K-TRIVIALS ARE NEVER CONTINUOUSLY RANDOM

GEORGE BARMPALIAS, NOAM GREENBERG, ANTONIO MONTALBÁN,
AND THEODORE A. SLAMAN

1. Introduction

In [RS07, RS08], Reimann and Slaman raise the question "For which infinite binary sequences X do there exist continuous probability measures μ such that X is effectively random relative to μ?". They defined the collection NCR_1 of binary sequences for which such measures do not exist (we give formal definitions below), and showed, for example, that NCR_1 is countable, indeed that every sequence in NCR_1 is hyperarithmetic. In this paper we contribute toward the understanding of NCR_1 by showing that it contains all sets which are Turing reducible to an incomplete, recursively enumerable set. In particular, NCR_1 contains all K-trivial sets.

1.1. **Randomness relative to continuous measures.**
We begin by reviewing the basic definitions needed to precisely formulate this question.

Notation 1.1.

- For $\sigma \in 2^{<\omega}$, $[\sigma]$ is the basic open subset of 2^ω consisting of those X's which extend σ. Similarly, for W a subset of $2^{<\omega}$, let $[W]$ be the open set given by the union of the basic open sets $[\sigma]$ such that $\sigma \in W$.
- For $U \subseteq 2^\omega$, $\lambda(U)$ denotes the measure of U under the uniform distribution. Thus, $\lambda([\sigma])$ is $1/2^\ell$, where ℓ is the length of σ.

Definition 1.2. A *representation* m of a probability measure μ on 2^ω provides, for each $\sigma \in 2^{<\omega}$, a sequence of intervals with rational endpoints, each interval containing $\mu([\sigma])$, and with lengths converging monotonically to 0.

Definition 1.3. Suppose that $Z \in 2^\omega$. A *test relative to* Z, or Z*-test*, is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in Z. For $X \in 2^\omega$, X *passes* a test W if and only if there is an n such that $X \notin [W_n]$.

Definition 1.4. Suppose that m represents the measure μ on 2^ω and that W is an m-test.

- W is *correct for* μ if and only if for all n, $\mu([W_n]) \leq 2^{-n}$.
- W is *Solovay-correct for* μ if and only if $\sum_{n\in\omega} \mu([W_n]) < \infty$.

Definition 1.5. $X \in 2^\omega$ is 1*-random relative to a representation* m *of* μ if and only if X passes every m-test which is correct for μ. When m is understood, we say that X is 1-random relative to μ.

By an argument of Solovay, see [Nie09], X is 1-random relative to a representation m of μ if an only if for every m-test which is Solovay-correct for μ, there are infinitely many n such that $X \notin [W_n]$.

Definition 1.6. $X \in \mathrm{NCR}_1$ if and only if there is no representation m of a continuous measure μ such that X is 1-random relative to the representation m of μ.

In [RS08], Reimann and Slaman show that if X is not hyperarithmetic, then there is a continuous measure μ such that X is 1-random relative to μ. Conversely, Kjøs-Hanssen and Montalbán, see [Mon05], have shown that if X is an element of a countable Π^0_1-class, then there is no continuous measure for which X is 1-random. As the Turing degrees of the elements of countable Π^0_1-classes are cofinal in the Turing degrees of the hyperarithmetic sets, the smallest ideal in the Turing degrees that contains the degrees represented in NCR_1 is exactly the Turing degrees of the hyperarithmetic sets.

In **(author?)** [RSte], Reimann and Slaman pose the problem to find a natural Π^1_1-norm for NCR_1 and to understand its connection with the natural norm mapping a hyperarithmetic set X to the ordinal at which X is first constructed. As of the writing of this paper, this problem is open in general, but completed in [RSte] for $X \in \Delta^0_2$.

Suppose that $X \in \Delta^0_2$ and that for all n, $X(n) = \lim_{t\to\infty} X_t(n)$, where $X_t(n)$ is a computable function of n and t. Let g_X be the convergence function for this approximation, that is for all n, $g_X(n)$ is the least s such that for all $t \geq s$ and all $m \leq n$, $X_t(m) = X(m)$. Let f_X be function obtained by iterated application of g_X: $f_X(0) = g_X(0)$ and $f_X(n+1) = g_X(f_X(n))$.

For a representation m of a continuous measure μ, the granularity function s_m maps $n \in \omega$ to the least ℓ found in the representation of μ by m such that for all σ of length ℓ, $\mu([\sigma]) < 1/2^n$. Note that, s_m is well-defined by the compactness of 2^ω.

Theorem 1.7 (Reimann and Slaman [RSte])**.** *Let X be a Δ^0_2 set and let f_X be the function defined as above. If X is 1-random relative the representation m of μ, then the granularity function s_m for μ is eventually bounded by f_X.*

In the proof of Theorem 1.7, the possibility that s_m eventually bound f_X is ruled out since it would imply that X is recursive in m, contradicting X's being 1-random. The possibility that neither function bound the other is ruled out by the direct construction of a Martin-Löf test for μ, defined using g and the recursive approximation to X, which X would fail, again contradicting X's being 1-random.

It follows that, for Δ^0_2 sets X, there is a continuous measure relative to which X is 1-random if and only if there is a continuous measure whose granularity is eventually bounded by f_X. The latter condition is arithmetic, again by a compactness argument.

1.2. **K-triviality.** K-triviality is a property of sequences which characterizes another aspect of their being far from random. We briefly review this notion and the results surrounding it. A full treatment is given in Nies [Nie09].

For $\sigma \in 2^{<\omega}$, let $K(\sigma)$ denote the prefix-free Kolmogorov complexity of σ. Intuitively, given a universal computable U with domain an antichain in $2^{<\omega}$, $K(\sigma)$ is length of the shortest τ such that $U(\tau) = \sigma$. Similarly, for $X \in 2^\omega$, let $K^X(\sigma)$ denote the prefix-free Kolmogorov complexity of σ relative to X. That is, K^X is determined by a function universal among those computable relative to X.

Definition 1.8. A sequence $X \in 2^\omega$ is *K-trivial* if and only if there is a constant k such that for every ℓ, $K(X \upharpoonright \ell) \leq K(0^\ell) + k$, where 0^ℓ is the sequence of 0's of length ℓ.

By early results of Chaitin and Solovay and later results of Nies and others, there are a variety of equivalents to K-triviality and a variety of properties of the K-trivial sets. For example, X is K-trivial if and and only if for every sequence R, R is 1-random for λ if and only if R is 1-random for λ relative to X.

In the next section, we will apply the following.

Theorem 1.9 (Nies [Nie09], strengthening Chaitin [Cha76])**.** *If X is K-trivial, then there is a computably enumerable and K-trivial set which computes X.*

The following lemma follows from the work of Nies and others [Nie09]. Some versions of this property have been used by Kučera extensively, e.g. in [Kuč85].

Lemma 1.10. *Suppose that X is K-trivial and $\{U_e^X : e \in \omega\}$ a uniformly $\Sigma_1^{0,X}$ family of sets. Then, there is a computable function g and a Σ_1^0 set V of measure less than 1 such for every e, if $\lambda(U_e^Z) < 2^{-g(e)}$ for every oracle Z, then $U_e^X \subseteq V$.*

Proof. Let $\left((E_i^e)\right)_{e\in\mathbb{N}}$ be a uniform sequence of all oracle Martin-Löf tests. A standard construction of a universal oracle Martin-Löf test (T_i) (e.g. see [Nie09]) gives a recursive function f such that $\forall Z \subseteq \omega\ (E_{f(i,e)}^{e,Z} \subseteq T_i^Z)$ for all $e, i \in \mathbb{N}$. Let $T := T_2$ and $f(e) := f(2, e)$ for all $e \in \mathbb{N}$, so that $\mu(T^Y) \leq 2^{-2}$ for all $Y \in 2^\omega$ and $E_{f(e)}^e \subseteq T$ for all $e \in \mathbb{N}$. In [KH07] it was shown that X is K-trivial iff for some member T of a universal oracle Martin-Löf test, there is a Σ_1^0 class V with $T^X \subseteq V$ and $\mu(V) < 1$.

Now given a uniform enumeration (U_e) of oracle Σ_1^0 classes we have the following property of T:

> There is a recursive function g such that for each e,
> either $\exists Z \subseteq \omega\ (\mu(U_e^Z) \geq 2^{-g(e)-1})$, or $\forall Z \subseteq \omega\ (U_e^Z \subseteq T^Z)$.

To see why this is true, note that every U_e can be effectively mapped to the oracle Martin-Löf test (M_i) where $M_i^Z = U_e^Z[s_i]$ and s_i is the largest stage such that $\mu(U_e^Z[s_i]) < 2^{-i-1}$ (which could be infinity). Effectively in e we can get an index n of (M_i). It follows that if $\mu(U_e^Z) < 2^{-f(n)-1}$ for all Z, then $U_e^X = M_{f(n)}^X = E_{f(n)}^{n,X} \subseteq T^X \subseteq V$. So $g(e) = f(n)+1$ is as wanted. $\square$

1.3. **Our results.** Intuitively, $X \in \mathrm{NCR}_1$ asserts that X is not effectively random relative to any continuous measure and X is K-trivial asserts that relativizing to X does change the evaluation of randomness relative to the uniform distribution. In the next section, we connect the two notions.

Theorem 1.11. *Every K-trivial set belongs to* NCR_1.

A recursively enumerable (r.e.) set W is called *incomplete* if it does not compute the halting problem $\emptyset'$.

Theorem 1.12. *If W is an incomplete r.e. set and $X \leq_{\mathrm{T}} W$, then $X \in \mathrm{NCR}_1$.*

As we mentioned above, Theorem 1.12 implies Theorem 1.11, because every K-trivial set is computable from a r.e. K-trivial set, and every K-trivial set is incomplete. However we believe that the technique in the direct proof of Theorem 1.11 is of independent interest.

2. K-trivial sets and NCR_1

In this section we prove Theorem 1.11.

Let Y be K-trivial and let μ be a continuous measure with representation m; we want to show Y is not μ-random. By Theorem 1.9, let X be a computably enumerable K-trivial sequence that computes Y. Let f be the iterated convergence function as defined above for the computable approximation to Y given by approximating X's computation of Y. Since X is computably enumerable, X can compute the convergence function for its own enumeration and hence f is computable from X.

Let s_m be the granularity function for μ as represented by m. By Theorem 1.7, f eventually dominates s_m. By changing finitely many values of f, we may assume that f dominates s_m everywhere. So, we have that for every n

$$\mu([Y \restriction f(n)]) \leq 2^{-n}.$$

Further, we may assume that f can be obtained as the limit of a computable function $f(n,s)$ such that for all s, $f(n-1,s) \leq f(n,s) \leq f(n,s+1)$.

We will build an m-test $\{S_i : i \in \omega\}$ which is Solovay-correct for μ and which Y does not pass, thereby concluding that Y is not μ-random. That is, we plan to build $\{S_i : i \in \omega\}$ to be a uniformly $\Sigma_1^{0,m}$ sequence of sets such that $\sum_{i\in\omega} \mu(S_i)$ is bounded and such that there are co-finitely i for which $Y \in [S_i]$. Our construction will not be uniform.

X's K-triviality is exploited in the form of Lemma 1.10. Let V and g be given by Lemma 1.10 where $\{U_e^X : e \in \omega\}$ is a listing of all $\Sigma_1^{0,X}$ sets. We will build an oracle Σ_1^0 class U along the construction. We use the recursion theorem to assume that in advance we know an index e such that

$U = U_e$. During the construction we will make sure that for every oracle Z, $\lambda(U^Z) < 2^{-g(e)}$. Lemma 1.10 then implies that $U^X \subseteq V$ where V is a Σ^0_1 class of measure less than 1. To simplify our notation, let a denote $g(e)$. Furthermore, assume a is large enough so that $\lambda(V) + 2^{-a} < 1$.

We use the approximation to X as a computably enumerable set to enumerate approximations to initial segments of Y into the sets S_i; we rely on the K-triviality of X to keep the total μ-measure of the S_i's bounded.

For each $n > a$ we have a requirement R_n whose task is to enumerate $Y \upharpoonright f(n)$ into S_n. Let $y_{n,s} = Y_s \upharpoonright f(n,s)$ the stage s approximation to $Y \upharpoonright f(n)$. Let $x_{n,s}$ be the initial segment of X_s necessary to compute $y_{n,s}$ and $f(n,s)$. So, if $y_{n,s+1} \neq y_{n,s}$, it is because $x_{n,s+1} \neq x_{n,s}$. In this case, $x_{n,s+1}$ is not only different than $x_{n,s}$, but also incomparable. At stage s, R_n would like to enumerate $y_{n,s}$ into S_n, but before doing that it will *ask for confirmation* using the fact that $U^X \subseteq V$. Since we are constrained to keep $\lambda(U^X)$ less than or equal to 2^{-a}, we will restrict R_n to enumerate at most 2^{-n} measure into U^X. The reason why we need a bit of security before enumerating a string in S_n is that we have to ensure that $\sum_i \mu(S_i)$ is bounded. For this purpose, we will only enumerate mass into S_n when we see an equivalent mass going into V.

Action of requirement R_n:

(1) The first time after R_n is initialized, R_n chooses a clopen subset of 2^ω, σ_n, of m-measure 2^{-n}, that is disjoint form V_s and $U_s^{X_s}$. Note that since V and U^{X_s} have measure less than $\lambda(V) + 2^{-a} < 1$, we can always find such a clopen set. Furthermore we can chose σ_n to be different from the σ_i chosen by other requirements R_i, $i > a$. We note the value of σ_n might change if R_n is initialized.

(2) To *confirm* $x_{n,s}$, requirement R_n enumerates σ_n into $U^{x_{n,s}}$. Requirement R_n will not be allowed to enumerate anything else into U^{X_s} unless X_s changes below $x_{n,s}$. This way R_n is always responsible for at most 2^{-n} measure enumerated in U^{X_s}.

(3) Then, we wait until a stage $t > s$ such that
  (a) either $x_{n,s} \not\subseteq x_{n,t}$ (as strings),
  (b) or $\sigma_n \subseteq V_t$.

  Observe that if $x_{n,s}$ is actually an initial segment of X, then we will have $\sigma_n \subseteq U^X \subseteq V$. So, we will eventually find such a stage t.

  - In Case 3(a), we start over with R_n. Note that in this case σ_n has come out of U^{X_t}, and hence R_n is responsible for no measure inside U^{X_t} at stage t.
  - In Case 3(b), if $\mu([y_{n,t}]) \leq 2^{-n}$, enumerate $y_{n,t}$ into S_n. (Recall that we are allowed to use the representation of μ as an oracle when enumerating S_n.)

Since we only enumerate $y_{n,t}$ of μ-measure less than 2^{-n} when σ_n is enumerated in V, we have that

$$\sum_i \mu(S_i) \leq \lambda(V) < 1.$$

It is not hard to check that $\lambda(U^X) \leq \sum_{n=a+1}^{\infty} 2^{-n} = 2^{-a}$, so we actually have that $U^X \subseteq V$. Also notice that once $x_{n,s}$ is a initial segment of X, we will eventually enumerate σ_n into V and an initial segment of Y into S_n.

This completes the proof of Theorem 1.11.

3. Incomplete r.e. degrees and NCR$_1$

We turn to the proof of Theorem 1.12. Let W be an incomplete r.e. set, and let $X \leq_{\mathrm{T}} W$.

The fact that W is recursively enumerable and $X \leq_{\mathrm{T}} W$ implies that there is a recursive approximation $X = \lim_t X_t$ such that the modulus functions g_X is recursive in W, hence $f_X \leq_{\mathrm{T}} W$.

Suppose, for contradiction, that X is 1-random relative to a representation m of a continuous measure μ. By Theorem 1.7, by changing f_X at finitely many inputs, we obtain a function $f \leq_{\mathrm{T}} W$ which bounds the granularity function s_m. Let $h(n) = X \upharpoonright f(n)$. So $h \leq_{\mathrm{T}} W$, and for all n, $\mu([h(n)]) < 2^{-n}$.

Let J be a universal partial recursive function. For $n \in \mathbb{N}$, let $U_n = \{J(n)\}$ if $n \in \operatorname{dom} J$ and $J(n)$ is a binary string such that $\mu(J(n)) < 2^{-n}$. Otherwise, U_n is empty. Then the test U is recursively enumerable in m, and is correct for μ. Since X must pass U, we see that for all $n \in \operatorname{dom} J$, $h(n) \neq J(n)$.

The function h is diagonally nonrecursive. By Jockusch [Joc89], h computes a fixed-point-free function. This contradicts Arslanov's completeness criterion [Ars81], which states that an incomplete r.e. set cannot compute a fixed-point-free function.

This completes the proof of Theorem 1.12.

The question of which Δ^0_2 sets belong to NCR$_1$ remains open.

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George Barmpalias Institute for Logic, Language and Computation, Universiteit van Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands.

E-mail address: barmpalias@gmail.com

URL: http://www.barmpalias.net/

School of Mathematics, Statistics and Operations Research, Victoria University, Wellington, New Zealand

E-mail address: greenberg@msor.vuw.ac.nz

Department of Mathematics, University of Chicago, 5734 S. University ave., Chicago, IL 60637, USA

E-mail address: antonio@math.uchicago.edu

Department of Mathematics, University of California, Berkeley Berkeley, CA 94720-3840 USA

E-mail address: slaman@math.berkeley.edu