LOWNESS FOR KURTZ RANDOMNESS

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Abstract. We prove that degrees that are low for Kurtz randomness cannot be diagonally non-recursive. Together with the work of Stephan and Yu [16], this proves that they coincide with the hyperimmune-free non-DNR degrees, which are also exactly the degrees that are low for weak 1-genericity.

We also consider $\mathrm{Low}(\mathcal{M}, \mathrm{Kurtz})$, the class of degrees $\mathbf{a}$ such that every element of $\mathcal{M}$ is $\mathbf{a}$-Kurtz random. These are characterised when $\mathcal{M}$ is the class of Martin-Löf random, computably random, or Schnorr random reals. We show that $\mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$ coincides with the non-DNR degrees, while both $\mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz})$ and $\mathrm{Low}(\mathrm{Schnorr}, \mathrm{Kurtz})$ are exactly the non-high, non-DNR degrees.

1. Introduction

Lowness is a notion of feebleness. A Turing degree $\mathbf{a}$ is called *low* in a given context if it is indistinguishable from $\mathbf{0}$, the degree of recursive sets, using the tools being examined; in other words, it is useless as an oracle for the notion of computation under discussion. Technically, if $\mathcal{C}$ is a relativisable class of reals, we say that A is *low for* $\mathcal{C}$ if $\mathcal{C}^A = \mathcal{C}$.

The first notion of lowness to be examined was that of lowness for the Turing jump: $\mathbf{a}$ is *low* if the halting set relative to $\mathbf{a}$ (denoted by $\mathbf{a}'$) is computable from the unrelativised halting set $\mathbf{0}'$ (in the notation of the previous paragraph, $\mathcal{C} = \Delta^0_2$). Spector showed that there are non-recursive low degrees [15].

Recently, attention has been focused on lowness notions arising from effective randomness, genericity, and other classes from computability theory. Typically, the class of degrees that are low for a class $\mathcal{C}$ is shown to be definable using common notions of computability, thus shedding light on the class $\mathcal{C}$ itself.

The first such result is due to Slaman and Solovay [14], who showed that the degrees that are low for EX-learning are exactly the degrees of Δ^0_2, 1-generic sets. However, lowness has come to prominence with the increased interest in algorithmic randomness. The class of degrees that are low for Martin-Löf randomness has been shown [10, 2] to be robust and to posses various pleasing degree-theoretic qualities, such as being a Σ^0_3 ideal contained in the Δ^0_2 degrees and generated by its c.e. elements. This class, known as the *K-trivials*, is also unique in being the only well understood lowness class that has not yet been shown to be definable without appealing to measure or effective randomness. As can be seen in Table 1, most lowness classes that arise from other notions of randomness have either turned out

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to be identical to the K-trivials or have been characterised without reference to randomness, often using the discrete notion of traceability [17, 7].

On the genericity front, Greenberg, Miller and Yu (published in [18]) have shown that the sets that are low for 1-genericity are exactly the computable ones. It is, however, easy to construct non-computable degrees that are low for weak 1-genericity (a set is *weakly* 1-*generic* if it is an element of all dense Σ^0_1 sub-classes of Cantor space, 2^ω). In a remarkable paper [16], Stephan and Yu have characterised the class of degrees that are low for weak 1-genericity as exactly those degrees that are hyperimmune-free and not DNR. (The fact that such degrees are hyperimmune-free is due to Nitzpon [12]). We recall that a degree $\mathbf{a}$ is hyperimmune-free (or $\mathbf{0}$-*dominated*) if every function $f\colon \omega \to \omega$ computable from $\mathbf{a}$ is dominated by some recursive function; and a degree is *DNR* if it computes a diagonally non-recursive function f, namely a function such that, for all e, if $J(e) = \varphi_e(e)\downarrow$ then its value is different from $f(e)$ (where $\langle \varphi_e \rangle$ is an effective listing of all partial computable functions).

Stephan and Yu have also showed that each degree that is low for weak 1-genericity is low for Kurtz tests, hence low for Kurtz randomness. This connection between weak genericity and weak randomness may seem surprising; however, Kurtz randomness is so weak that it may actually be considered a notion of very weak genericity, as it only refers to open sets: a real is Kurtz-random if it lies in every Σ^0_1 class of measure one. It is known that the genericity notions themselves are distinct: there are reals that are Kurtz random and yet are not weakly 1-generic [8]. However, Stephan and Yu asked if the lowness notions coincided. It is easy to see that degrees that are low for Kurtz randomness have to be hyperimmune-free [4]; indeed, every hyperimmune degree contains a Kurtz random real [8]. In this paper we confirm Stephan and Yu's suspicion by showing the following:

Theorem 1.1. *No DNR degree is low for Kurtz randomness. Hence the degrees that are low for Kurtz randomness are precisely those that are low for weak* 1*-genericity: the hyperimmune-free, non-DNR degrees.*

We remark that another lowness result of a somewhat similar flavor has recently been established by Simpson and Cole [13], who showed that a degree is low for bounded limit recursiveness, a notion resembling ω-r.e.-ness, if and only if it is both superlow and jump-traceable.

Additional lowness notions can be defined by using two classes $\mathcal{M} \subseteq \mathcal{N}$ as parameters. Here a set A is in $\mathrm{Low}(\mathcal{M},\mathcal{N})$ if $\mathcal{M} \subseteq \mathcal{N}^A$. Such notions have been mainly investigated by Nies (see [9]). In this paper we add to our understanding by extending Theorem 1.1 to show:

Theorem 1.2. *No DNR degree is in* $\mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$.

As a corollary, we get that all intermediate lowness classes, $\mathrm{Low}(\mathrm{Schnorr}, \mathrm{Kurtz})$ and $\mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz})$, also consist of non-DNR degrees. Since every high degree contains a Schnorr random real, we get in fact:

Corollary 1.3. $\mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz}) = \mathrm{Low}(\mathrm{Schnorr}, \mathrm{Kurtz})$ *consists exactly of the degrees that are non-DNR and not high.*

Table 1 summarizes the current knowledge about lowness classes issuing from notions of randomness. Two slots currently remain uncomputed:

$\mathcal{M}$ \ $\mathcal{N}$	W2R	ML	CR	Schnorr	Kurtz
W2R	K-trivial [3, 6, 9]	K-trivial [3]	K-trivial [9]	?	?
ML		K-trivial [10]	K-trivial [10]	r.e. traceable [7]	non-DNR
CR			recursive [10]	recursively traceable [7]	non-high and non-DNR
Schnorr				recursively traceable [17, 7]	non-high and non-DNR
Kurtz					hyperimmune-free and non-DNR

TABLE 1. $\mathrm{Low}(\mathcal{M}, \mathcal{N})$ for various randomness classes. Note that each class is contained in the classes above it and to its right. The gray entries are from this paper.

Question 1.4. What are $\mathrm{Low}(\mathrm{W2R}, \mathrm{Schnorr})$ and $\mathrm{Low}(\mathrm{W2R}, \mathrm{Kurtz})$?

We note that this last class contains $\mathbf{0}'$.

The main theorem (5.2) will follow from the seemingly weaker Theorem 5.1, which states that no DNR degree is low for Kurtz tests; that is, if f is a DNR function then there is some null $\Pi^0_1(f)$ class which is not contained in any null Π^0_1 class. The general plan for the proof of this theorem (and the organization of this paper) is as follows. We define (in Section 3) a continuous operator P that maps functions $f\colon \omega \to \omega$ to a null $\Pi^0_1(f)$ class P^f. The aim is to make P^f and P^g, for distinct functions f and g, sufficiently independent, so that even though the measure of each P^f is zero, the union of P^f for non-negligible collections of functions f will be far from null. The technical notion of negligibility that we employ is that of *svelte trees* (Definition 3.2).

Suppose that C is a null Π^0_1 class. Since P is a continuous operator, the collection of functions f for which $P^f \subseteq C$ is a closed subset of Baire space ω^ω, so it is the set of paths through a tree $T \subseteq \omega^{<\omega}$. Essentially, we show that this tree can be covered by sufficiently few basic clopen sets $[\sigma]$ so that we can make none of the paths of T DNR, dispensing of those clopen sets σ by defining $J(x) = \sigma(x)$ for some x in some column of the jump function J over which we have control (using the recursion theorem, of course). The notion of svelteness is exactly what is needed to carry out this plan.

As mentioned, in Section 3 we define the notion of svelte trees (which applies to finite trees); we use various calculations (which we make in Section 2) to show that if $\bigcup P^f$, where f ranges over a *clopen* subset of ω^ω, has small measure, then the corresponding finite tree which generates this clopen set is svelte (Theorem 3.3). In Section 4 we generalise the situation to the case where the union ranges over (roughly speaking) a closed class of functions, which is what appears in the actual construction. We remark that up to and including Section 4, the paper is completely combinatorial, and does not mention any computability.

In Section 5 we carry out the plan and provide the proof of Theorem 5.2. In Section 6 we generalize this result and characterise the classes $\mathrm{Low}(\mathcal{C}, \mathrm{Kurtz})$ for other randomness notions $\mathcal{C}$.

We remark that since the discovery of the proof described in this paper, work by the second author, motivated by results of Kjos-Hanssen's regarding infinite subsets of random sets, has resulted in a completely different proof of our results.

2. Calculations and inequalities

Lemma 2.1. *The function*

$$f(x) = \left(\frac{1+x}{2}\right)^{1/\ln x}$$

is increasing on the interval $(0,1)$.

Proof. This is routine. We show that $\ln f$ is increasing on $(0,1)$. Note that

$$(\ln f(x))' = \frac{x \ln x - (1+x)\ln(\frac{1+x}{2})}{x(x+1)(\ln x)^2}.$$

Since the denominator is positive when $x \in (0,1)$, it is enough to prove that $x \ln x > (1+x)\ln(\frac{1+x}{2})$ for $x \in (0,1)$. Both sides are equal when $x = 1$, so it suffices to prove that the right hand side grows faster than the left hand side. Differentiating and then simplifying, we need to show that $\ln x < \ln(\frac{1+x}{2})$, which holds when $x < 1$. $\square$

Lemma 2.2. *Let* $\delta_0, \dots, \delta_{n-1} \in (0,1)$ *and* $\varepsilon = \min_{i<n} \delta_i$. *Then*

$$\prod_{i<n} \left(\frac{1+\delta_i}{2}\right) \leqslant \left(\frac{1+\varepsilon}{2}\right)^{\ln\left(\prod_{i<n}\delta_i\right)/\ln\varepsilon}.$$

Proof. We proceed by induction on n. We get equality when $n = 1$. Assume that the lemma holds for n, and let $\delta_0, \dots, \delta_n \in (0,1)$. By rearranging, we may assume that $\delta_n = \min_{i\leqslant n} \delta_i$. Let $\eta = \min_{i<n} \delta_i$. Since $\delta_n \leqslant \eta$, Lemma 2.1 ensures that

$$\left(\frac{1+\eta}{2}\right)^{1/\ln\eta} \geqslant \left(\frac{1+\delta_n}{2}\right)^{1/\ln\delta_n}.$$

Since $\ln\left(\prod_{i<n}\delta_i\right)$ is negative,

$$\left(\frac{1+\eta}{2}\right)^{\ln\left(\prod_{i<n}\delta_i\right)/\ln\eta} \leqslant \left(\frac{1+\delta_n}{2}\right)^{\ln\left(\prod_{i<n}\delta_i\right)/\ln\delta_n}.$$

Now by induction,

$$\prod_{i\leqslant n} \left(\frac{1+\delta_i}{2}\right) \leqslant \left(\frac{1+\eta}{2}\right)^{\ln\left(\prod_{i<n}\delta_i\right)/\ln\eta} \left(\frac{1+\delta_n}{2}\right) \leqslant$$
$$\left(\frac{1+\delta_n}{2}\right)^{\ln\left(\prod_{i<n}\delta_i\right)/\ln\delta_n} \left(\frac{1+\delta_n}{2}\right) = \left(\frac{1+\delta_n}{2}\right)^{\ln\left(\prod_{i\leqslant n}\delta_i\right)/\ln\delta_n}. \quad \square$$

The following lemma is again an exercise in elementary calculus, although it is worth noting that the left hand side approaches $1/2$ as ε approaches either 0 or 1, so the inequality is tight at both extremes.

Lemma 2.3. *If $\varepsilon \in (0,1)$, then*

$$\left(\frac{1+\varepsilon}{2}\right)^{1-\ln 4/\ln\varepsilon} < \frac{1}{2}.$$

Proof. Taking the logarithm of both sides, we will show that

$$\left(1-\frac{\ln 4}{\ln\varepsilon}\right)\ln\left(\frac{1+\varepsilon}{2}\right) < -\ln 2.$$

Multiplying by $\ln\varepsilon$, which is negative, it is sufficient to show that

$$(\ln\varepsilon - \ln 4)\ln\left(\frac{1+\varepsilon}{2}\right) > -\ln 2\ln\varepsilon.$$

Both sides are equal when $\varepsilon = 1$, so it is enough to prove that the right hand side grows faster than the left hand side. Differentiating and then simplifying, we need to show that

$$\ln\varepsilon - \ln 4 + \frac{1+\varepsilon}{\varepsilon}\ln(1+\varepsilon) < 0.$$

The value of the left hand side is 0 for $\varepsilon = 1$, so the inequality will follow from the fact that the left hand side is increasing on $(0,1)$. Again differentiating and rearranging, it suffices to show that

$$2\varepsilon > \ln(1+\varepsilon).$$

However, for all ε, we have $\ln(1+\varepsilon) \leqslant \varepsilon$ so we are done. □

Raising to the n^{th} power, we get:

Corollary 2.4. *For all $\varepsilon \in (0,1)$ and $n > 0$,*

$$\left(\frac{1+\varepsilon}{2}\right)^{n+\ln\left(2^{-2n}\right)/\ln\varepsilon} < 2^{-n}.$$

3. Svelte trees

We begin by defining the operator taking an $f \in \omega^{\leqslant\omega}$ to a closed subset P^f of Cantor space, 2^ω. The operator depends on an increasing sequence of natural numbers $\bar{n} = \langle n_1, n_2, \dots\rangle$, which we fix for now (and which we will eventually compute using the recursion theorem).

Fix an injection $I\colon \omega^{<\omega} \to \omega$. For $n \in \omega$, let $U_n = \{X \in 2^\omega : X(n) = 1\}$. For $f \in \omega^{\leqslant\omega}$, define

$$P^f = \bigcap_{m\,:\,n_m \leqslant |f|} U_{I(f\restriction n_m)}.$$

The reason for using the U_n is that they yield independent sets. Recall that two sets $A, B \subseteq 2^\omega$ are *independent* if $\mu(A\cap B) = \mu(A)\mu(B)$. For $I \subset \omega$, let $\mathcal{A}_I$ be the algebra of sets obtained from U_n, for $n \in I$, by using finite Boolean combinations.

Fact 3.1. Suppose that $I, J \subset \omega$ are disjoint, $A \in \mathcal{A}_I$ and $B \in \mathcal{A}_J$. Then A and B are independent.

We will use this observation repeatedly, but for now, note that it implies that $\mu(P^f) = 0$ for every $f \in \omega^\omega$.

Definition 3.2. Let $T \subset \omega^{<\omega}$ be a finite tree. We say that T is *k-svelte* if there is a sequence $\langle S_{k+1}, S_{k+2}, S_{k+3}, \dots\rangle$ of subsets of T such that:

- $S_m \subseteq T \cap \omega^{n_m}$;
- $|S_m| \leqslant 2^{m-(k+1)}$; and
- every leaf of T extends some string in $\bigcup S_m$.

For a finite tree $T \subset \omega^{<\omega}$, let P^T be the *union* of P^σ for all leaves σ of T. The following theorem, which is the main computational theorem in this paper, establishes a dichotomy: either the measure of P^T is large, or T is svelte (and so can be covered "cheaply").

Theorem 3.3. *If $T \subset \omega^{<\omega}$ is a finite tree such that $\mu(P^T) \leqslant 2^{-(k+1)}$, then T is k-svelte.*

Proof. For $m \in \omega$, let $T_{n_m} = T \cap \omega^{n_m}$ and let $S = \bigcup_m T_{n_m}$. Let L be the collection of strings in S that do not have proper extensions in S. We note that

$$P^T = \bigcup_{\sigma \in L} P^\sigma.$$

For $\tau \in S$ and $\sigma \in L$ such that $\sigma \supseteq \tau$, let

$$P^\sigma_\tau = \bigcap_{m\,:\,|\tau|<n_m\leqslant|\sigma|} U_{I(\sigma\restriction n_m)}$$

(so for $\sigma = \tau$ we have $P^\sigma_\tau = 2^\omega$) and for $\tau \in S$ let

$$Q_\tau = \bigcup_{\sigma\in L\,:\,\sigma\supseteq\tau} P^\sigma_\tau.$$

For $\tau \in S$, let $\delta_\tau = 1 - \mu\left(Q_\tau\right)$.

By induction on $m \geqslant k+1$, define sets $S_m \subseteq T_{n_m}$. Let $S_{k+1} = \{\tau\}$ where δ_τ is minimal among the strings of $T_{n_{k+1}}$. Suppose that $S_{k+1}, \dots, S_{m-1}$ have been chosen. Let $S_{<m}$ be the collection of strings in T_{n_m} that extend some string in $S_{k+1} \cup S_{k+2} \cup \dots \cup S_{m-1}$. Order $T_{n_m} \smallsetminus S_{<m}$ so that δ_τ is non-decreasing, and let S_m be the first $2^{m-(k+1)}$ many strings on the list (or if the list is shorter, let S_m be all of $T_{n_m} \smallsetminus S_{<m}$). We halt the process when we get to some m such that $S_{<m} = T_{n_m}$. Figure 1 shows an example.

If every string in L (and so every leaf of T) extends some string in $\bigcup_{m>k} S_m$ then T is k-svelte. Otherwise, we show that $\mu(P^T) > 2^{-(k+1)}$.

First, if there is some $\sigma \in L \cap T_{n_m}$ for some $m \leqslant k$, then we note that $\mu(P^\sigma) \geqslant 2^{-k}$ and $P^\sigma \subseteq P^T$. Assume from now that is not the case.

For $m > k$, let $S_{\leqslant m} = S_{<m} \cup S_m$. Let m^* be least such that $L \cap T_{n_{m^*}} \smallsetminus (S_{\leqslant m^*})$ is nonempty. By reverse induction on m, $k \leqslant m < m^*$, we show that

$$\prod_{\tau\in T_{n_m}\smallsetminus S_{\leqslant m}} \delta_\tau < 2^{-2^{m-k}}. \tag{1}$$

First, we show that for all $m \in [k, m^*-1]$,

$$\prod_{\tau\in T_{n_m}\smallsetminus S_{\leqslant m}} \delta_\tau = \prod_{\sigma\in T_{n_{m+1}}\smallsetminus S_{<m+1}} \left(\frac{1+\delta_\sigma}{2}\right). \tag{2}$$

Because $m < m^*$, for every $\tau \in T_{n_m} \smallsetminus S_{\leqslant m}$, the set

$$\tau\uparrow = \{\sigma \in T_{n_{m+1}} \,:\, \tau \subset \sigma\}$$

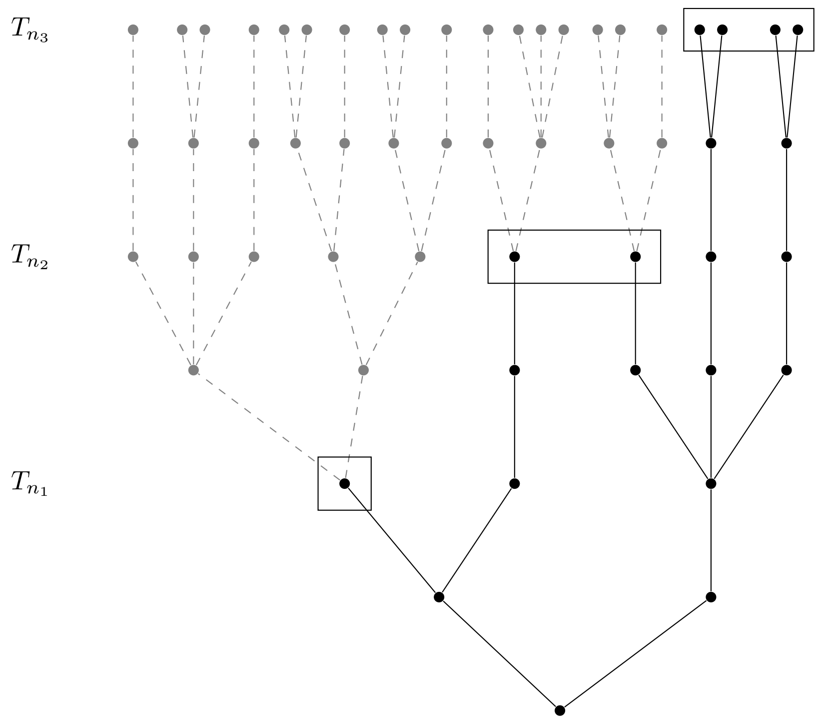

FIGURE 1. The covering process shows that T is 0-svelte. The elements of S_m are the boxed nodes on level n_m of T; the grey nodes are the ones that have been "knocked off" by $S_1 \cup S_2$. Note that we first choose the weightiest nodes τ (among the non-grey ones) to put in S_m - those for which $\mu(Q_\tau)$ is the greatest. This is not necessarily the same as having the largest number of leaves (in L) above τ, since if, for example, two leaves σ and σ' are close to each other (they have a long common initial segment) then there is significant overlap between P_τ^σ and $P_\tau^{\sigma'}$, so the joint contribution $P_\tau^\sigma \cup P_\tau^{\sigma'}$ to Q_τ is smaller than the contribution of leaves which lie further apart.

is nonempty (and disjoint from $S_{<m+1}$), and

$$Q_\tau = \bigcup_{\sigma \in \tau\uparrow} Q_\sigma \cap U_{I(\sigma)}.$$

Further, $\{\tau\uparrow \colon \tau \in T_{n_m} \setminus S_{\leqslant m}\}$ is a partition of $T_{n_{m+1}} \setminus S_{<m+1}$. For every $\sigma \in \tau\uparrow$ there is some $J_\sigma \subset \omega$ such that $Q_\sigma \in \mathcal{A}_{J_\sigma}$, $I(\sigma) \notin J_\sigma$ and such that the $J_\sigma \cup \{I(\sigma)\}$ are pairwise disjoint. It follows that all the sets involved are independent (Fact 3.1). Thus $\mu(Q_\sigma \cap U_{I(\sigma)}) = \mu(Q_\sigma)/2$ and so $\mu\left(2^\omega \setminus (Q_\sigma \cap U_{I(\sigma)})\right) = (1+\delta_\sigma)/2$; and

$$\mu\left(2^\omega \setminus Q_\tau\right) = \prod_{\sigma \in \tau\uparrow} \mu\left(2^\omega \setminus \left(Q_\sigma \cap U_{I(\sigma)}\right)\right);$$

this establishes Equation 2. See Figure 2 for an illustration.

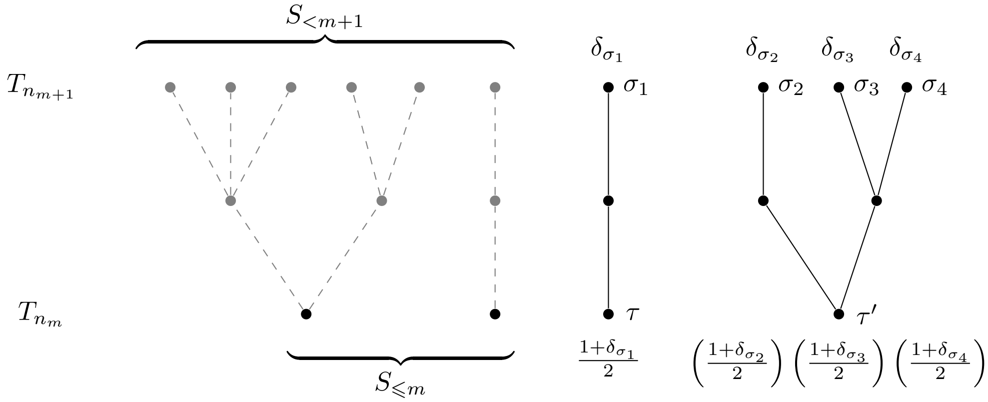

FIGURE 2. Illustrating Equation 2: in this tree, we have two nodes τ and τ' in $T_{n_m} \smallsetminus S_{\leqslant m}$. We have $\tau\uparrow = \{\sigma_1\}$ and $\tau'\uparrow = \{\sigma_2, \sigma_3, \sigma_4\}$, a partition of $T_{n_{m+1}} \smallsetminus S_{<m+1}$. This gives $\delta_\tau = (1+\delta_{\sigma_1})/2$ and $\delta_{\tau'} = (1+\delta_{\sigma_2})(1+\delta_{\sigma_3})(1+\delta_{\sigma_4})/8$.

Now let $m = m^* - 1$. Then since $T_{n_{m^*}} \smallsetminus S_{\leqslant m^*}$ is nonempty, we know that $|S_{m^*}| = 2^{m^*-(k+1)}$. By the definition of m^*, in fact, there is some $\sigma^* \in T_{n_{m^*}} \smallsetminus S_{\leqslant m^*}$ in L; we have $\delta_{\sigma^*} = 0$ and $\delta_\sigma \leqslant \delta_{\sigma^*}$ for all $\sigma \in S_{m^*}$. Thus $\delta_\sigma = 0$ for all $\sigma \in S_{m^*}$. So,

$$\prod_{\sigma \in T_{n_{m^*}} \smallsetminus S_{<m^*}} \left(\frac{1+\delta_\sigma}{2}\right) \leqslant \frac{1}{2} 2^{-2^{m^*-(k+1)}} < 2^{-2^{m-k}}.$$

Equation 2 now ensures that the Inequality 1 holds for $m = m^* - 1$.

Assume that Inequality 1 holds for m where $k < m < m^*$; we prove it for $m-1$. Again because $m \leqslant m^*$, we know that $T_{n_m} \smallsetminus S_{\leqslant m}$ is nonempty, and so $|S_m| = 2^{m-(k+1)}$. Because $m < m^*$, we know that for all $\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}$, $\delta_\sigma > 0$. Let $\varepsilon = \min_{\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}} \delta_\sigma$. Then for all $\sigma \in S_m$, $\delta_\sigma \leqslant \varepsilon$. Thus

$$\prod_{\sigma \in S_m} \left(\frac{1+\delta_\sigma}{2}\right) \leqslant \left(\frac{1+\varepsilon}{2}\right)^{2^{m-(k+1)}}.$$

By Lemma 2.2,

$$\prod_{\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}} \left(\frac{1+\delta_\sigma}{2}\right) \leqslant \left(\frac{1+\varepsilon}{2}\right)^{\ln\left(\prod_{\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}} \delta_\sigma\right)/\ln\varepsilon}.$$

By induction, we know that

$$\prod_{\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}} \delta_\sigma < 2^{-2^{m-k}} = 2^{-2 \cdot 2^{m-(k+1)}},$$

so (as $\varepsilon < 1$),

$$\frac{\ln\left(\prod_{\sigma \in T_{n_m} \smallsetminus S_{\leqslant m}} \delta_\sigma\right)}{\ln\varepsilon} > \frac{\ln\left(2^{-2 \cdot 2^{m-(k+1)}}\right)}{\ln\varepsilon}.$$

Since $(1+\varepsilon)/2 < 1$, the function $y \mapsto \left(\frac{1+\varepsilon}{2}\right)^y$ is decreasing, so

$$\left(\frac{1+\varepsilon}{2}\right)^{\ln\left(\prod_{\sigma\in T_{n_m}\smallsetminus S_{\leqslant m}} \delta_\sigma\right)/\ln\varepsilon} < \left(\frac{1+\varepsilon}{2}\right)^{\ln\left(2^{-2\cdot 2^{m-(k+1)}}\right)/\ln\varepsilon}.$$

Putting it all together, since $T_{n_{m-1}\smallsetminus S_{<m-1}} = S_{m-1} \cup (T_{n_{m-1}} \smallsetminus S_{\leqslant m-1})$, we get

$$\prod_{\tau\in T_{n_{m-1}}\smallsetminus S_{<m-1}} \delta_\tau \leqslant \left(\frac{1+\varepsilon}{2}\right)^{2^{m-(k+1)}+\ln\left(2^{-2\cdot 2^{m-(k+1)}}\right)/\ln\varepsilon}.$$

Corollary 2.4 shows that this quantity is strictly smaller than $2^{-2^{m-(k+1)}} = 2^{-2^{(m-1)-k}}$, as required.

Thus (noting that $S_{\leqslant k} = \emptyset$) we get that

$$\prod_{\tau\in T_{n_k}} \delta_\tau < \frac{1}{2}.$$

Now, by reverse induction on $m \leqslant k$, we show that

$$\prod_{\tau\in T_{n_m}} \delta_\tau < 1 - 2^{-(k+1-m)} \tag{3}$$

(where we let $n_0 = 0$). Assume that this is true for $m \leqslant k$. Note that Equation 2 still holds for the levels below k:

$$\prod_{\tau\in T_{n_{m-1}}} \delta_\tau = \prod_{\sigma\in T_{n_m}} \left(\frac{1+\delta_\sigma}{2}\right).$$

Let $\varepsilon = \min_{\sigma\in T_{n_m}} \delta_\sigma$. The key observation here is that $\varepsilon \geqslant \prod_{\sigma\in T_{n_m}} \delta_\sigma$. Thus Lemma 2.1 says that

$$\left(\frac{1+\varepsilon}{2}\right)^{1/\ln\varepsilon} \geqslant \left(\frac{1+\prod_{\sigma\in T_{n_m}}\delta_\sigma}{2}\right)^{1/\ln\left(\prod_{\sigma\in T_{n_m}}\delta_\sigma\right)};$$

raising both sides to the power of $\ln\left(\prod_{\sigma\in T_{n_m}}\delta_\sigma\right)$, which is negative, we get that

$$\left(\frac{1+\varepsilon}{2}\right)^{\ln\left(\prod_{\sigma\in T_{n_m}}\delta_\sigma\right)/\ln\varepsilon} \leqslant \left(\frac{1+\prod_{\sigma\in T_{n_m}}\delta_\sigma}{2}\right).$$

Now Lemma 2.2 shows that

$$\prod_{\sigma\in T_{n_m}} \left(\frac{1+\delta_\sigma}{2}\right) \leqslant \left(\frac{1+\prod_{\sigma\in T_{n_m}}\delta_\sigma}{2}\right) < \frac{1+\left(1-2^{-(k+1-m)}\right)}{2} = 1-2^{-(k+1-(m-1))},$$

as required. We note that the worst case ($\prod\delta_\tau$ being maximal) is when $\varepsilon = \prod\delta_\sigma$, which happens exactly when there is only one branch below level n_k, i.e., when $|T_{n_k}| = 1$. In that case, $\mu(P^T) = 2^{-k}\mu(Q_\tau)$, where τ is the unique string on level n_k.

For $m = 0$, we get $\delta_{\langle\rangle} < 1 - 2^{-(k+1)}$, so $\mu\left(Q_{\langle\rangle}\right) = \mu(P^T) > 2^{-(k+1)}$ and the proof is complete. $\square$

4. Infinite trees

In this section, instead of using finite trees, we use infinite trees that have no dead ends. We maintain our notation from the previous section. Our trees have a particular form:

Definition 4.1. A tree $T \subseteq \omega^{<\omega}$ is *full-by-finite* if there is a finite tree $S \subset \omega^{<\omega}$, each leaf of which lies on some level n_m, such that

$$T = S \cup \{\sigma \in \omega^{<\omega} : \sigma \text{ extends some leaf of } S\}.$$

Let $T \subseteq \omega^{<\omega}$ be a tree with no dead ends. We let $P^T = \bigcup_{f \in [T]} P^f$.

Theorem 4.2. *Suppose that T is a full-by-finite tree, witnessed by the finite tree S. If $\mu\left(P^T\right) \leqslant 2^{-(k+1)}$ then S is k-svelte.*

Proof. Let L be the set of leaves of S. Assume that $\mu\left(P^T\right) \leqslant 2^{-(k+1)}$. We will show that $\mu\left(\bigcup_{\tau \in L} P^\tau\right) \leqslant 2^{-(k+1)}$, from which will follow by Theorem 3.3 that S is k-svelte.

For any $m \in \omega$, for any $\sigma \in \omega^{n_m}$, let

$$B_\sigma = P^\sigma \setminus \bigcup_{\rho \in \omega^{n_{m+1}} : \rho \supset \sigma} P^\rho.$$

Then since $\{U_{I(\rho)} : \rho \in \omega^{n_{m+1}}\}$ are all independent,

$$\mu\left(B_\sigma\right) = \mu\left(P^\sigma \setminus \bigcup_{\rho \in \omega^{n_{m+1}} : \rho \supset \sigma} U_{I(\rho)}\right) = 0.$$

Let $\tau \in L$ and suppose that $|\tau| = n_{m_\tau}$.

$$P^\sigma \setminus \bigcup_{f \in \omega^\omega : f \supset \tau} P^f \subseteq \bigcup_{m \geqslant m_\tau} \bigcup_{\sigma \in \omega^{n_m} : \sigma \supseteq \tau} B_\sigma$$

and so the measure of the set on the left is 0. Since every $f \in [T]$ extends some $\tau \in L$, we get that

$$\mu\left(\bigcup_{\tau \in L} P^\tau \setminus P^T\right) = 0.$$

□

Note that we did not actually need T to be full above the leaves of S; it is sufficient that every string of length n_m extending a leaf of S has infinitely many extensions of length n_{m+1}. However, this generality is not useful because the full-by-finite trees are the ones that arise naturally:

Lemma 4.3. *Suppose that $C \subseteq 2^\omega$ is clopen. Then there is some full-by-finite tree T such that*

$$[T] = \left\{f \in \omega^\omega : P^f \subseteq C\right\}.$$

Proof. Let $C \subseteq 2^\omega$ be clopen; then there is some $d \in \omega$ such that C is the union of $[\rho]$ for certain $\rho \in 2^d$.

Let R be the set of strings $\sigma \in \omega^\omega$ such that $P^\sigma \subseteq C$, and let L be the set of minimal strings in R. Then L is finite: if $\tau \in L$ then $|\tau| = n_m$ for some m and $I(\tau) \leqslant d$. Let S be the downward closure of L. Then S is a finite tree (and each leaf of S has length n_m for some m); noting that R is closed upwards, we can let $T = S \cup R$. □

Remark 4.4. The witnessing tree S can be obtained effectively from C.

Proof. To find L we only need to search over the finitely many strings σ such that $I(\sigma) \leqslant d$, and tell for which such strings we have $P^\sigma \subseteq C$. The operation mapping σ to (a canonical index for the clopen set) P^σ is effective, so given C, we can answer that question, for each σ, effectively. The number d can also be obtained effectively from C. □

5. Characterising Low(Kurtz)

In this section we prove that no DNR function is low for Kurtz randomness. First we show that DNR functions cannot be low for Kurtz tests. In particular, there is a (recursive) increasing sequence $\bar{n}$ as above such that using the derived operator P, if f is a DNR function, then P^f (which is a $\Pi^0_1(f)$ class of measure 0) is not contained in any Π^0_1 class of measure 0.

Theorem 5.1. *A DNR degree is not low for Kurtz tests.*

Proof. We use the recursion theorem. Recall that J, the universal partial recursive function, is defined by letting $J(e) = \varphi_e(e)$ if the latter converges, and is left undefined otherwise; $f \in \omega^\omega$ is a DNR function if for all $e \in \operatorname{dom} J$, $f(e) \neq J(e)$. The function J is universal in the sense that for every $c \in \omega$ there is some (uniformly) recursive order function α_c such that $J \circ \alpha_c = \varphi_c$.

During the construction we define a partial recursive function ψ; by the recursion theorem, we can assume that we know a $c \in \omega$ such that $\psi = \varphi_c$ and so we know the "column" of J over which we "have control". Partition ω into finite intervals $X_{m,k}$ for $m > 0$ and $k < m$ so that $|X_{m,k}| = 2^{m-(k+1)}$ (and so that if $m < m'$, $k < m$ and $k' < m'$ then $\max X_{m,k} < \min X_{m',k'}$).

We can now define the sequence $\langle n_m \rangle$ so that for all $m > 0$ and $k < m$, $\alpha_c[X_{m,k}]$ is contained in the interval $[n_{m-1}, n_m)$.

We now describe how to define ψ. Let $k \in \omega$ and let Q_k be the k^{th} Π^0_1 class. If at some stage s of the construction we have enumerated enough of (the complement of) Q_k to see that $\mu\left(Q_k[s]\right) < 2^{-(k+1)}$, then we compute a finite tree S which is k-svelte and such that the upward closure T of S is the tree of paths f such that $P^f \subseteq Q_k[s]$. Let $\langle S_{k+1}, S_{k+2}, \dots, S_l \rangle$ witness that S is k-svelte. For every $m \in [k+1, l]$ and every $\sigma \in S_m$, we pick a distinct $x \in X_{m,k}$ and define $\psi(x) = \sigma(\alpha_c(x))$. Thus for such σ, $\sigma(\alpha_c(x)) = J(\alpha_c(x))\downarrow$ and so σ is not a DNR string. Since every $f \in [T]$ extends some σ in some S_m, no $f \in [T]$ is DNR. This concludes the proof. □

The uniformity of P^f yields the main theorem:

Theorem 5.2. *Suppose that f is DNR. Then P^f contains a Kurtz random real. Hence,* Low(Kurtz) $\subseteq \neg$DNR.

Proof. The main idea is that for any $f \in \omega^\omega$, if some nonempty clopen subclass $[\rho^*] \cap P^f$ is covered by a Π^0_1 class of measure 0, then so is all of P^f. For suppose that $\rho^* \in 2^{<\omega}$, $[\rho^*] \cap P^f \neq \emptyset$, and $[\rho^*] \cap P^f \subseteq Q$ where Q is Π^0_1. For all $\rho \in 2^{<\omega}$ of the same length as ρ^*, let

$$Q_\rho = \{\rho^\frown X \,:\, \rho^{*\frown} X \in Q\}$$

and let

$$Q' = \bigcup_{\rho\,:\,|\rho|=|\rho^*|} Q_\rho.$$

Then Q' is a Π^0_1 class, and if Q has measure 0, then so does Q'. Suppose that $X \in P^f$, and let $\rho = X \upharpoonright |\rho^*|$; let $X^* = \rho^{*\frown} X \upharpoonright [|\rho^*|, \infty)$. Recall that

$$P^f = \{Y \in 2^\omega : (\forall m \in \omega)\, Y(I(f \upharpoonright n_m)) = 1\}.$$

Let $m \in \omega$ and let $i = I(f \upharpoonright n_m)$. If $i \geqslant |\rho|$, then $X^*(i) = X(i) = 1$ because $X \in P^f$. If $i < |\rho|$ then $X^*(i) = \rho(i) = 1$ because $[\rho] \cap P^f \neq \emptyset$. Hence $X^* \in P^f$, so $X^* \in Q$. It follows that $X \in Q'$ so $P^f \subseteq Q'$.

Now suppose that f is a DNR function. We know that P^f is not contained in any Π^0_1 class of measure 0, and so for all $\rho \in 2^{<\omega}$, if $[\rho] \cap P^f$ is nonempty, then that class is also not contained in any Π^0_1 class of measure 0. By induction, we define an increasing sequence $\langle \xi_k \rangle$ of strings. List all Π^0_1 classes of measure 0 as $R_1, R_2, \dots$. Let $\xi_0 = \langle\rangle$. Given ξ_{k-1}, assuming that $[\xi_{k-1}] \cap P^f \neq \emptyset$, we know that $[\xi_{k-1}] \cap P^f$ is not contained in R_k, and so we can find some ξ_k extending ξ_{k-1} such that $[\xi_k] \cap P^f \neq \emptyset$ but $[\xi_k] \cap R_k = 0$. Then $X = \bigcup_k \xi_k$ is a Kurtz random real contained in P^f. □

This completes the proof that the degrees that are low for Kurtz randomness are low for weak 1-genericity. Since every degree that is low for weak 1-genericity is low for Kurtz tests [16], all three notions coincide.

6. Characterising $\mathrm{Low}(\mathcal{M}, \mathrm{Kurtz})$

We now consider the classes $\mathrm{Low}(\mathcal{M}, \mathrm{Kurtz})$ for notions of randomness stronger than Kurtz randomness. We start with Martin-Löf randomness; we show that a degree is in $\mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$ iff it is not DNR. Note that, as there is a universal Martin-Löf test, lowness for the pair $(\mathrm{ML}, \mathrm{Kurtz})$ is the same as lowness for $(\mathrm{ML}, \mathrm{Kurtz})$-tests.

One direction of the characterisation of $\mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$ was provided by Bjørn Kjos-Hanssen in [5].

Theorem 6.1 (Kjos-Hanssen)**.** $\neg\mathrm{DNR} \subseteq \mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$.

Proof. Let $\mathbf{a}$ be a degree that does not compute a DNR function. Let P be any $\Pi^0_1(\mathbf{a})$ class of measure 0. We want to show that P is contained in an unrelativised ML-test.

There is an $\mathbf{a}$-recursive, nested sequence $\langle C_n \rangle$ of clopen sets such that $\mu(C_n) = 2^{-n}$ and $P = \bigcap_n C_n$. By the assumption on $\mathbf{a}$, there are infinitely many numbers n such that $J(n) = C_n$ (or more precisely, $J(n)$ is a code for C_n). Now let U_n be the union of all $J(e)$ for $e > n$ such that $J(e)$ codes a clopen set of measure at most 2^{-e}. Then $\langle U_n \rangle$ is uniformly r.e. and indeed a ML-test, and $P \subseteq \bigcap U_n$ because for all n there is some $e > n$ such that $C_e \subseteq U_n$. □

The other direction extends Theorem 5.2 and uses the tools developed in earlier sections.

Theorem 6.2. $\mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz}) \subseteq \neg\mathrm{DNR}$.

Proof. As in the proof of Theorem 5.1, we consider the operator $f \mapsto P^f$; again we define a partial function ψ and, by the recursion theorem, we get an index c such that $\psi = \varphi_c$. We partition ω into intervals $X_{m,k}$ for $m > 0$ and $k < m$ such that $|X_{m,k}| = 2^{m-(k+1)}$ and define the sequence $\langle n_m \rangle$ so that $\alpha_c[X_{m,k}] \subset [n_{m-1}, n_m)$ for all $k < m$.

Let $\langle U_k \rangle$ be the universal Martin-Löf test; at stage s, for all k, let $U_k[s]$ be the clopen set that has so far been enumerated as part of U_k. At stage s, suppose that a (unique) number k has been enumerated into $\emptyset'$. Since $\mu(U_{k+1}[s]) < 2^{-(k+1)}$, we can compute a finite k-svelte tree whose upwards closure is the tree of paths f such that $P^f \subseteq U_{k+1}[s]$. We ensure none of these paths are DNR by defining ψ as before.

Now suppose that f is DNR but that $P^f \subseteq \bigcap_k U_k$. We show that $\emptyset' \leqslant_T f$. This will prove the theorem: no complete degree is low for (ML,Kurtz) because there is an ML-random set $R \leqslant_T \emptyset'$. Let $k \in \omega$. Since $P^f \subseteq U_{k+1}$ and P^f is compact, there is some s such that $P^f \subseteq U_{k+1}[s]$. Such a stage s can be effectively obtained from f. Then we know that if $k \notin \emptyset'[s]$ then $k \notin \emptyset'$, for otherwise we would ensure that f is not DNR at the stage at which k enters $\emptyset'$. □

We now turn to the intermediate classes $\mathrm{Low}(\mathrm{Schnorr}, \mathrm{Kurtz})$ and $\mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz})$. Recall that a degree $\mathbf{a}$ is called *high* if $\mathbf{a}' \geqslant \mathbf{0}''$.

Theorem 6.3. $\mathrm{Low}(\mathrm{Schnorr}, \mathrm{Kurtz}) = \mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz}) = \neg\mathrm{High} \cap \neg\mathrm{DNR}$. *Furthermore, the test notions are also equivalent.*

Proof. For a degree $\mathbf{a}$,

$$\begin{array}{ccc} \mathbf{a} \text{ is low for (Schnorr, Kurtz)-tests} & \Longrightarrow & \mathbf{a} \text{ is low for (CR, Kurtz)-tests} \\ \Downarrow & & \Downarrow \\ \mathbf{a} \text{ is low for (Schnorr, Kurtz)} & \Longrightarrow & \mathbf{a} \text{ is low for (CR, Kurtz).} \end{array}$$

Because $\mathrm{Low}(\mathrm{CR}, \mathrm{Kurtz}) \subseteq \mathrm{Low}(\mathrm{ML}, \mathrm{Kurtz})$, all of these properties imply that $\mathbf{a}$ is non-DNR. Nies, Stephan and Terwijn [11] proved that every high degree contains a computably random real. Hence no high degree can be low for $(\mathrm{CR}, \mathrm{Kurtz})$, so again, all four properties imply that $\mathbf{a}$ is not high.

All that remains is to prove that if $\mathbf{a}$ is not high and not DNR, then it is low for $(\mathrm{Schnorr}, \mathrm{Kurtz})$-tests. We will use a result of Kjos-Hanssen (see [1]): if $\mathbf{a}$ is not high and not DNR, then for any $\mathbf{a}$-recursive function h, there is a (total) recursive function f such that $f(n) = h(n)$ for infinitely many n.

Let P be a $\Pi^0_1(\mathbf{a})$ class of measure 0, and again consider the $\mathbf{a}$-recursive, nested sequence $\langle C_n \rangle$ of clopen sets such that $\mu(C_n) = 2^{-n}$ and $P = \bigcap_n C_n$. Define an $\mathbf{a}$-recursive function h such that $h(n)$ is a code for C_n. Take the recursive function f guaranteed by Kjos-Hanssen's result; we may assume that $f(n)$ codes a clopen set of measure 2^{-n} for all n. Now by padding, we can increase $\bigcup_{e>n} f(e)$ to a nested sequence of uniformly r.e. classes $\langle U_n \rangle$ of measure exactly 2^{-n}—the point is that after taking $\bigcup_{e \in (n,m]} f(e)$ we know that the contribution of the rest is at most 2^{-m}. Therefore, we have a Schnorr test and $P \subseteq \bigcap_n U_n$. □

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