# A RANDOM SET WHICH ONLY COMPUTES STRONGLY JUMP-TRACEABLE C.E. SETS

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Abstract. We prove that there is a $\Delta^0_2$, 1-random set $Y$ such that every computably enumerable set which is computable from $Y$ is strongly jump-traceable.

We also show that for every order function $h$ there is an $\omega$-c.e. random set $Y$ such that every computably enumerable set which is computable from $Y$ is $h$-jump-traceable. This establishes a correspondence between rates of jump-traceability and computability from $\omega$-c.e. random sets.

## 1. Introduction

The study of the relationship between computably enumerable (c.e.) sets and the $\Delta^0_2$ Martin-Löf random sets which compute them started with Kučera's seminal [7], in which he proved that every $\Delta^0_2$ random set computes a promptly simple c.e. set. This led to his injury-free solution to Post's problem.

More recently, this interaction has been found to be fundamental to the study of lowness notions that arise from algorithmic randomness. The main notion of lowness in this area is that of $K$-triviality, originally due to Solovay [10], but deeply investigated only recently. In a sequence of results (see [9]), this class was shown to be robust, as it could be characterised alternatively as the class of sets which are far from random ($K$-trivial); the class of sets which do not have oracular power of compression (low for $K$); or the class of sets which cannot detect patterns in random sets (low for Martin-Löf randomness).

Another such characterisation – being a base for randomness – appeared in [6]. In that paper, Hirschfeldt, Nies and Stephan show the following:

(1) If $Y$ is random, $A$ is c.e., and $Y$ computes $A$, then $Y$ is random relative to $A$.
(2) If $Y$ is incomplete and random relative to $A$, and also computes $A$, then $A$ is $K$-trivial.

The second result implies that a set $A$ is $K$-trivial if and only if it is computable from a set $Y$ which is random relative to $A$, that is, if $A$ is too weak to realise that it is not computable in the sense of measure (as any cone in the Turing degrees has measure 0). The conjunction of both results shows that every c.e. set which is computable from an incomplete random set, such as Kučera's promptly simple sets, is $K$-trivial. The question whether every $K$-trivial is computable from an incomplete random set is considered one of the main open problems of the field of algorithmic randomness (see [8]).

Another notion of lowness which has recently gained attention is that of *strong jump-traceability*, defined by Nies, Figueira and Stephan [3]. Restricted to the c.e.

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degrees, this collection has been shown to be an ideal, strictly contained in the ideal of $K$-trivial degrees [1]. In recent work, Nies and Greenberg showed that the main question above can be solved in the affirmative if one restricts its domain to the collection of strongly jump-traceable c.e. sets. Indeed, they showed that every strongly jump-traceable c.e. set is computable from many incomplete random sets, for instance, all $\omega$-c.e. random sets, and all LR-hard random sets [5]. Even more recently, Hirschfeldt, Greenberg and Nies [4] showed that the c.e., strongly jump-traceable sets can in fact be characterised as those that are computable from all $\omega$-c.e. random sets, thus showing this class too is robust.

The focus of the main question, "which c.e. sets can be computed from incomplete random sets", is on the c.e. part of the equation; it is natural to also focus on the random sets and investigate what kind of c.e. sets can be computed from a given incomplete random set. The result mentioned above, that every $\omega$-c.e. random set computes all strongly jump-traceable c.e. sets, leads us to pay special attention to incomplete $\Delta^0_2$ random sets. In [5], it was shown that a single $\Delta^0_2$-random set cannot characterise strong jump-traceability; in particular, if $Y$ is an $\omega$-c.e. random set, then apart from all strongly jump-traceable c.e. sets, there are c.e. sets that are computable from $Y$, but are not strongly jump-traceable. Can we drop the condition that $Y$ be $\omega$-c.e.?

In this paper we show that we cannot.

**Theorem 1.1.** *There is a $\Delta^0_2$ random set $Y$ such that every c.e. set which is computable from $Y$ is strongly jump-traceable.*

Inspecting the proof of Theorem 1.1, we show a correspondence between rates of jump-traceability and computablity from $\omega$-c.e. random sets. Strong jump-traceability is the culmination of a hierarchy of notions of traceability, indexed by computable rates of growth, or "order functions". For every order function $h$ we have the class of sets that are $h$-jump-traceable (we give precise definitions below); this class shrinks the more slowly $h$ grows. A set is strongly jump-traceable if it is $h$-jump-traceable for every order function $h$. The result quoted from [5] was established by showing that for every $\omega$-c.e. random set $Y$ there is an order function $h$ such that every $h$-jump-traceable c.e. set is computable from $Y$. Here we show the converse:

**Theorem 1.2.** *For every order function $h$, there is an $\omega$-c.e. random set $Y$ such that every c.e. set computable from $Y$ is $h$-jump-traceable.*

The random set of Theorem 1.1, which cannot be $\omega$-c.e., can be seen as some kind of limit of the random sets guaranteed by Theorem 1.2.

## 2. Traceability and lowness

We define a *pc functional* to be a partial computable function $\Gamma\colon 2^{<\omega} \times \omega \to \omega$, such that for all $x < \omega$, the domain of $\Gamma(-, x)$ is an antichain of $2^{<\omega}$ (in other words, that domain is prefix-free). The idea is that the functional is the collection of minimal oracle computations of an oracle Turing machine. For any $A \in 2^{\leqslant\omega}$ and $x < \omega$, we let $\Gamma^A(x) = y$ if there is some initial segment $\tau$ of $A$ such that $\Gamma(\tau, x) = y$. Then $\Gamma^A$ is an $A$-partial computable function, and every $A$-partial computable function is of the form $\Gamma^A$ for some pc functional $\Gamma$. We write $\Gamma^A(x)\downarrow$ if $x \in \operatorname{dom} \Gamma^A$; otherwise we write $\Gamma^A(x)\uparrow$; we also write $\Gamma^A(x) = \uparrow$. The *use* of a computation $\Gamma^A(x) = y$ is the length of the unique initial segment $\tau$ of $A$ such that $\Gamma(\tau, x) = y$.

Among all pc functionals we fix one, denoted by $J$, which is universal; say $J^A(e) = \Psi_e^A(e)$, where $\langle \Psi_e \rangle$ is an enumeration of all pc functionals. We let $A' = \operatorname{dom} J^A$.

**Definition 2.1.**

(1) A *trace* is a uniformly c.e. sequence of finite sets. That is, a collection $\langle S_x \rangle$ such that each $S_x$ is finite, and there is a computable function $g$ such that for all $x$, $S_x = W_{g(x)}$.
(2) A *trace for a partial function* $\psi\colon \omega \to \omega$ is a trace $\langle S_x \rangle$ such that for all $x \in \operatorname{dom} \psi$, $\psi(x) \in S_x$.
(3) An *order function* is a computable, non-decreasing and unbounded function $h\colon \omega \to \omega$ such that $h(0) > 0$.
(4) A trace $\langle S_x \rangle$ is *bounded* by an order function $h$ if for all $x$, $|S_x| \leqslant h(x)$.

We say that a set $A$ is *$h$-jump-traceable* if every function which is partial computable in $A$ has a trace which is bounded by $h$. We say that a set is *strongly jump-traceable* if it is $h$-jump-traceable for every order function $h$. Because of the existence of the universal $A$-partial computable function, a set $A$ is strongly jump-traceable if and only if for every order function $h$, $J^A$ has a trace bounded by $h$.

A *computable approximation* for a function $f \in \omega^\omega$ is a uniformly computable sequence $\langle f_s \rangle_{s<\omega}$ of functions such that for every $n$, for almost all $s$, $f_s(n) = f(n)$. Associated to every computable approximation $\langle f_s \rangle$ is the mind-change function

$$m_{\langle f_s \rangle} = \lambda n. \#\{s \,:\, f_{s+1}(n) \neq f_s(n)\}.$$

Shoenfield's limit lemma states that a function has a computable approximation if and only if it is $\Delta^0_2$-definable if and only if it is computable from $\emptyset'$.

We also approximate partial functions. To this end, we add a symbol $\uparrow$ which denotes that an input is not in the domain of the partial function. We allow our approximations to take the value $\uparrow$.

Let $h$ be an order function. We say that the approximation $\langle f_s \rangle$ is *$h$-c.e.* if the mind-change function $m_{\langle f_s \rangle}$ is majorised by $h$. We say that a function $f$ is $h$-c.e. if it has some computable approximation which is $h$-c.e. We say that a function is *$\omega$-c.e.* if it is $h$-c.e. for some order function $h$. A function $f$ is $\omega$-c.e. if and only if $f \leqslant_{\mathrm{wtt}} \emptyset'$.

We note that Figueira, Nies and Stephan have showed [3] that a c.e. set $A$ is strongly jump-traceable if and only if $A'$ is $h$-c.e. for every order function $h$.

Let $\langle A_s \rangle$ be a computable enumeration of a c.e. set $A$, and $\langle \Gamma_s \rangle$ be a computable enumeration of a pc functional $\Gamma$. For any $s$, we can apply the functional $\Gamma_s$ to the set $A_s$; by convention, we assume that the function $\Gamma_s^{A_s}$ has *computable* domain and in fact only contains numbers smaller than $s$, with uses smaller than $s$. We thus obtain a uniformly computable sequence $\left\langle \Gamma_s^{A_s} \right\rangle$ of partial functions. We modify this sequence according to the following "hat trick". Suppose that $x \in \operatorname{dom} \Gamma_s^{A_s}$; let $u$ be the use of this computation. We say that the computation $\Gamma_s^{A_s}(x) = y$ is *destroyed* at stage $s+1$ if $A_s\upharpoonright_u \neq A_{s+1}\upharpoonright_u$. We now define

$$\Gamma^A(x)\,[s] = \begin{cases} \uparrow, & \text{if } x \in \operatorname{dom} \Gamma^{A_{s-1}}_{s-1} \text{ and that computation is destroyed at stage } s; \\ \Gamma^{A_s}_s(x), & \text{otherwise.} \end{cases}$$

In general, if $A$ is c.e. and $\Gamma$ is a pc functional, then $\operatorname{dom}\Gamma^X$ may not be $\Delta^0_2$, and so the sequence $\langle \Gamma^A\,[s] \rangle$ may not be a computable approximation of any partial function. However, the hat trick ensures that if $\langle \Gamma^A\,[s] \rangle$ is a computable approximation, then it is an approximation of $\Gamma^A$.

Recall that a set $A$ is *low* if $A' \leqslant_{\mathrm{T}} \emptyset'$. The following is well-known:

**Lemma 2.2.** *A c.e. set $A$ is low if and only if there is an enumeration $\langle A_s \rangle$ of $A$ and an enumeration $\langle \Gamma_s \rangle$ of a pc functional $\Gamma$ such that $\Gamma^A = J^A$ and such that the sequence $\langle \Gamma^A\,[s] \rangle$ is a computable approximation. Computable indices for $\langle A_s \rangle$ and $\langle \Gamma_s \rangle$ can be effectively obtained from a c.e. index of $A$ and a lowness index of $A$ (that is, an index of a Turing reduction of $A'$ to $\emptyset'$).*

Just as $\omega$-c.e.-ness is a strengthening of being $\Delta^0_2$, we can use order functions to obtain superlowness, which is a strengthening of being low (but still much weaker than strong jump-traceability). A c.e. set $A$ is *superlow* if $A'$ is $\omega$-c.e. Lemma 2.2 extends to characterise superlowness: a c.e. set $A$ is superlow if and only if there are enumerations $\langle A_s, \Gamma_s \rangle$ of $A$ and a pc functional $\Gamma$ such that $\Gamma^A = J^A$ and the approximation $\langle \Gamma^A\,[s] \rangle$ is $\omega$-c.e.

## 3. The proof of Theorem 1.1

We first mention that we make no special use of randomness: our constructions work in arbitrary non-empty $\Pi^0_1$ classes. We actually prove the following:

**Proposition 3.1.** *Let $\mathcal{P}$ be a nonempty $\Pi^0_1$ class. There is a $\Delta^0_2$ set $Y \in \mathcal{P}$ such that every superlow c.e. set $W \leqslant_{\mathrm{T}} Y$ is strongly jump-traceable.*

To see that Theorem 1.1 follows from Proposition 3.1, we apply the proposition to any $\Pi^0_1$ class of random sets, and obtain a random $\Delta^0_2$ set $Y$. Since there are superlow c.e. sets that are not strongly jump-traceable, $Y$ must be Turing incomplete (in fact, the proof of the proposition can be modified to make $Y$ low). It follows by the result from [6] mentioned in the introduction that every c.e. set that is computable from $Y$ is $K$-trivial. Since every $K$-trivial set is superlow, it follows that every c.e. set computable from $Y$ is strongly jump-traceable.

We present the proof of Proposition 3.1 in a modular fashion. The proof will involve guessing for superlowness, as appears in [2]. We first ignore this aspect of the construction and assume that lowness (which is all that will be used) is given to us uniformly. Recall that a uniformly c.e. collection of sets $\langle A_i \rangle$ is *uniformly low* if effectively in $i$ we can find an index for a Turing reduction of $A'_i$ to $\emptyset'$. We first prove the following, in some ways weaker, version of Proposition 3.1.

**Proposition 3.2.** *If $\mathcal{P}$ is a nonempty $\Pi^0_1$ class, and $\mathcal{A} = \langle A_i \rangle$ is a uniformly c.e. collection of uniformly low sets, then there is some $\Delta^0_2$ set $Y \in \mathcal{P}$ such that for all $A \in \mathcal{A}$, if $A \leqslant_{\mathrm{T}} Y$ then $A$ is strongly jump-traceable.*

## 4. Proof of Proposition 3.2

4.1. **Discussion.** The proof of Proposition 3.2 paradoxically builds on the proof from [4] of the fact that every set computable from every $\omega$-c.e. random set is strongly jump-traceable. The difference is that in that construction, an attempt is made to build a superlow random set $Z$ which does not compute $A$; the trace for $J^A$ was manufactured from the necessary failure of such a construction. Now we will need to construct a set $Y$ that does exist. For some $A \in \mathcal{A}$ we may have $A \leqslant_{\mathrm{T}} Y$, and for such $A$, we hope to indeed build traces for $J^A$. We need to ensure though that manufacturing such an approximation, for some $A$, does not prevent weaker requirements, which deal with some other $A$'s, to get their turn and try to construct their own traces. The priority tree mechanism utilised in $\Pi^0_2$ constructions is employed. A strategy will have a $\Pi^0_2$ outcome, indicating the success of building traces, along with finitary outcomes, which tell how we fail to obtain such an approximation, witnessing the failure of $Y$ to compute the current $A$; these outcomes impose finitary restraint on weaker strategies.

The $\Pi^0_2$ mechanism can also guess which partial computable functions are order functions. However, for notational convenience, mostly, we adopt a trick from [3], using an unbounded function that is approximable from above and dominated by all order functions.

*Fact* 4.1. There is a nondecreasing, unbounded $\Delta^0_2$ function $h$ such that

- $h$ has a computable approximation $\langle h_s \rangle$ such that for all $s$, $h_s$ is an order function, and for all $x$, the sequence $\langle h_s(x)\rangle_{s<\omega}$ is nonincreasing; and
- for every order function $g$, for almost all $x$, $h(x) < g(x)$.

Now using the fact that there is a uniformly c.e., uniformly low enumeration of the sets in $\mathcal{A}$, quoting Lemma 2.2, we can effectively enumerate a sequence of pairs $\langle\langle A_{e,s}, \Gamma_{e,s}\rangle, \Phi_e\rangle$ such that for all $e$,

(1) $\langle A_{e,s}\rangle$ is an effective enumeration of a c.e. set $A_e \in \mathcal{A}$;
(2) $\langle \Gamma_{e,s}\rangle$ is an enumeration of a pc functional $\Gamma_e$ such that $J^{A_e} = \Gamma_e^{A_e}$;
(3) $\left\langle \Gamma_e^{A_e}[s]\right\rangle$ is a computable approximation; and
(4) $\Phi_e$ is a Turing functional;

and such that for every $A \in \mathcal{A}$ and every Turing functional $\Phi$, there is some $e$ such that $A = A_e$ and $\Phi = \Phi_e$.

The requirement $R_e$ is: if $\Phi_e(Y) = A_e$, then there is a run of a strategy associated with $R_e$ which can build traces for $J^{A_e}$.

4.2. **The tree of strategies.** As mentioned above, we work with a tree of strategies. The $e^{\text{th}}$ level of the tree consists of nodes that work for $R_e$. To define the tree, we mention the possible outcomes of a node $\alpha$ that works for a requirement $R_e$. The possible outcomes are:

- $\infty$, indicating that $\alpha$ succeeds in building approximations; and
- for every pair $(\tau, \sigma)$ of finite binary strings, the outcome $(\tau, \sigma)$, indicating that some attempt to trace $J^{A_e}$ fails. Here $\sigma$ is assumed to be an initial segment of $Y$, and $\tau$ an initial segment of $A_e$ that is not computed correctly by $\Phi_e(Y)$.

To each node $\alpha$ on the tree of strategies, we associate a $\Pi^0_1$-subclass of $\mathcal{P}$, defined by recursion on the height of $\alpha$:

(1) $\mathcal{P}^{\langle\rangle} = \mathcal{P}$;
(2) $\mathcal{P}^{\alpha\frown\infty} = \mathcal{P}^{\alpha}$;
(3) $\mathcal{P}^{\alpha\frown(\tau,\sigma)} = \{X \in \mathcal{P}^{\alpha} \cap [\sigma] \,:\, \Phi_e(X) \not\supseteq \tau\}$,

where $[\sigma] = \{X : X \supset \sigma\}$. Note that if $\alpha \subseteq \beta$, then $\mathcal{P}^{\beta} \subseteq \mathcal{P}^{\alpha}$.

We assume that every $\Pi^0_1$ class $\mathcal{P}^{\alpha}$ in the construction comes with a co-enumeration $\langle \mathcal{P}^{\alpha}_s \rangle$: a computable, decreasing sequence of clopen classes such that $\bigcap_s \mathcal{P}^{\alpha}_s = \mathcal{P}^{\alpha}$. We may also assume that these enumerations respect the recursive definitions of the classes $\mathcal{P}^{\alpha}$: so for example, if we define $\mathcal{P}^{\beta} \subseteq \mathcal{P}^{\alpha} \cap [\sigma]$ for some immediate predecessor $\alpha$ of $\beta$, then for all $s$, $\mathcal{P}^{\beta}_s \subseteq [\sigma]$.

4.3. **Construction.** At every stage $s$ we define a sequence of nodes on the tree of strategies that are *accessible* at that stage. At the end of the stage we define a (finite) set $Y_s \in 2^{\omega}$.

The sequence of accessible nodes is defined by recursion. The root $\langle\rangle$ is always accessible.

Suppose that a node $\alpha$, working for requirement $R_e$, is accessible at stage $s$. If $e < s$, we need to determine which outcome of $\alpha$ is next accessible. There are several cases.

(1) If $s$ is the first stage greater than $e$ at which $\alpha$ is accessible, we let $\infty$ be the accessible outcome.

Otherwise, let $t$ be the previous stage at which $\alpha$ was accessible.

(2) Suppose that at stage $t$, some $(\tau, \sigma)$ was the accessible outcome of $\alpha$. If either
 (a) $\mathcal{P}^{\alpha\frown(\tau,\sigma)}_s$ is empty, or
 (b) $\tau \not\subset A_{e,s}$,
 then we let $\infty$ be the accessible outcome.
 Otherwise, we let $(\tau, \sigma)$ be accessible again.
(3) Suppose that at stage $t$, $\infty$ was the accessible outcome of $\alpha$.
 For $x \in \operatorname{dom} \Gamma^{A_e}_e [s]$ with use $u$, let $\tau = A_{e,s}\restriction_u$ and $\sigma = Y_t\restriction_{h_s(x)}$.
 We look for some such $x > \alpha$ such that $\mathcal{P}^{\alpha\frown(\tau,\sigma)}_s$ is nonempty, for $\tau$ and $\sigma$ corresponding to $x$ as above. If there is such $x$, we pick the least one, and let $(\tau, \sigma)$ be $\alpha$'s accessible outcome. We say that the outcome $(\tau, \sigma)$ is chosen *due to* $x$. If there is no such $x$, we let $\infty$ be the accessible outcome.

When an accessible node $\delta_s$ of length $s$ is reached, we halt the stage. Note that by induction on the length of the accessible nodes, if $\alpha$ is accessible at stage $s$, then $\mathcal{P}^{\alpha}_s$ is nonempty.

We then let $Y_s$ be the leftmost branch of $\mathcal{P}^{\delta_s}_s$. This ends stage $s$, and the description of the construction.

4.4. **Verification.** The instructions for which outcomes are accessible imply that the *true path* — the path of nodes that are leftmost with respect to being accessible at infinitely many stages — is infinite. If $\alpha$ is a node that is accessible infinitely often, then either $\alpha\frown\infty$ is accessible infinitely often, or there is some outcome $(\tau, \sigma)$ such that for almost all $s$, if $\alpha$ is accessible at stage $s$, then $\alpha\frown(\tau, \sigma)$ is also accessible at stage $s$.

Let $\alpha$ be a node, and suppose that $\alpha\frown\infty$ is accessible infinitely often. We can show that for each finitary outcome $(\tau, \sigma)$, the node $\alpha\frown(\tau, \sigma)$ is accessible at only finitely many stages.

**Lemma 4.2.** *Let $\alpha$ be a node and let $\tau$ and $\sigma$ be strings. There is at most one stage $s$ at which $\alpha^\frown(\tau,\sigma)$ is accessible, such that at the previous stage $t$ at which $\alpha$ was accessible, $\alpha^\frown\infty$ was accessible.*

*Proof.* Suppose, for contradiction, that there are two such stages $s_0 < s_1$. By the instructions, we know that there is a stage $t \in (s_0, s_1)$ such that $\alpha^\frown\infty$ is accessible at stage $t$, such that if we let $s$ be the last stage before $t$ at which $\alpha$ was accessible, $\alpha^\frown(\tau,\sigma)$ is accessible at stage $s$.

There are two possible reasons that $(\tau,\sigma)$ is not accessible at stage $t$: either $\tau \not\subset A_t$, or $\mathcal{P}_t^{\alpha^\frown(\tau,\sigma)} = \emptyset$. Since $A$ is c.e., and $\tau \subseteq A_s$, we know that if $\tau \not\subset A_t$, then $\tau \not\subset A_{s_1}$. If $\mathcal{P}_t^{\alpha^\frown(\tau,\sigma)} = \emptyset$, then certainly $\mathcal{P}_{s_1}^{\alpha^\frown(\tau,\sigma)} = \emptyset$. So in either case, we cannot choose $(\tau,\sigma)$ as an outcome at stage $s_1$. □

Hence there are two possibilities for an outcome $(\tau,\sigma)$ of a node $\alpha$ which is first chosen at some stage $s_0$: either it is always accessible whenever $\alpha$ is accessible at a stage $s \geqslant s_0$; or at the least stage $s > s_0$ at which $\alpha$ is accessible and $\alpha^\frown(\tau,\sigma)$ is not, we have $\alpha^\frown\infty$ accessible, and $\alpha^\frown(\tau,\sigma)$ is never accessible after stage $s$.

**Lemma 4.3.** *Let $\alpha$ be a node and let $x < \omega$. There are only finitely many outcomes $(\tau,\sigma)$ which are ever chosen due to $x$.*

*Proof.* Let $R_e$ be the requirement for which $\alpha$ works. Suppose, for contradiction, that there are infinitely many outcomes $(\tau,\sigma)$ of $\alpha$ which are chosen due to $x$. Of course, each such outcome is chosen at a distinct stage, so there are infinitely many stages $s$ at which a new outcome $(\tau,\sigma)$ is chosen for $\alpha$ due to $x$.

Let $s$ be any such stage. Then we have $x \in \operatorname{dom}\Gamma_e^{A_e}[s]$. Since there are infinitely many such stages $s$, and since $\lim \Gamma_e^{A_e}[s] = \Gamma_e^{A_e} = J^{A_e}$, we have $x \in \operatorname{dom}\Gamma_e^{A_e}$. Let $u$ be the use of this computation, and let $\tau^* = A_e\restriction_u$. There is some stage $s_0$ such $\tau^* \subset A_{s_0}$; so for all $s \geqslant s_0$, $\tau^* \subset A_s$.

Hence if $(\tau,\sigma)$ is chosen due to $x$ at stage $s \geqslant s_0$, we have $\tau = \tau^*$. Also, we have $|\sigma| = h_s(x) \leqslant h_{s_0}(x)$, so there are at most $2^{h_{s_0}(x)+1} - 1$ many such strings $\sigma$. That is, there are only finitely many outcomes that can be chosen due to $x$ after stage $s_0$. Of course, there are only finitely many outcomes which are chosen prior to stage $s_0$. This is the contradiction. □

**Corollary 4.4.** *Suppose that $\alpha^\frown\infty$ is accessible infinitely often. For every $n$, there are at most finitely many stages at which some $\alpha^\frown(\tau,\sigma)$ with $|\sigma| \leqslant n$ is accessible.*

*Proof.* Because $h$ is unbounded, there are only finitely many inputs $x$ such that for any stage $s$ we have $h_s(x) \leqslant n$. The corollary then follows from Lemmas 4.2 and 4.3. □

**Lemma 4.5.** *If $\alpha^\frown(\tau,\sigma)$ is on the true path, then $\sigma \subset Y_s$ for almost all $s$.*

*Proof.* Let $s^*$ be a stage such that:

- $\alpha^\frown(\tau,\sigma)$ is accessible at stage $s^*$.
- If $\beta^\frown(\tau',\sigma') \subseteq \alpha^\frown(\tau,\sigma)$, then for all $s \geqslant s^*$, if $\beta$ is accessible at stage $s$, then $\beta^\frown(\tau',\sigma')$ is also accessible at stage $s$.
- If $\beta^\frown\infty \subseteq \alpha$, then for all $s \geqslant s^*$, if $\beta^\frown(\tau',\sigma')$ is accessible at stage $s$, then $|\sigma'| \geqslant |\sigma|$.

Corollary 4.4 ensures that a stage that meets the third condition can be found. By induction on $s \geqslant s^*$, we show that $\sigma \subset Y_s$. Certainly at $s = s^*$, and any other stage at which $\alpha^\frown(\tau,\sigma)$ is accessible, we have $\alpha^\frown(\tau,\sigma) \subseteq \delta_s$ and so

$$\mathcal{P}_s^{\delta_s} \subseteq \mathcal{P}_s^{\alpha^\frown(\tau,\sigma)} \subseteq [\sigma],$$

so since $Y_s \in \mathcal{P}_s^{\delta_s}$, we have $\sigma \subset Y_s$. Now suppose that that $\alpha^\frown(\tau,\sigma)$ is not accessible at stage $s > s^*$. Since $s > s^*$, there must be some $\beta$ and some $(\tau',\sigma')$ such that $\beta^\frown\infty \subseteq \alpha$, but $\beta^\frown(\tau',\sigma')$ is accessible at stage $s$. Let $t$ be the last stage prior to $s$ at which $\beta^\frown\infty$ is accessible; $t \geqslant s^*$. Let $s'$ be the least stage after stage $t$ at which $\beta$ was accessible; so $s' \leqslant s$ and $\beta^\frown(\tau',\sigma')$ was accessible at stage $s'$. By induction, $\sigma \subset Y_t$. Since $s > s^*$, we have $|\sigma'| \geqslant |\sigma|$.

Now by the instructions of case (3) of the construction, at stage $s'$ we choose $\sigma' \subset Y_t$. Since $\sigma \subset Y_t$, $\sigma$ and $\sigma'$ are comparable; since $|\sigma| \leqslant |\sigma'|$, we must have $\sigma \subseteq \sigma'$.

Since $\beta^\frown(\tau',\sigma')$ is accessible at stage $s$, we must have $\sigma' \subset Y_s$. Hence $\sigma \subset Y_s$ as required. □

**Lemma 4.6.** *For any $n$, there is some node $\alpha^\frown(\tau,\sigma)$ on the true path such that $|\sigma| \geqslant n$.*

*Proof.* It is sufficient to show that there are infinitely many nodes $\alpha$ on the true path such that $\alpha^\frown\infty$ is not on the true path, the reason being that if the true outcome $(\tau,\sigma)$ of $\alpha$ is chosen due to $x$, then $x > \alpha$ and so $|\sigma| \geqslant h(x) \geqslant h(\alpha)$; and $h$ is unbounded.

By fudging our functionals, we may assume that the use of any computation $\Gamma_e^{A_e}(x) = y$ is never 0. For any set $A \in \mathcal{A}$, let $\Phi$ be a Turing functional such that for all $X$, $\Phi^X \perp A$, for example, we can have $\Phi^X(0)\downarrow \neq A(0)$ for all $X$. Let $e$ be an index such that $A_e = A$ and $\Phi_e = \Phi$. Let $\alpha$ on the true path work for $R_e$.

For every nonzero $\tau \subset A_e$, for any $\sigma$, we have $\mathcal{P}^{\alpha^\frown(\tau,\sigma)} = \mathcal{P}^\alpha \cap [\sigma]$. The set $\operatorname{dom} J^{A_e} = A_e'$ is infinite; let $x$ be the least element of $\operatorname{dom} J^{A_e}$ which is greater than $\alpha$. Let $u$ be the use of the computation $x \in \operatorname{dom} \Gamma_e^{A_e}$; by assumption, $u > 0$. Let $\tau = A_e\restriction_u$.

We call a stage $s$ *late* if:

- $\tau \subset A_{e,s}$ and $x \in \operatorname{dom} \Gamma_e^{A_e}[s]$;
- for all $x' \in (\alpha, x)$, $x' \notin \operatorname{dom} \Gamma_e^{A_e}[s]$;
- $h_s(x) = h(x)$;
- for all $\sigma$ of length $h(x)$, if $\mathcal{P}^\alpha \cap [\sigma] = \emptyset$, then $\mathcal{P}_s^\alpha \cap [\sigma] = \emptyset$.

Suppose that $\alpha^\frown\infty$ is accessible at some late stage $t$; let $s$ be the next stage at which $\alpha$ is accessible. Let $\sigma = Y_t\restriction_{h(x)}$. We have $Y_t \in \mathcal{P}_t^\alpha$, so $\mathcal{P}_t^\alpha \cap [\sigma] \neq \emptyset$; so $\mathcal{P}^\alpha \cap [\sigma] \neq \emptyset$; so $\mathcal{P}_s^\alpha \cap [\sigma] \neq \emptyset$. Then at stage $s$, we will choose the outcome $(\tau,\sigma)$ for $\alpha$ because of $x$, and never revert to $\infty$. □

**Corollary 4.7.** *The sequence $\langle Y_s \rangle$ has a limit $Y$, which is a $\Delta_2^0$ element of $\mathcal{P}$.*

*Proof.* Let $n < \omega$. By Lemmas 4.5 and 4.6, there is some $\sigma$ of length $n$ such that for almost all $n$, $\sigma \subset Y_s$. Then $Y = \lim_s Y_s$ exists, and is, of course, $\Delta_2^0$. For each $n$, for $\sigma = Y\restriction_n$, we have, for almost all $s$, $\sigma \subset Y_s$, and $Y_s \in \mathcal{P}_s^{\delta_s} \subseteq \mathcal{P}_s^{\langle\rangle} = \mathcal{P}_s$, so for almost all $s$ we have $[\sigma] \cap \mathcal{P}_s \neq \emptyset$; it follows that $[\sigma] \cap \mathcal{P} \neq \emptyset$. Since $\mathcal{P}$ is closed, we have $Y \in \mathcal{P}$. □

In fact, the same argument shows that if $\alpha$ is on the true path, then $Y \in \mathcal{P}^\alpha$.

**Lemma 4.8.** *Let $\alpha$ be the node on the true path that works for requirement $R_e$. If $\Phi_e(Y) = A_e$ then $\alpha^\frown\infty$ is on the true path.*

*Proof.* Otherwise, we have $\alpha^\frown(\tau,\sigma)$ on the true path, and so we have $Y \in \mathcal{P}^{\alpha^\frown(\tau,\sigma)}$. But then $\tau \subset A_e$, and for every $X \in \mathcal{P}^{\alpha^\frown(\tau,\sigma)}$ we have $\Phi_e(X) \not\supseteq \tau$, which contradicts the assumption that $\Phi_e(Y) = A_e$. □

The following lemma completes the verification, and so the proof of Proposition 3.2.

**Lemma 4.9.** *Let $\alpha$ be the node on the true path that works for requirement $R_e$, and suppose that $\alpha^\frown\infty$ is on the true path. Then $A_e$ is strongly jump-traceable.*

*Proof.* We verify that for every order function $g$, $J^{A_e}$ has a trace bounded by $g$. Fix an order function $g$. Let $x^* \geqslant \alpha$ such that for all $x \geqslant x^*$, $2^{h(x)} < g(x)$.

We enumerate a trace $\langle S_x \rangle_{x \geqslant x^*}$. Let $x \geqslant x^*$. Suppose that at a stage $s$:

(1) $\alpha$ is accessible at stage $s$, and at the previous stage $t$ at which $\alpha$ was accessible, $\alpha^\frown\infty$ was accessible.
(2) $\Gamma_e^{A_e}(x)\downarrow\ [s]$, with a use $u$.
(3) $2^{h_s(x)} < g(x)$.
(4) Letting $\sigma = Y_t\restriction_{h_s(x)}$ and $\tau = A_{e,s}\restriction_u$, we have $\mathcal{P}_s^{\alpha^\frown(\tau,\sigma)} = \emptyset$.

At such a stage, we enumerate $\Gamma_e^{A_e}(x)\,[s]$ into $S_x$.

**Claim 4.10.** *For all $x \geqslant x^*$, we have $|S_x| \leqslant g(x)$.*

*Proof.* Suppose that a value $y$ is enumerated into $S_x$ at a stage $s$. Let $\tau$ and $\sigma$ be the associated strings. For all $X \in \mathcal{P}_s^\alpha \cap [\sigma]$, we have $\Phi_e(X) \supseteq \tau$, and $\Gamma_e(\tau, x) = y$. Hence every $\sigma$ can be responsible for at most one value $y$ in $S_x$. Since $|\sigma| = h_s(x)$ and $2^{h_s(x)} < g(x)$, there are at most $g(x)$ many such strings $\sigma$. Hence $|S_x| \leqslant g(x)$. □

**Claim 4.11.** *For all $x \geqslant x^*$, if $x \in \operatorname{dom} J^{A_e}(x)$, then $J^{A_e}(x) \in S_x$.*

*Proof.* Let $x \geqslant x^*$.

Let $u$ be the use of $J^{A_e}(x)$, and let $\tau = A_e\restriction_u$. Of course, for almost all $s$, we have $\tau \subset A_{e,s}$. By Lemma 4.3, for almost all stages $s$, we do not pick an outcome $(\tau,\sigma)$ for $\alpha$ due to $x$. For large enough $s$, we have $2^{h_s(x)} < g(x)$. So if $t$ is a late stage at which $\alpha^\frown\infty$ is accessible, and $s$ is the next stage at which $\alpha$ is accessible, then the conditions (1), (2), and (3) for enumerating $J^{A_e}(x) = \Gamma_e^{A_e}(x)\,[s]$ at stage $s$ hold. But also, condition (4) holds, for otherwise we could choose $(\tau,\sigma)$ as an outcome at stage $s$ due to $x$, as $x > \alpha$. Hence, the correct value $J^{A_e}(x)$ gets enumerated into $S_x$. □

The two claims together prove Lemma 4.9, and so complete the proof of Proposition 3.2. □

## 5. Proof of Proposition 3.1

We discuss the additions we need to make in the previous proof to obtain a proof of Proposition 3.1. Let $\langle A_e, g_e, \langle C_{e,s}\rangle, \Phi_e\rangle$ be an effective enumeration of all tuples such that $A_e$ is a c.e. set, $g_e$ is a partial computable function, $\langle C_{e,s}\rangle$ is a uniformly computable sequence of sets, and $\Phi_e$ is a Turing functional, such that for all $n$,

- $C_{e,0}(n) = 0$;

- If $n \in \operatorname{dom} g_e$, then $\#\{s : C_{e,s+1}(n) \neq C_{e,s}(n)\} \leqslant g_e(n)$;
- If $n \notin \operatorname{dom} g_e$, then for all $s$, $C_{e,s}(n) = 0$.

We construct a pc functional $\Theta$ – we view $\Theta^{A_e}$ as a column of $J^{A_e}$ that we control. If $A$ is a superlow c.e. set, then for every Turing functional $\Phi$ there will be some $e$ such that:

- $A = A_e$ and $\Phi = \Phi_e$;
- $g_e$ is total and $\lim_s C_{e,s} = \operatorname{dom} \Theta^{A_e}$.

Then, we will be able to verify apparent computations $J^{A_e}(x)\downarrow [s]$ to ensure that only finitely many such computations will be believable (for any $x$). Essentially, using $C_{e,s}$ and $\Theta$, we obtain a pc functional $\Gamma_e$ such that $\Gamma_e^{A_e} = J^{A_e}$ and such that $\left\langle \Gamma_e^{A_e}[s] \right\rangle$ is a computable approximation, indeed an $\omega$-c.e. approximation.

Even though we did not use superlowness in the proof of Proposition 3.2, as lowness sufficed, we need to guess for superlowness in the current proof. It is crucial that the mind-changes of $\langle C_{e,s} \rangle$ are bounded by $g_e$; without such a bound, we cannot effectively enumerate sequences such as $\langle C_{e,s} \rangle$ that we know necessarily converge to some limit. This enumeration is needed to verify a lemma analogous to Lemma 4.3.

Given the list above, and the "super slow order function" $h$ from the previous section, we can describe a construction. The requirement $R_e$ is now: if $g_e$ is total and $\lim_s C_{e,s} = \operatorname{dom} \Theta^{A_e}$, and $\Phi_e(Y) = A$, then there is a run of a strategy associated with $R_e$ which succeeds in building traces for $J^{A_e}$.

If $\alpha$ is a node that works for requirement $R_e$, then in addition to the outcomes $\infty$ and $(\tau, \sigma)$ which were described above, we have new outcomes $\mathtt{fin}_\sigma$, one for each binary string $\sigma$, indicating that $g_e$ is not total, or that a permanent disagreement between $\lim_s C_{e,s}$ and $\operatorname{dom} \Theta^{A_e}$ can be found.

The definition of $\mathcal{P}^\alpha$ for all nodes $\alpha$ proceeds by recursion as before, with the added clause that $\mathcal{P}^{\alpha^\frown \mathtt{fin}_\sigma} = \mathcal{P}^\alpha \cap [\sigma]$.

5.1. **Construction.** For every $e$, let $\langle A_{e,s} \rangle$ be an enumeration of $A_e$. We also fix an enumeration $\langle J_s \rangle$ of the universal pc functional $J$.

To avoid clashes, we appoint, for every node $\alpha$, a column $\omega^{[\alpha]}$ of numbers. The node $\alpha$ will make definitions of $\Theta^{A_e}(m)$ only for numbers $m \in \omega^{[\alpha]}$. For any $x < \omega$, let $m_{\alpha,x} = \langle \alpha, x \rangle$ be the $x^{\text{th}}$ element of $\omega^{[\alpha]}$.

We say that a node $\alpha$ is *active* at stage $s > |\alpha|$ if it is accessible at stage $s$, and $\alpha$'s outcome at stage $s$ is either $\infty$ or some $(\tau, \sigma)$ (so not $\mathtt{fin}_\sigma$ for any $\sigma$).

Suppose that a node $\alpha$, working for $R_e$, is accessible at stage $s > |\alpha|$. We let $t$ be the last stage before $s$ at which $\alpha$ was *active*. (If $s$ is the least stage after stage $|\alpha|$ at which $\alpha$ is accessible, then we let $\alpha^\frown\infty$ be accessible at stage $s$.)

We first determine whether $\alpha$ is active at stage $s$: it is active if the following two conditions hold:

(i) For all $x < t$, $g_e(m_{\alpha,x})\downarrow$ at stage $s$; and
(ii) for all $x < t$, $C_{e,s}(m_{\alpha,x}) = 1$ iff $\Theta^{A_e}(m_{\alpha,x})\downarrow [s]$.

If $\alpha$ is inactive at stage $s$, then the outcome if $\mathtt{fin}_\sigma$, where $\sigma$ is the longest initial segment of $Y_t$ of length at most $t$ such that $\mathcal{P}^\alpha_s \cap [\sigma] \neq \emptyset$.

Suppose that $\alpha$ was determined to be active at stage $s$. First, we determine the outcome of $\alpha$.

(1) Suppose that at stage $t$, some $(\tau,\sigma)$ was the accessible outcome of $\alpha$. If either $\mathcal{P}_s^{\alpha\frown(\tau,\sigma)}$ is empty or $\tau \not\subset A_{e,s}$, then we let $\infty$ be the accessible outcome; otherwise, we let $(\tau,\sigma)$ again be the accessible outcome.

(2) Suppose that at stage $t$, $\infty$ was the accessible outcome of $\alpha$. We search for $x \in (\alpha, t)$ such that $\Theta^{A_e}(m_{\alpha,x})\downarrow [s]$. For such $x$, let $u$ be the use of that computation, let $\tau = A_{e,s}\restriction_u$, and let $\sigma = Y_t\restriction_{h_s(x)}$.

If there is such an $x$ such that $\mathcal{P}_s^{\alpha\frown(\tau,\sigma)}$ is non-empty, we pick the least such $x$, and let $(\tau,\sigma)$ be the outcome. If there is no such $x$, then we let $\infty$ be the outcome.

Before yielding control to the next accessible node, we update $\Theta$: for every $x < s$ in $\operatorname{dom} J^{A_e}[s]$, we let $\Theta^{A_e}(m_{\alpha,x})\downarrow$ with the same use (by enumerating the axiom $(\tau, m_{\alpha,x}) \mapsto 0$ into $\Theta$, where $u$ is the use of $J^{A_e}(x)\downarrow [s]$ and $\tau = A_{e,s}\restriction_u$).

We halt the stage when we get to $\delta_s$, the node, accessible at stage $s$, of length $s$. We let $Y_s$ be the leftmost element of $\mathcal{P}_s^{\delta_s}$.

5.2. **Verification.**

**Lemma 5.1.** *Suppose that $A$ is a superlow c.e. set and $\Phi$ is a Turing functional. Then there is some $e$ such that $A_e = A$, $\Phi_e = \Phi$, $g_e$ is total, and $\lim_s C_{e,s} = \operatorname{dom}\Theta^{A_e}$.*

*Proof.* Immediate, since $\operatorname{dom}\Theta^A$ is $A$-computably enumerable, so is $\omega$-c.e. □

The following is also immediate:

**Lemma 5.2.** *If $g_e$ is total, and $\lim C_{e,s} = \operatorname{dom}\Theta^{A_e}$, and $\alpha$, working for $R_e$, is accessible infinitely often, then $\alpha$ is active infinitely often.*

*Proof.* We only make new computations $\Theta^{A_e}(m_{\alpha,x})$ when $\alpha$ is active. □

Lemma 5.2 implies that a true path exists. If $\alpha$ is active infinitely many times, and no outcome $(\tau,\sigma)$ is eventually accessible whenever $\alpha$ is active, then $\infty$ is accessible infinitely many times. If there is a last stage $t$ at which $\alpha$ is active, then at every stage $s > t$ at which $\alpha$ is active, the outcome of $\alpha$ is $\texttt{fin}_{\sigma_s}$ where $\sigma_s \subset Y_t\restriction_t$; and if $s < s'$ are two such stages, then $\sigma_s \supseteq \sigma_{s'}$. Hence, if $\alpha$ is not active infinitely often, there is an outcome $\texttt{fin}_\sigma$ which eventually is accessible whenever $\alpha$ is accessible.

We also mention that at any stage $s$, if $\Theta^{A_e}(m_{\alpha,x})\downarrow [s]$, then $J^{A_e}(x)\downarrow [s]$ with the same use; we use the fact that $A_e$ is c.e.

**Lemma 5.3.** *Suppose that $\alpha$, working for $R_e$, is active infinitely often. Let $x < \omega$, and suppose that there are infinitely many stages at which $\alpha\frown(\tau,\sigma)$ is accessible, due to $x$. Then $x \in \operatorname{dom} J^{A_e}$.*

*Proof.* Let $m = m_{\alpha,x}$. Let $s^*$ be a stage such that for all $s \geqslant s^*$, $C_{e,s}(m) = C_{e,s^*}(m)$. If $\alpha\frown(\tau,s)$ is accessible at a stage $s$ due to $x$, then $\Theta^{A_e}(m)\downarrow [s]$ and $C_{e,s}(m) = 1$. Hence $C_{e,s^*}(m) = 1$.

Let $t \geqslant s^*$ be a stage at which some $\alpha\frown(\tau,\sigma)$ is accessible due to $x$. Then $x \in \operatorname{dom} J^\tau$. To show that $x \in \operatorname{dom} J^{A_e}$, we show that $\tau \subset A_e$. If not, let $s > t$ be the least stage at which $\alpha$ is active such that $\tau \not\subset A_{e,s}$. Then $\Theta_e^{A_e}(m)\uparrow [s]$; since $\alpha$ is active at stage $s$, we have $C_{e,s}(m) = 0$, contradicting $s > s^*$. □

Using Lemma 5.3, the proof of Lemma 4.3 carries over to the current construction. The same holds for Corollary 4.4.

Lemma 4.5 holds too:

**Lemma 5.4.** *If $\alpha^\frown(\tau,\sigma)$ is on the true path, then for almost all $s$, $\sigma \subset Y_s$.*

*Proof.* We need to update the proof of Lemma 4.5, by considering the outcomes $\mathtt{fin}_\sigma$. We find a stage $s^*$ satisfying:

- $\alpha^\frown(\tau,\sigma)$ is accessible at stage $s^*$.
- If $\beta^\frown(\tau',\sigma') \subseteq \alpha^\frown(\tau,\sigma)$, then for all $s \geqslant s^*$, if $\beta$ is *active* at stage $s$, then $\beta^\frown(\tau',\sigma')$ is also accessible at stage $s$.
- If $\beta^\frown\infty \subseteq \alpha$, then for all $s \geqslant s^*$, if $\beta^\frown(\tau',\sigma')$ is accessible at stage $s$, then $|\sigma'| \geqslant |\sigma|$.
- If $\beta^\frown\mathtt{fin}_\sigma \subseteq \alpha$, then for all $s \geqslant s^*$, if $\beta$ is accessible at stage $s$, then $\beta^\frown\mathtt{fin}_\sigma$ is accessible at stage $s$.

We argue by induction on $s \geqslant s^*$ that $\sigma \subset Y_s$. The argument of the proof of Lemma 4.5 goes through, but there is one new possibility: that there is some $\beta$ such that $\beta^\frown\infty \subseteq \alpha$ or $\beta^\frown(\tau',\sigma') \subseteq \alpha^\frown(\tau,\sigma)$, but $\beta^\frown\mathtt{fin}_\rho$ is accessible at stage $s$. In this case, we let $t$ be the previous stage at which $\beta$ was *active*. By induction, $\sigma \subset Y_t$. The fact that $\alpha^\frown(\tau,\sigma)$ is on the true path implies that $[\sigma] \cap \mathcal{P}^\beta \neq \emptyset$. Hence $\sigma \subseteq \rho$, and $\rho \subset Y_s$. □

The proof of Lemma 4.6 goes through if we choose (by Lemma 5.1) $e$ such that $A_e$ is superlow, and such that $g_e$ and $\langle C_{e,s}\rangle$ indeed witness $A_e$'s superlowness. By Lemma 5.2, if $\alpha$ on the true path works for $R_e$, then the outcome of $\alpha$ is either $\infty$ or some $(\tau,\sigma)$; the proof above of Lemma 4.6 now shows that indeed the outcome must be $(\tau,\sigma)$ for some $\sigma$ which can be made as long as we like. Hence we again get $Y = \lim_s Y_s$ exists, and is an element of $\mathcal{P}$, and indeed of $\mathcal{P}^\alpha$ for every $\alpha$ on the true path.

The conclusion of the verifications follows the proof of Lemma 4.9, noting that if $\Phi_e(Y) = A_e$, and if $g_e$ is total and $\lim_s C_{e,s} = \operatorname{dom}\Theta^{A_e}$, then for the corresponding $\alpha$, $\alpha^\frown\infty$ is on the true path. In the instructions for enumerating $\langle S_x\rangle$, we replace $\Gamma_e^{A_e}(x)\downarrow [s]$ with $\Theta^{A_e}(m_{\alpha,x})\downarrow [s]$. To ensure that if $x \in \operatorname{dom} J^{A_e}$ then $J^{A_e}(x) \in S_x$, we note that in this case we will have $\Theta^{A_e}(m_{\alpha,x})\downarrow$ with the correct use. The rest follows in the same way.

## 6. Proof of Theorem 1.2

To prove Theorem 1.2, we adapt the construction of the proof of Proposition 3.1.

**Definition 6.1.** Let $g$ be an order function. We say that a set $A$ is *$g$-superlow* if every set which is c.e. in $A$ is $g$-c.e.

We make use of the following fact, which follows from [9]:

*Fact* 6.2. There is a computable function $g$ such that for every $K$-trivial c.e. set $A$ is $g$-superlow.

Unfortunately, a $g$-c.e. approximation of $J^A$ for a $K$-trivial c.e. set $A$ cannot be obtained effectively from $A$, even if a $K$-triviality constant for $A$ is given. We thus will need to guess for an approximation as we did in the proof of Proposition 3.1.

We will prove the following:

**Proposition 6.3.** *Let $h$ and $g$ be order functions, and let $\mathcal{P}$ be a nonempty $\Pi^0_1$ class. There is an $\omega$-c.e. set $Y \in \mathcal{P}$ such that for every $g$-superlow c.e. set $A \leqslant_{\mathrm{T}} Y$, $J^A$ has a trace which is bounded by $2^h$.*

Theorem 1.2 follows by the argument proving Theorem 1.1, using a $\Pi^0_1$ class of random sets and the order function $g$ given by Fact 6.2. We recall (see, for example, [1]) that for every order function $h$ there is an slower order function $\widetilde{h}$ with the following property: for every set $A$, if $J^A$ has a trace bounded by $2^{\widetilde{h}}$, then every set which is c.e. in $A$ has a trace bounded by $h$.

The rest of the section is devoted to the proof of Proposition 6.3.

6.1. **Discussion; the strategies.** We mostly follow the proof of Proposition 3.1. Of course, we use the given order function $h$, rather than the "super slow order function" that was used before. Also, since $g$ is fixed, we do not need to wait for convergence of $g_e$.

The main difference is that in comparison to showing that $\langle Y_s \rangle$ is a computable approximation (in particular, Lemma 4.6), we need to be more proactive in order to show that in the current construction, $\langle Y_s \rangle$ will be an $\omega$-c.e. approximation. We add a new collection of requirements, $S_n$, which seek to stabilise $Y_s \restriction_n$. We also add a little information to the tree of strategies by giving more detailed outcomes.

We enumerate the tuples $\langle A_e, \langle C_{e,s} \rangle, \Phi_e \rangle$ where $A_e$ is a c.e. set, $\langle C_{e,s} \rangle$ is a $g$-c.e. approximation of some set $C_e$, and $\Phi_e$ is a Turing functional.

For a node (a strategy) $\alpha$, we define, by induction, which are the outcomes of $\alpha$, and which $\Pi^0_1$ classes are associated to them. Nodes of length $2e$ work for $R_e$, and nodes of length $2n+1$ work for $S_n$. As usual, we start with $\mathcal{P}^{\langle\rangle} = \mathcal{P}$. Assume that $\alpha$ is a node on the tree, and that $\mathcal{P}^\alpha$ has been defined.

- If $\alpha$ is a strategy which works for $R_e$, then the outcomes and trees associated with the immediate descendants of $\alpha$ are as follows:
  - An outcome $\infty$, with $\mathcal{P}^{\alpha\frown\infty} = \mathcal{P}^\alpha$;
  - For binary strings $\sigma$ and $\tau$, an outcome $(\tau, \sigma)$, with
  $$\mathcal{P}^{\alpha\frown(\tau,\sigma)} = \{X \in \mathcal{P}^\alpha \cap [\sigma] \,:\, \Phi_e(X) \not\supseteq \tau\}\,;$$
  - For $t < \omega$ and a binary string $\sigma$, an outcome $\mathtt{fin}_{t,\sigma}$, with
  $$\mathcal{P}^{\alpha\frown\mathtt{fin}_{t,\sigma}} = \mathcal{P}^\alpha \cap [\sigma].$$
- If $\alpha$ is a strategy which works for $S_n$, then the outcomes are the binary strings of length $n$. For $\sigma$ of length $n$, $\mathcal{P}^{\alpha\frown\sigma} = \mathcal{P}^\alpha \cap [\sigma]$.

Again, we enumerate a pc functional $\Theta$, and for $\alpha$ on the tree of strategies, let $m_{\alpha,x} = \langle \alpha, x \rangle$.

6.2. **Construction.** At stage $s$ we define the (infinite) path of nodes which are accessible at stage $s$, by recursion, starting at the root. We also define a (computable) set $Y_s$. By recursion, if $\alpha$ is accessible at stage $s$, then $\mathcal{P}^\alpha_s \neq \emptyset$.

Suppose that $\alpha$ is a node which is accessible at stage $s$ and works for $S_n$. We let the accessible outcome of $\alpha$ be the leftmost string $\sigma$ of length $n$ such that $\mathcal{P}^\alpha_s \cap [\sigma]$ is nonempty.

Suppose that $\alpha$ is a node which is accessible at stage $s$, and works for $R_e$. If $\alpha$ was not accessible at any stage $t < s$, then let $\infty$ be the accessible outcome (and declare $\alpha$ to be active).

Otherwise, let $t < s$ be the last stage before $s$ at which $\alpha$ was *active*. We check if for all $x < t$, $C_{e,s}(m_{\alpha,x}) = 1$ iff $\Theta^{A_e}(m_{\alpha,x})\downarrow [s]$. If not, the we declare $\alpha$ to be inactive at stage $s$, and let the outcome be $\mathtt{fin}_{t,\sigma}$, where $\sigma$ is the longest initial segment of $Y_t\restriction_t$ such that $\mathcal{P}^\alpha_s \cap [\sigma] \neq \emptyset$.

Otherwise, we declare $\alpha$ to be active at stage $s$. The rest of the instructions for choosing the accessible outcome of $\alpha$ and adding axioms to $\Theta$ are precisely as are described in the construction of the proof of Proposition 3.1 in Section 5.

We let $Y_s$ be the unique element of $\bigcap \mathcal{P}^\alpha$, as $\alpha$ ranges over the nodes which are accessible at stage $s$. Note that for every $n$, the determination of $Y_s\restriction_n$ is made after only finitely many steps of the stage, so the sequence $\langle Y_s \rangle$ is uniformly computable.

6.3. **Verification.** Our main task is to show that $\langle Y_s \rangle$ is an $\omega$-c.e. approximation. To do this, we need to improve the "finiteness" lemmas (4.3 and 5.3) into a "counting" lemma.

*Counting short accessible nodes.* The following notation will be useful.

(1) For any non-empty node $\beta$, let $p(\beta)$ be the last bit of $\beta$, and let $\bar\beta$ be the immediate predecessor of $\beta$. Thus $\beta = \bar\beta^\frown p(\beta)$.
(2) Let $\mathcal{N}$ be the collection of all non-empty nodes $\beta$ such that $p(\beta) \neq \infty$.
(3) If $\beta \in \mathcal{N}$ then there is some string $\sigma$ such that $p(\beta)$ is of the form $(\tau, \sigma)$, $\mathtt{fin}_{t,\sigma}$ or $\sigma$. We let $\rho(\beta) = \sigma$.
    If $\beta \in \mathcal{N}$, then $\mathcal{P}^\beta \subseteq \mathcal{P}^{\bar\beta} \cap [\rho(\beta)]$, so we may assume that $\mathcal{P}^\beta_s \subseteq [\rho(\beta)]$ at every stage $s$. Thus, if $\beta \in \mathcal{N}$ is accessible at a stage $s$, then $\rho(\beta) \subset Y_s$.
(4) Let $\mathcal{A}$ be the collection of all nodes which are accessible at some stage. It follows from (3) that if $\beta \in \mathcal{A} \cap \mathcal{N}$ and $\alpha \subset \beta$ is also in $\mathcal{N}$, then $\rho(\alpha)$ and $\rho(\beta)$ are comparable.
    We thus let for $\beta \in \mathcal{A}$,
$$\widehat{\rho}(\beta) = \bigcup \{\rho(\alpha) \,:\, \alpha \subseteq \beta \;\&\; \alpha \in \mathcal{N}\}.$$
(5) For $n < \omega$, we let
$$\mathcal{A}_n = \{\beta \in \mathcal{A} \,:\, |\widehat{\rho}(\beta)| \leqslant n\}.$$

**Lemma 6.4.** *The function $n \mapsto \#\mathcal{A}_n$ is bounded by a computable function.*

*Proof.* We note that every node of length $2n+3$ works for $S_{n+1}$, so for every node $\beta$ of length $2n+4$ we have $|\rho(\beta)| = n+1$. Hence every node in $\mathcal{A}_n$ has length at most $2n+3$. By definition of $\widehat{\rho}$, if $\beta \neq \langle\rangle$ is in $\mathcal{A}_n$, then $\bar\beta \in \mathcal{A}_n$. We show that there is a computable function $f$ such that every $\alpha \in \mathcal{A}_n$ has at most $f(n)$ many immediate (one-bit) extensions in $\mathcal{A}_n$. This suffices, because we then get that $\#\mathcal{A}_n \leqslant \sum_{k \leqslant 2n+3} f(n)^k$.

We need, thus, to define the function $f(n)$ appropriately. If $\alpha$ works for $S_m$ and has proper extensions in $\mathcal{A}_n$, then $m \leqslant n$, and so $\alpha$ has $2^m \leqslant 2^n$ many immediate extensions. So to take care of such $\alpha$'s, we just need to ensure that $f(n) \geqslant 2^n$.

Now, consider some $\alpha \in \mathcal{A}_n$ which works for $R_e$. We first investigate the number of outcomes $(\tau, \sigma)$ such that $\alpha^\frown(\tau, \sigma) \in \mathcal{A}_n$. Suppose that $(\tau, \sigma)$ is such an outcome. This outcome is chosen on account of some $x$; we have $|\sigma| = h(x)$, so $h(x) \leqslant n$. Also, we required that $x > \alpha$.

The proof of Lemma 5.3 yields that for each $x$, there are at most $g(m_{\alpha,x})$ many strings $\tau$ such that for any $\sigma$, an outcome $(\tau, \sigma)$ for $\alpha$ is obtained due to $x$. Let $x_n$ be the least number such that $h(x_n) > n$; $x_n$ can be found effectively from $n$. Since

$h$ is non-decreasing, if $h(x) \leqslant n$ then $x < x_n$. Hence, the total number of outcomes $(\tau, \sigma)$ such that $|\sigma| \leqslant n$ and $\alpha^\frown(\tau,\sigma) \in \mathcal{A}_n$ is bounded by

$$\max_{\alpha < x_n} \sum_{x < x_n} 2^{h(x)} g(m_{\alpha,x}),$$

which is computable given $n$.

Next, we investigate outcomes of the form $\mathtt{fin}_{t,\sigma}$. Fix $\sigma \in 2^{\leqslant n}$. Suppose that for $i < 2$, $\alpha^\frown\mathtt{fin}_{t_i,\sigma}$ is accessible at stage $s_i$, and that $|\sigma| < t_0 < t_1$. Then $t_0 < s_0 < t_1 < s_1$. For both $i < 2$ we have $\sigma \subset Y_{t_i}$. Let $m = |\sigma|$. We claim that $Y_{t_0}(m) \neq Y_{t_1}(m)$. For the fact that the outcome at stage $s_0$ does not extend $Y_{t_0}\restriction_{m+1}$ implies that $[Y_{t_0}\restriction_{m+1}] \cap \mathcal{P}^\alpha_{s_0} = \emptyset$, and so $[Y_{t_0}\restriction_{m+1}] \cap \mathcal{P}^\alpha_{t_1} = \emptyset$. However, $Y_{t_1} \in \mathcal{P}^\alpha_{t_1}$, so we must have $Y_{t_1}\restriction_{m+1} \neq Y_{t_0}\restriction_{m+1}$. Since $Y_{t_0}\restriction_m = Y_{t_1}\restriction_m = \sigma$, we must have $Y_{t_0}(m) \neq Y_{t_1}(m)$ as required.

It follows that there are at most two stages $t > |\sigma|$ such that $\alpha^\frown\mathtt{fin}_{t,\sigma} \in \mathcal{A}$. But if $\alpha^\frown\mathtt{fin}_{t,\sigma} \in \mathcal{A}$, then $|\sigma| \leqslant t$. Hence in total, there are at most three stages $t$ such that $\alpha^\frown\mathtt{fin}_{t,\sigma} \in \mathcal{A}$. Hence the total number of pairs $(t,\sigma)$ such that $\alpha^\frown\mathtt{fin}_{t,\sigma} \in \mathcal{A}_n$ is bounded by $3 \cdot 2^{n+1}$.

Thus, letting

$$f(n) = 1 + 3 \cdot 2^{n+1} + \max_{\alpha < x_n} \sum_{x < x_n} 2^{h(x)} g(m_{\alpha,x})$$

suffices. □

*Initial and final stages.* For $\beta \in \mathcal{A} \cap \mathcal{N}$, we define the notion of *initial* and *final* stages for $\beta$. We utilise the following notation: if $\alpha$ is accessible at stage $s$, we let $o(\alpha)[s]$ be $\alpha$'s outcome at stage $s$, and let $\alpha^+[s] = \alpha^\frown o(\alpha)[s]$ be the immediate extension of $\alpha$ which is accessible at stage $s$.

**Definition 6.5.** Let $\beta \in \mathcal{A} \cap \mathcal{N}$. We define a set $T(\beta)$ of stages:

- If $p(\beta) = \sigma$ for some $\sigma$, or $p(\beta) = \mathtt{fin}_{t,\sigma}$ for some $t$ and $\sigma$, then $T(\beta)$ is the collection of stages at which $\bar\beta$ is accessible.
- If $p(\beta) = (\tau, \sigma)$ for some $\tau$ and $\sigma$, then $T(\beta)$ is the set of stages at which $\bar\beta$ is active.

Note that if $\beta$ is accessible at stage $s$, then $s \in T(\beta)$.

For $s \in T(\beta)$ such that $s \neq \min T(\beta)$, we let $s^-(\beta) = \max(T(\beta) \cap s)$, that is, the greatest stage in $T(\beta)$ which is smaller than $s$.

(1) We say that $s$ is *initial* for $\beta$ if $\beta$ is accessible at stage $s$, $s \neq \min T(\beta)$, and $\beta$ is not accessible at stage $s^-(\beta)$.
(2) We say that $s$ is *final* for $\beta$ if $s \in T(\beta)$, $\beta$ is not accessible at stage $s$, $s \neq \min T(\beta)$, and $\beta$ is accessible at stage $s^-(\beta)$.

**Lemma 6.6.** *Let $\beta \in \mathcal{A} \cap \mathcal{N}$. If $s$ is a final stage for $\beta$, then $s^-(\beta)$ is the last stage at which $\beta$ is accessible.*

*Proof.* Let $t = s^-(\beta)$. Since $\beta$ is accessible at stage $t$, we have $p(\beta) = o(\bar\beta)[t]$; but $p(\beta) \neq o(\bar\beta)[s]$.

Let $r > t$. Suppose, for contradiction, that $\beta$ is accessible at stage $r$. Then $r \in T(\beta)$; so $r \geqslant s$.

There are three cases.

Suppose that $p(\beta) = \sigma$. Let $<_L$ be the lexicographic ordering on $2^{<\omega}$. We know that for all $u < v$ in $T(\beta)$, $o(\bar\beta)[u] \leqslant_L o(\bar\beta)[v]$. Since $\beta$ is accessible at stage

$t$ but not at stage $s$, we have $p(\beta) = o(\bar\beta)[t] <_L o(\bar\beta)[s]$. Since $r \geqslant s$, we have $o(\bar\beta)[s] \leqslant_L o(\bar\beta)[r]$. Hence $p(\beta) <_L o(\bar\beta)[r]$, so $\beta$ cannot be accessible at stage $r$.

Suppose that $p(\beta) = \mathtt{fin}_{u,\sigma}$. Let $v$, if it exists, be the least stage $v \geqslant s$ at which $\bar\beta$ is active. If $r < v$ (or if $v$ does not exist), then $o(\bar\beta)[s] = \mathtt{fin}_{u,\sigma'}$ and $o(\bar\beta)[r] = \mathtt{fin}_{u,\sigma''}$ where $\sigma \supsetneq \sigma' \supseteq \sigma''$, so $p(\beta) \neq o(\bar\beta)[r]$, so $\beta$ is not accessible at stage $r$. If $r > v$ and $o(\bar\beta)[r] = \mathtt{fin}_{w,\rho}$ then $w \geqslant v > u$ so $\beta$ is not accessible at stage $r$.

Finally, suppose that $p(\beta) = (\tau, \sigma)$. The argument of the proof of Lemma 4.2 is relevant here. The reason that $o(\bar\beta)[s] \neq (\tau,\sigma)$ is that either $\tau \not\subset A_s$, or $\mathcal{P}^\beta_s = \emptyset$. In the first case, since $A$ is c.e., $\tau \not\subset A_r$. In the second case, since $\mathcal{P}^\beta$ is $\Pi^0_1$, $\mathcal{P}^\beta_r = \emptyset$. Hence $\beta$ cannot be accessible at stage $r$. □

**Lemma 6.7.** *Let $\beta \in \mathcal{A} \cap \mathcal{N}$. If $s$ is an initial stage for $\beta$, then $s$ is the first stage at which $\beta$ is accessible.*

*Proof.* Suppose that there is some stage $r < s$ at which $\beta$ is accessible. Let $r$ be the greatest such stage. Then $r \in T(\beta)$. Since $\beta$ is not accessible at stage $s^-(\beta)$, we have $r < s^-(\beta)$. By maximality of $r$, then, $\beta$ is not accessible at the next stage $t$ in $T(\beta)$ beyond $r$. Then $r$ is final for $\beta$; but $\beta$ is accessible at stage $s > r$, contradicting Lemma 6.6. □

Lemmas 6.6 and 6.7 immediately imply the following:

**Corollary 6.8.** *Let $\beta \in \mathcal{A} \cap \mathcal{N}$. There is at most one initial stage for $\beta$, and at most one final stage for $\beta$.*

We let $\mathcal{S}_n$ be the set of all stages which are either initial or final for some $\beta \in \mathcal{A}_n \cap \mathcal{N}$. By Corollary 6.8, $\#\mathcal{S}_n \leqslant 2\#\mathcal{A}_n$. By Lemma 6.4, the function $n \mapsto \#\mathcal{S}_n$ is bounded by some computable function.

*Controlling initial segments of $Y$.* Let $\alpha$ be a node which works for some $R_e$. We define a partial ordering $<^*$ on the outcomes of $\alpha$:

- $\infty <^* o$ for any outcome $o \neq \infty$ of $\alpha$;
- $(\tau, \sigma) <^* \mathtt{fin}_{t,\rho}$ for all $\tau, \sigma, t$ and $\rho$.

This partial ordering extends to a partial ordering of nodes: $\beta_0 <^* \beta_1$ if $\beta_i = \alpha^\frown o_i$, $\alpha$ works for some $R_e$ and $o_0 <^* o_1$.

For the rest of this subsection, fix $n < \omega$. For each stage $s$, we define nodes $\beta_s = \beta_s(n)$ and $\gamma_s = \gamma_s(n)$, and a stage $t(s) = t(s,n) \leqslant s$.

We let $\beta_s = \beta_s(n)$ be the shortest node $\beta$ which is accessible at stage $s$ such that $\beta \notin \mathcal{A}_{n-1}$, that is, the shortest node $\beta$ which is accessible at stage $s$ such that $\beta \in \mathcal{N}$ and $|\rho(\beta)| \geqslant n$.

- If $\bar\beta_s$ works for $S_n$, then we say that $s$ is a *direct stage* (for $n$), and we let $\gamma_s = \beta_s$. In this case, we define $t(s) = s$.
- Otherwise, $\bar\beta_s$ works for some $R_e$. We let $t(s)$ be the last stage $t$ prior to stage $s$ at which $\bar\beta_s$ was accessible, and at which $o(\bar\beta_s)[t] <^* p(\beta_s)$. (Such a stage exists because at the least stage at which $\bar\beta_s$ is accessible, $\bar\beta_s{}^\frown\infty$ is accessible). By recursion, we let $\gamma_s = \gamma_{t(s)}$.

We think of $\gamma_s$ as the node which "controls" $Y_s\restriction_n$ – this is justified by Lemma 6.10. The node $\beta_s$ is the node which witnesses this control, and this control is passed on from stage $t(s)$ to stage $s$.

**Lemma 6.9.** *For all $s < \omega$, $|\rho(\gamma_s)| \geqslant n$, but $|\widehat{\rho}(\bar{\gamma}_s)| < n$.*

*Proof.* By definition of $\beta_s$, for all $s$, $|\rho(\beta_s)| \geqslant n$ but $|\widehat{\rho}(\bar{\beta}_s)| < n$. If $s$ is a direct stage for $n$, then the lemma holds at stage $s$ because $\gamma_s = \beta_s$. If $s$ is an indirect stage for $n$, then the lemma holds by induction on $s$. □

**Lemma 6.10.** *For all $s < \omega$, $\rho(\gamma_s)\restriction_n \subset Y_s$.*

*Proof.* If $s$ is a direct stage for $n$, then $\gamma_s = \beta_s$ is accessible at stage $s$, and so $\rho(\gamma_s) \subset Y_s$. If $s$ is an indirect stage for $n$, let $t = t(s)$. At stage $s$, the instructions of the construction tell us that the outcome $(\tau, \sigma)$ or $\mathtt{fin}_{t,\sigma}$ for $\bar{\beta}_s$ is chosen such that $\sigma \subset Y_t$. By induction, $\rho(\gamma_s)\restriction_n = \rho(\gamma_t)\restriction_n \subset Y_t$. Hence $\sigma$ and $\rho(\gamma_s)\restriction_n$ are comparable. By definition of $\beta_s$, we have $|\sigma| \geqslant n$. Hence $\rho(\gamma_s)\restriction_n \subseteq \sigma$. Because $\beta_s$ is accessible at stage $s$, and $\sigma = \rho(\beta_s)$, we have $\sigma \subset Y_s$, so $\rho(\gamma_s)\restriction_n \subset Y_s$. □

**Lemma 6.11.** *For all $s$, $\bar{\beta}_s \subsetneq \gamma_s$. If $s$ is an indirect stage, then $\gamma_s <^* \beta_s$.*

*Proof.* If $s$ is a direct stage, then $\gamma_s = \beta_s$, so certainly $\bar{\beta}_s \subset \gamma_s$. Suppose that $s$ is an indirect stage; assume, by induction, that the lemma holds at stage $t = t(s)$.

At stage $t$, $\bar{\beta}_s$ is accessible, and by minimality of $\beta_s$, $|\widehat{\rho}(\bar{\beta}_s)| < n$; but $\beta_t$ is the shortest node which is accessible at stage $t$ such that $|\widehat{\rho}(\beta_t)| \geqslant n$. Hence $\bar{\beta}_s \subset \beta_t$, so $\bar{\beta}_s \subseteq \bar{\beta}_t$. By induction, $\bar{\beta}_t \subset \gamma_t$, and by definition, $\gamma_s = \gamma_t$. Hence $\bar{\beta}_s \subset \gamma_s$.

By definition of $t$, $o(\bar{\beta}_s)[t] <^* p(\beta_s)$; since $(\bar{\beta}_s)^+[t] = \bar{\beta}_s{}^\frown o(\bar{\beta}_s)[t] \subseteq \beta_t$, we have $\beta_t <^* \beta_s$. If $t$ is a direct stage, then $\beta_t = \gamma_t = \gamma_s$, so $\gamma_s <^* \beta_s$. If $t$ is an indirect stage, then by induction, $\gamma_t <^* \beta_t$, and so $\gamma_s = \gamma_t <^* \beta_s$. □

**Lemma 6.12.** *Let $s < \omega$ and let $\alpha \subseteq \gamma_s$. Let $r$ be the greatest stage $r \leqslant s$ at which $\alpha$ was accessible. Then $\gamma_r = \gamma_s$.*

*Proof.* Let $t^{(k)}(s)$ be the $k^{\text{th}}$ iteration of the function $t$. For every $k$, we have $\gamma_{t^{(k)}(s)} = \gamma_s$; for some $k$, $t^{(k)}(s)$ is direct for $n$, so $\gamma_s$, and so $\alpha$, is accessible at stage $t^{(k)}(s)$. Hence there is some $k$ such that $r \geqslant t^{(k)}(s)$.

Let $k$ be the least number such that $r \geqslant t^{(k)}(s)$. If $k = 0$ then $t^{(k)}(s) = s$ and so $r = s$, so $\gamma_r = \gamma_s$. Suppose that $k > 0$. Since $\alpha$ is not accessible at stage $t^{(k-1)}(s)$, and $\alpha, \bar{\beta}_{t^{(k-1)}(s)} \subseteq \gamma_s$ (Lemma 6.11, using the fact that $\gamma_s = \gamma_{t^{(k-1)}(s)}$), we have $\bar{\beta}_{t^{(k-1)}(s)} \subsetneq \alpha$; again by Lemma 6.11, since $t^{(k-1)}(s)$ is not direct, we have $\gamma_s <^* \beta_{t^{(k-1)}(s)}$, which implies that $\alpha <^* \beta_{t^{(k-1)}(s)}$.

Now by definition, $t^{(k)}(s)$ is the greatest stage prior to $t^{(k-1)}(s)$ at which some node $\delta \supset \bar{\beta}_{t^{(k-1)}(s)}$ such that $\delta <^* \beta_{t^{(k-1)}(s)}$ is accessible. Since $\alpha$ is such a node, we must have $r \leqslant t^{(k)}(s)$. Hence $r = t^{(k)}(s)$ and so $\gamma_r = \gamma_{t^{(k)}(s)} = \gamma_s$. □

The following is the main lemma of the verification.

**Lemma 6.13.** *Suppose that $\gamma_{s-1} \neq \gamma_s$. Then $s \in \mathcal{S}_n$.*

*Proof.* Let $\alpha$ be the longest common initial segment of $\gamma_{s-1}$ and $\gamma_s$.

**Claim 6.14.** *$\alpha$ is accessible at stage $s$.*

*Proof.* Let $r$ be the greatest stage $r \leqslant s$ at which $\alpha$ is accessible. By Lemma 6.12, since $\alpha \subset \gamma_s$, $\gamma_r = \gamma_s$. Suppose, for contradiction, that $r < s$. Then $r$ is the greatest stage before stage $s - 1$ at which $\alpha$ is accessible. Then again by Lemma 6.12, since $\alpha \subset \gamma_{s-1}$, $\gamma_r = \gamma_{s-1}$, which contradicts $\gamma_{s-1} \neq \gamma_s$. □

We note that since $\alpha \subsetneq \gamma_s$, we have $|\widehat{\rho}(\alpha)| < n$ (Lemma 6.9). By minimality of $\beta_s$, $\alpha \subsetneq \beta_s$, so $\alpha \subseteq \bar{\beta}_s$.

We let $r$ be the greatest stage $r < s$ at which $\alpha$ was accessible. By Lemma 6.12, $\gamma_r = \gamma_{s-1}$. The definition of $\beta_r$ implies that $\alpha \subseteq \bar{\beta}_r$.

Suppose that $\alpha$ works for some $S_m$ (for $m \leqslant n$). If $m = n$, then $\gamma_s$ and $\gamma_r = \gamma_{s-1}$ are immediate successors of $\alpha$, and both $r$ and $s$ are direct stages. Since $\gamma_r \neq \gamma_s$, we have $p(\gamma_r) <_L p(\gamma_s)$. We note that $r = s^{-1}(\gamma_s)$, so $s$ is final for $\gamma_r$ and initial for $\gamma_s$, so $s$ is in $\mathcal{S}_n$.

Suppose that $m < n$. Then for every immediate extension $\delta$ of $\alpha$, we have $|\widehat{\rho}(\delta)| < n$. Hence, $\alpha^+[r] \subseteq \bar{\beta}_r$, and $\alpha^+[s] \subseteq \bar{\beta}_s$. As $\bar{\beta}_r \subset \gamma_r$, and $\bar{\beta}_s \subset \gamma_s$ (Lemma 6.11), we must have $\alpha^+[r] \subset \gamma_r$ and $\alpha^+[s] \subset \gamma_s$. By the definition of $\alpha$ (since $\gamma_r = \gamma_{s-1}$), it follows that $\alpha^+[r] \neq \alpha^+[s]$. By definition of $r$, $r$ is the predecessor of $s$ in $T(\alpha^+[r]) = T(\alpha^+[s])$. Hence $s$ is final for $\alpha^+[r]$ and initial for $\alpha^+[s]$, so $s \in \mathcal{S}_n$.

So from now, we assume that $\alpha$ works for some $R_e$.

**Claim 6.15.** *If $\alpha = \bar{\beta}_s$ then $s \in \mathcal{S}_n$.*

*Proof.* Suppose that $\alpha = \bar{\beta}_s$, so $o(\alpha)[s]$ is of the from $(\tau, \sigma)$ or $\mathtt{fin}_{u,\sigma}$ where $|\sigma| \geqslant n$. Then $s$ is an indirect stage, and $t(s)$ is the greatest stage $t$ before stage $s$ at which $\alpha$ is accessible and $o(\alpha)[t] <^* o(\alpha)[s]$; so $t(s) \leqslant r$. Since $\gamma_s = \gamma_{t(s)}$ and $\gamma_r = \gamma_{s-1}$, we have $t(s) < r$. This shows that $o(\alpha)[r] \not<^* o(\alpha)[s]$.

We argue that $o(\alpha)[s] = (\tau, \sigma)$ for some $\tau$ and $\sigma$. Otherwise, $o(\alpha)[s] = \mathtt{fin}_{t(s),\sigma}$ for some $\sigma$ such that $|\sigma| \geqslant n$. Since $o(\alpha)[r] \not<^* o(\alpha)[s]$, we must have $o(\alpha)[r] = \mathtt{fin}_{t(s),\sigma'}$ for some $\sigma' \supseteq \sigma$; so $|\sigma'| \geqslant n$. So $\alpha = \bar{\beta}_r$, and $t(r) = t(s)$, yielding $\gamma_r = \gamma_{t(r)} = \gamma_{t(s)} = \gamma_s$ which is not true.

Next, we show that $o(\alpha)[r] = \mathtt{fin}_{u,\rho}$ for some $\rho$, where $u = s^-(\alpha^+[s])$ is the last stage before $s$ at which $\alpha$ was active. Otherwise, since $o(\alpha)[r] \not<^* o(\alpha)[s]$, we would have to have $o(\alpha)[r] = o(\alpha)[s] = (\tau, \sigma)$, so $|\rho(\beta^+[r])| \geqslant n$. But then again $\alpha = \bar{\beta}_r$ and $t(r) = t(s)$ which is impossible.

Finally, we argue that $|\rho| < n$, that is, that $\alpha \neq \bar{\beta}_r$. For otherwise, we would have $t(r) = u$. If $o(\alpha)[u] = o(\alpha)[s] = (\tau, \sigma)$, then $t(u) = t(s)$, so we would have $\gamma_r = \gamma_{t(r)} = \gamma_u = \gamma_{t(u)} = \gamma_{t(s)} = \gamma_s$ which is wrong. Otherwise, $o(\alpha)[u] = \infty$, in which case $u = t(s)$ so $t(r) = t(s)$, contradiction again.

We see that $\alpha^+[r] \in \mathcal{A}_n$ and certainly $\alpha^+[r] \in \mathcal{N}$; that $r$ and $s$ are successive stages in $T(\alpha^+[r])$, and that $\alpha^+[r]$ is accessible at stage $r$ but not at stage $s$. Hence $s$ is a final stage for $\alpha^+[r]$, so $s \in \mathcal{S}_n$. ☐

Hence, we assume from now that $\alpha^+[s] \in \mathcal{A}_n$. We would be done if $s$ is initial for $\alpha^+[s]$. There are two possible reasons for why this would not be the case:

(1) $\alpha^+[s] \notin \mathcal{N}$, that is, $o(\alpha)[s] = \infty$; or
(2) $\alpha^+[s]$ is accessible at stage $u = s^-(\alpha^+[s])$.

The proof of the lemma will be complete when we show that in either case, $s$ is final for $\gamma_r\restriction_{|\alpha|+1}$. We first note that $\alpha^+[s] \subseteq \bar{\beta}_s$ implies that $\alpha^+[s] \subset \gamma_s$.

We first tackle the first case. Suppose that $\alpha^+[s] = \alpha^\frown\infty$. Let $\widehat{\alpha} = \gamma_{s-1}\restriction_{|\alpha|+1}$ be the immediate extension of $\alpha$ which is extended by $\gamma_{s-1}$. By the definition of $\alpha$, we have $\widehat{\alpha} \neq \alpha^\frown\infty$. Let $w$ be the greatest stage $w \leqslant r$ at which $\widehat{\alpha}$ is accessible. If $w < r$ then $\alpha = \bar{\beta}_r$, and $w = t(r)$; for if $\alpha = \bar{\beta}_{t(r)}$, then $\alpha^\frown\infty$ is accessible at stage $t(t(r))$, implying that $\alpha^\frown\infty \subset \gamma_{t(t(r))} = \gamma_r$, which is not the case. In either case, since $\alpha^\frown\infty$ is accessible at stage $s$, we see that $w$ and $s$ are successive stages in $T(\widehat{\alpha})$. Certainly $\widehat{\alpha} \in \mathcal{A}_n \cap \mathcal{N}$. Hence $s$ is final for $\widehat{\alpha}$, and $s \in \mathcal{S}_n$.

Finally, suppose that the second, but not the first case, holds: $\alpha^+[s] \in \mathcal{N}$, but $\alpha^+[s]$ is accessible at stage $u = s^-(\alpha^+[s])$. Again by the definition of $\alpha$, we cannot have $\alpha^+[r] = \alpha^+[s]$, because then $\alpha \neq \bar{\beta}_r$ and we would have $\alpha^+[s] \subset \gamma_r$. This means that $u < r$, which in turn implies that $o(\alpha)[s] = (\tau, \sigma)$ for some $\tau$ and some $\sigma$ such that $|\sigma| < n$; and that $o(\alpha)[r] = \mathtt{fin}_{u,\rho}$ for some $|\rho|$. We must have $|\rho| < n$, that is, $|\rho(\alpha^+[r])| < n$. Otherwise, $\alpha = \bar{\beta}_r$, and $u = t(r)$, so $\alpha^+[s] = \alpha^+[u] \subset \gamma_{t(r)} = \gamma_r$, which is not the case. Thus, $\alpha^+[r] \in \mathcal{A}_n$, $r$ and $s$ are successive stage in $T(\alpha^+[r])$, and certainly $\alpha^+[r] \in \mathcal{N}$; so $s$ is final for $\alpha^+[r]$, and so $s \in \mathcal{S}_n$. This concludes the proof of Lemma 6.13. □

**Corollary 6.16.** *The sequence $\langle Y_s \rangle$ is an $\omega$-c.e. approximation of a set $Y$.*

*Proof.* Let $n < \omega$ and let $s$ be a stage. By Lemma 6.10, if $Y_s\restriction_n \neq Y_{s-1}\restriction_n$, then $\gamma_s(n) \neq \gamma_{s-1}(n)$. By Lemma 6.13, if $\gamma_s(n) \neq \gamma_{s-1}(n)$, then $s \in \mathcal{S}_n$. As mentioned above, By Lemma 6.4 and Corollary 6.8, there is a computable bound on $\#\mathcal{S}_n$. □

The rest of the verification proceeds as in the previous sections.

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