficiently large stages t, we have $\lambda(\mathcal{U}^\sigma \cap (\mathcal{W}^\sigma \cup \mathcal{E})[t]) > \delta_\sigma$. This includes some stage at which σ is accessible, which contradicts Rule 3. □

If τ extends σ then $\mathcal{W}^\sigma \subseteq \mathcal{W}^\tau$. It follows that $\langle \mathcal{U}^\sigma \backslash (\mathcal{W}^\sigma \cup \mathcal{E})\rangle$, as σ ranges over the true path, is a decreasing sequence of closed, nonempty sets, and so the intersection contains some real X. On page 10, just before Section 3.1, we detailed three properties of X which are sufficient for the proof of the theorem. We verify that X has these properties. The first is immediate from the definition of X: $X \notin \mathcal{E}$.

The second is not too difficult.

Claim 3.29. Γ^X is total.

Proof. Fix $e > 0$ and find σ on the true path of length e. Let τ be the child of σ on the true path. Since $X \in \mathcal{U}^\tau$, there is some stage s at which X is added to $\mathcal{U}^\tau$. At that stage we enumerate into Γ an axiom which makes $\operatorname{dom} \Gamma^X \geqslant e$. □

The third is also easy.

Claim 3.30. X is Demuth random.

Proof. Suppose that g^e is total. Let σ on the true path have length e. We showed that the child ρ of σ on the true path is in F. We have $g^\rho = g^e$ and $\langle \mathcal{V}^\rho_n \rangle_{n<\omega} = \langle \mathcal{V}^e_n \rangle_{n<\omega}$. Let τ be the child of ρ on the true path. Then $\bigcup_{n \geqslant n_\rho} \mathcal{V}^\rho_n$ is a subset of $\mathcal{W}^\tau$. Since $X \notin \mathcal{W}^\tau$, X passes the test $\langle \mathcal{V}^e_n \rangle$. □

4. Proof of Theorem 1.2

We enumerate a c.e. set A. To ensure that A is strongly jump-traceable, we meet the following requirements:

N_e**:** if h_e is an order function, then J^A has an h_e-trace $\langle T^e_x \rangle_{x<\omega}$.

Here $\langle h_e \rangle$ is an effective list of all partial computable functions whose domain is an initial segment of ω and which are nondecreasing on their domain, and J^A denotes a universal A-partial computable function.

We will find the following approximation to the use function helpful: if $J^A(x)\,[s]$ converges, we define $j_s(x)$ to be the use of the computation $J^A(x)\,[s]$. Otherwise, we define $j_s(x) = 0$.

To ensure that A is not computable from an A-Demuth random set, we meet the following requirements:

P_e**:** Every $X \in 2^\omega$ such that $\Phi_e(X) = A$ fails an A-Demuth test $\langle \mathcal{U}_n \rangle_{n<\omega}$.

This will be a single Demuth test shared by all P_e. To meet P_e, we meet subrequirements:

$P_{e,m}$**:** If $\Phi_e(X) = A$ then there is some $n > m$ such that $X \in \mathcal{U}_n$.

To guess which functions h_e are in fact order functions, we use a tree of strategies. To define the tree, we list the possible outcomes of each strategy (node on the tree). Let σ be a node. If σ works for N_e, then the possible outcomes of σ are `inf` and `fin` (denoting whether or not h_e is an order). Otherwise, σ has only one outcome.

For a node σ, accessible at stage s and of length $\leqslant s$, we describe what actions σ takes at stage s, and if $|\sigma| < s$, which outcome of σ is next accessible.

4.1. Strategy for N_e-requirements.

Let $\ell_s(e)$ be the greatest n such that there is some $x \leqslant s$ such that $h_{e,s}(x)\!\downarrow= n$.

At stage s, σ attends to all x such that $h_e(x) < \ell_s(e)$: if $J^A(x)\!\downarrow= y\,[s]$ and $y \notin T^\sigma_{x,s}$, then we enumerate y into T^σ_x and initialise all $\tau \supseteq \sigma\texttt{fin}$.

Also, σ aggregates restraint: for all $n < \ell_s(e)$, we let $R_s(\sigma, n)$ be the maximum of $j_s(x)$ where $h_e(x) \leqslant n$.

Let $t < s$ be the last stage at which $\sigma\texttt{inf}$ was accessible ($t = 0$ if there is no such stage). If $\ell_s(e) > 2^{t+2}$ and $\ell_s(e) > 2^{n_t(\tau)}$ for every $\tau \supseteq \sigma\texttt{inf}$ such that $n_t(\tau)$ is defined[9], let $\sigma\texttt{inf}$ be accessible at stage s. Otherwise, $\sigma\texttt{fin}$ is accessible at stage s.

4.2. A Basic Strategy for $P_{e,m}$.

When we see measure that appears to compute A (using Φ_e), we have two possible ways in which we can satisfy $P_{e,m}$: we can cover that measure with our test, or we can change A. We employ a combination of the two.

First, we choose an unclaimed test component $\mathcal{U}_n$ with $n > m$ and a large y. We keep y out of A. We study the open set

$$\mathcal{V}_s = \{X \in 2^\omega \mid \Phi^X_{e,s} \supseteq A_s \restriction_{y+1}\}.$$

[9] $n_t(\tau)$ will be defined in the $P_{e,m}$-strategy

While $\lambda(\mathcal{V}_s) \leqslant 2^{-n}$, we can cover it with $\mathcal{U}_n$. When $\mathcal{V}_s$ grows to be too large, we can enumerate y into A_{s+1}. Then all of $\mathcal{V}_s$ is wrong (it is in $\mathcal{E}$, to use the notation of the previous proof), so $P_{e,m}$ need no longer concern itself with it. We can then empty $\mathcal{U}_n$, choose a new y, and start again. This can happen at most 2^n many times (since at least 2^{-n} measure goes bad each time it happens), so we have a computable bound on the number of times we empty $\mathcal{U}_n$.

This strategy is insufficient, however, because the strongly jump-traceable strategies $N_{e'}$ act to ensure the trace at x by putting restraint on A. If a higher priority strategy places restraint that prevents y from entering A, it will interfere with the $P_{e,m}$-strategy.

Our response is to modify the strategy slightly. If y is restrained from entering A, we empty $\mathcal{U}_n$, choose a new large y, and start again. Every $N_{e'}$-strategy will only impose restraint for x at most $h_e(x)$ many times, so eventually this stops occurring.

It would seem that we have just constructed a strongly jump traceable c.e. set A which is not computable from a Demuth random, in contradiction with the previous theorem. There is a complication, however, in the bound on the number of changes to $\mathcal{U}_n$; specifically, how many x are there with higher priority?

When a $P_{e,m}$-strategy is initialised, it chooses a test component $\mathcal{U}_n$ to work with. This indicates that it will enumerate at most n many elements y into A. This then determines which x are higher priority than $P_{e,m}$; those pairs (e', x) such that $h_{e'}(x)$ is large enough to tolerate n many changes are lower priority, and the rest are higher. Thus this choice of n would seem to indicate how many higher priority pairs there are. However, we will not actually know how many such x there are until all the $h_{e'}$ have grown sufficiently large.

To guess which functions $h_{e'}$ are in fact order functions, we use a tree of strategies. The first time a $P_{e,m}$-strategy is accessible, the value n is chosen. We will not let this strategy be accessible again until every $h_{e'}$ which it guesses to be an order grows large relative to n (this is why we required $\ell_s(e) > n_t(\tau)$ in the N_e-strategy in order for $\sigma\mathtt{inf}$ to be accessible). So the second time the $P_{e,m}$-strategy is accessible, we know how many higher priority (e', x) pairs there are, and thus what the bound on the number of changes to $\mathcal{U}_n$ is. Unfortunately, this means that if a $P_{e,m}$-strategy is accessible precisely once, the computable bound we define will not be defined at n.

Our strategy then is to define the bound on the number of changes to $\mathcal{U}_n$ to be 0 when the $P_{e,m}$-strategy is first accessible. The second time the strategy is visited, we cause a change in A and redefine the bound to be whatever we now know it should be. Because our redefinition accompanied a change in A, the resulting function is A-computable. Hence $\langle \mathcal{U}_n \rangle$ will be an A-Demuth test. Indeed, the only part of our A-test which requires the oracle is the bound on the number of changes.

4.3. **The Full Strategy for $P_{e,m}$.** σ is associated with a *test component* $n_s(\sigma) > m$, a *coding marker* $x_s(\sigma)$ and a *witness* $y_s(\sigma)$. These become undefined whenever σ is initialised. Whenever σ changes the definitions of any of these or undefines them, all $\tau \supset \sigma$ are initialised. Let s_0 be the stage at which σ was last initialised. There are three cases.

Case 1. $n_s(\sigma)$ is not defined.

We set $n_s(\sigma)$ and $x_s(\sigma)$ to be large and pass to the next accessible node.

Case 2. $n_s(\sigma)$ and $x = x_s(\sigma)$ are defined and $x \notin A$

Let $b = 2^{s_0+2}$. If there is some $\beta\,\mathrm{inf} \subseteq \sigma$ such that β works for some N_d-requirement[10], and $R_s(\beta, b) \geqslant x_s(\sigma)$, then we undefine both $x_s(\sigma)$ and $n_s(\sigma)$. Otherwise, we enumerate x into A. Either way, we then pass to the next accessible node.

Case 3. $n = n_s(\sigma)$ and $x = x_s(\sigma)$ are defined and $x \in A$.

Let $k = 2^{n+2}$. If there is some $\beta\,\mathrm{inf} \subseteq \sigma$ such that β works for some N_d-requirement[10], and $R_s(\beta, k) \geqslant y_s(\sigma)$ (or if $y_s(\sigma)$ is not defined), then we set $y_s(\sigma)$ to be some large number and declare $\mathcal{U}_n = \varnothing$.

Let

$$\mathcal{V}_s(\sigma) = \left\{ X \in 2^\omega \ : \ \Phi^X_{e,s}(x) \supseteq A_s \restriction_{y+1} \right\}.$$

If $\lambda \mathcal{V}_s(\sigma) \geqslant 2^{-n}$, we:

- enumerate y into A, and declare $y_s(\sigma)$ to be undefined;
- declare $\mathcal{U}_n = \varnothing$.

Otherwise, we declare $\mathcal{U}_n = \mathcal{V}_s(\sigma)$. We then pass to the next accessible node.

4.4. **Construction.** We build a tree of strategies by devoting each level to a single requirement. Every strategy at level $2e$ is devoted to the N_e-requirement, while every strategy at level $2\langle e, m\rangle + 1$ is devoted to the $P_{e,m}$-requirement.

At stage s, we begin by letting the root be accessible and then proceed to let every accessible node σ with $|\sigma| < s$ act in order of length. At the end of stage s, for every σ with $|\sigma| < s$, we let $x_{s+1}(\sigma) = x_s(\sigma)$, $y_{s+1}(\sigma) = y_s(\sigma)$ and $n_{s+1}(\sigma) = n_s(\sigma)$.

4.5. **Verification.** We perform the verification as a sequence of claims.

Claim 4.1. Let σ work for some $P_{e,m}$. Let $t > s$, and suppose that $n = n_s(\sigma) = n_t(\sigma)$. Then between stages s and t, σ enumerates at most $2^{n_s(\sigma)}$ many witnesses into A.

Proof. Let $s_0 \leqslant s$ be the stage at which the location $n = n_s(\sigma) = n_{s_0}(\sigma)$ was chosen. Let $s_1 < s_2 < \dots$ be the stages, after stage s_0, at which a new witness $y_i = y_{s_i}(\sigma)$ is chosen. Since each y_i is chosen large, we have $y_1 < y_2 < \dots$.

Let $\mathcal{V}_i = \mathcal{V}_{s_{i+1}}(\sigma)$. We claim that if $n_{s_{i+1}}(\sigma) = n$ (so in particular, if $s_{i+1} < t$), then $\mathcal{V}_i$ is disjoint from every $\mathcal{V}_j$ for $j < i$. This is because for all $X \in \mathcal{V}_j$ we have $\Phi^X_e(y_j) = 0$, as $y_j \notin A_{s_{j+1}}$, but for all $X \in \mathcal{V}_i$ we have $\Phi^X_e(y_j) = 1$, as $y_j \in A_{s_{i+1}}$.

Since, for all j such that s_{j+1} is defined, we have $\lambda\mathcal{V}_j \geqslant 2^{-n}$, we see that s_{2^n+1} cannot exist. □

Claim 4.2. Let σ work for some N_e, and suppose that $m < \ell_s(e)$. Then there are fewer than m many stages $s' \geqslant s$ at which some $\tau \supseteq \sigma\mathtt{inf}$ enumerates an element into A below $R_{s'}(\sigma, m)$.

Proof. Such elements come in two sorts: $x_{s'}(\tau)$ and $y_{s'}(\tau)$. We count these separately.

By construction, in order for $x_{s'}(\tau) < R_{s'}(\sigma, m)$ to be enumerated into A at stage s', it must be that $2^{s_0+2} < m$, where $s_0 < s'$ is the last stage at which τ was initialised. But since $|\tau| < s_0$ and the priority tree is at most binary branching,

[10] Since σ is accessible, we know that $\ell_s(d)$ is greater than b and k, and thus that $R_s(\beta, b)$ and $R_s(\beta, k)$ are defined.

there are at most 2^{s_0} many strategies τ which were initialised at stage s_0. Thus a bound on the number of such $x_{s'}(\tau)$ is

$$\sum_{2^{s_0+2}<m} 2^{s_0} < m/2.$$

By construction, in order for $y_{s'}(\tau) < R_{s'}(\sigma, m)$ to be enumerated into A at stage s', it must be that $2^{n+2} < m$, where $n = n_{s'}(\tau)$. Since strategies always choose their n large, the same n never occurs more than once. For a fixed n, by Claim 4.1, at most 2^n many witnesses are enumerated. Thus a bound on the number of such $y_{s'}(\tau)$ is

$$\sum_{2^{n+2}<m} 2^{n} < m/2.$$

So there are fewer than m many such elements enumerated in total. Since no element is enumerated more than once, there are fewer than m many such stages. □

Claim 4.3. Let σ work for some N_e. Let $t > s$, and suppose that $h_{e,s}(x)\downarrow < \ell_s(e)$, σ is not initialised between stages s and t. Then between stages s and t, at most $h_e(x)$ many elements are enumerated into T^e_x.

Proof. Let $s_1 < s_2 < \dots$ be the stages between s and t at which σ enumerates an element into T^e_x. Then by construction, $J^A(x)\,[s_i] \neq J^A(x)\,[s_{i+1}]$, and so between stages i and $i+1$ some accessible node τ must have enumerated an element into A below $j_{s_i}(x)$.

By assumption, $\tau \nsubseteq \sigma$. If τ is to the left of σ, then when τ was accessible, σ would have been initialised, contrary to hypothesis. If τ is to the right of σ or $\tau \supseteq \sigma\texttt{fin}$, then τ was initialised at stage s_i, and so the element $x_{s'}(\tau)$ or $y_{s'}(\tau)$ which was enumerated would have been chosen after stage s_i, and thus would be larger than $j_{s_i}(x)$.

So it must be that $\tau \supseteq \sigma\texttt{inf}$. But the number of stages at which this can happen is less than $h_e(x)$ by Claim 4.2. Thus there can be no $s_{h_e(x)+1}$. □

Claim 4.4. Let σ be working for some $P_{e,m}$-requirement. Let $t > s$ be such that $n = n_s(\sigma) = n_t(\sigma)$. Then if $\mathcal{U}_n$ is not declared empty between stages s and t, $\mathcal{U}_{n,s} \subseteq \mathcal{U}_{n,t}$.

Proof. By hypothesis, $\mathcal{U}_{n,t} = \mathcal{V}_t(\sigma)$, $\mathcal{U}_{n,s} = \mathcal{V}_s(\sigma)$, $y = y_s(\sigma) = y_t(\sigma)$ and $y \notin A_t$. If $\mathcal{V}_s(\sigma) \nsubseteq \mathcal{V}_t(\sigma)$, then $A_s \upharpoonright_{y+1} \neq A_t \upharpoonright_{y+1}$. So some element less than y was enumerated into A by some accessible strategy ρ between stages s and t.

If $\rho \subset \sigma$ or ρ is to the left of σ, then σ would have been initialised between stages s and t when ρ was accessible, contradicting $n_s(\sigma) = n_t(\sigma)$.

If $\rho \supset \sigma$ or ρ is to the right of $|s$, then ρ would have been initialised when σ chose y, so any values chosen by ρ would be larger than y. □

Claim 4.5. There is an A-computable total function $g(n)$ bounding the number of times U_n is declared empty.

Proof. By construction, if n is not selected by some $P_{e,m}$-strategy by stage n, it will never be selected, and thus $g(n)$ can be set to 0.

Otherwise, let σ be the $P_{e,m}$-strategy which selects n, let s be the stage at which σ selects n, and let $x = x_s(\sigma)$. Note that by construction, if σ is accessible at stage $t > s$, $\ell_t(d) > 2^{s_0+2}$ and $\ell_t(d) > 2^{n+2}$ for all N_d-strategies $\beta\texttt{inf} \subseteq \sigma$.

If $x \notin A$, there are two possibilities: either σ was never again accessible after stage s, or x and n were undefined before the next time σ was accessible after stage s. In both cases, $g(n) = 0$ suffices.

If $x \in A$, then σ was accessible at some stage $t > s$. At this stage, for every N_d-strategy β with $\beta\texttt{inf} \subseteq \sigma$, we can compute $\#\{x \mid h_d(x) \leqslant 2^{n+2}\}$. By Claim 4.3, each such x can cause $R(\beta, 2^{n+2})$ to change at most 2^{n+2} many times.

By construction, whenever $\mathcal{U}_n$ is declared empty, either a new y was chosen because the previous y was below some restraint, or because the previous y was enumerated into A. We can use the previous paragraph to bound the first number, and Claim 4.1 to bound the second. Thus if $x \in A$,

$$g(n) = 2^n + \sum_{\tau\ \mathrm{inf} \subseteq \sigma} 2^{n+2} \cdot \#\{x \mid h_d(x) \leqslant 2^{n+2}\}$$

suffices. □

Claim 4.6. $\langle \mathcal{U}_n \rangle$ is an A-Demuth test.

Proof. Claims 4.4 and 4.5. □

Define the True Path in the usual fashion.

Claim 4.7. Every strategy along the true path is initialised only finitely many times.

Proof. Proof by induction.

Let σ be along the true path, and let s_0 be a stage such that for every $\tau\texttt{fin} \subseteq \sigma$ with τ an N_e-strategy, $\ell_s(e)$ will never change after stage s_0, and for every $\rho \subset \sigma$, ρ will never again enumerate an element into A. Then by construction, σ will never again be initialised. □

Claim 4.8. Every strategy along the true path guarantees its requirement.

Proof. By construction, if h_e is an order, T^e_x traces J^A. By Claims 4.3 and 4.7, T^e_x is eventually smaller than the order h_e, which suffices to meet the N_e-requirement.

Let σ be a $P_{e,m}$-strategy along the true path. Let s_0 be the final stage at which σ is initialised. The next time σ is accessible after s_0, we will choose an n and x, and from then after never again consider Case 1.

By Claim 4.3, $R_s(\beta, b)$ will eventually stabilise for every $\beta\texttt{inf} \subseteq \sigma$. Thus we will eventually enumerate some x into A and never again consider Case 2.

By Claim 4.3 again, $R_s(\beta, k)$ will eventually stabilise for every $\beta\texttt{inf} \subseteq \sigma$. Thus we will eventually stop rechoosing y because of restraint.

By Claim 4.1, we enumerate only finitely many of these y into A. After we have enumerated the last one, $\mathcal{U}_n$ will cover all X which compute A. □

This completes the proof of Theorem 1.2.

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