LOWNESS FOR COMPUTABLE MACHINES

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Abstract. Two lowness notions in the setting of Schnorr randomness have been studied (lowness for Schnorr randomness / tests, by Terwijn and Zambella [18], and by Kjos-Hanssen, Stephan, and Nies [7]; and Schnorr triviality, by Downey, Griffiths and LaForte [3, 4] and Franklin [6].) We introduce lowness for computable machines, which by results of Downey and Griffiths [3], is an analog of lowness for K. We show that the reals that are low for computable machines are exactly the computably traceable ones, and so this notion coincides with that of lowness for Schnorr randomness / tests.

1. Introduction

One of the most remarkable set of results in the theory of algorithmic randomness are those by Nies and his co-authors proving the coincidence of a number of natural "anti-randomness" classes associated with prefix-free Kolmogorov complexity. Recall that A is called *low for* K if for all x, $K^A(x) \geq K(x) - O(1)$,[1] A is called K*-trivial* if for all n, $K(A \restriction n) \leq K(n) + O(1)$, and that A is called *low for Martin-Löf randomness* if the collection of reals Martin-Löf random relative to A is the same as the collection of Martin-Löf random reals. We have the following.

Theorem 1.1 (Nies,Hirschfeldt,[11, 12]). *The following classes of reals coincide.*

(i) K*-low.*
(ii) K*-trivial.*
(iii) *Low for Martin-Löf randomness.*

The situation for other notions of randomness is less clear. In this paper we look at the situation for Schnorr randomness. Recall that a real A is said to be *Schnorr random* iff for all Schnorr tests $\{U_n : n \in \mathbb{N}\}$, $A \notin \cap_n U_n$, where a Schnorr test is a Martin Löf test such that $\mu(U_n) = 2^{-n}$ for all n. (Of course 2^{-n} is a convenience. As Schnorr [15] observed, any uniformly computable sequence of reals with effective limit 0 would do.)

The reader might note that there are two possible lowness notions associated with Schnorr randomness. A real A is *low for Schnorr randomness* if no Schnorr random real became non-Schnorr-random relative to A. But since there is no universal Schnorr test, we can also define the stronger (and more technical) notion of lowness

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[1] In this paper K will denote prefix-free Kolmogorov complexity and we will refer to members $A = a_0a_1\dots$ of Cantor space as *reals*, with $A \restriction n$ being the first n bits of A. We assume that the reader is familiar with the theory of algorithmic randomness. For details we refer to the monographs of Li and Vitányi [9], of Downey and Hirschfeldt [5], and of Nies [13].

for *tests*; a real A is *low for Schnorr tests* if for every A-Schnorr test $\{U_n^A : n \in \mathbb{N}\}$, there is a Schnorr test $\{V_n : n \in \mathbb{N}\}$ such that $\cap_n U_n^A \subseteq \cap_n V_n$.

Terwijn and Zambella [18] proved that there were reals that were low for Schnorr tests. In fact, they classified the collection of reals which were low for Schnorr tests.

Definition 1.2 (Terwijn and Zambella [18])**.** *We say that a real A is* computably traceable *if there is a computable function $h(x)$ such that for all functions $g \leq_T A$, there is a computable collection of canonical finite sets $D_{r(x)}$ with $|D_{r(x)}| \leq h(x)$ and such that $g(x) \in D_{r(x)}$.*

We remark that (as noticed by Terwijn and Zambella) if A is computably traceable then for the witnessing function h we can choose any computable, non-decreasing and unbounded function.

Terwijn and Zambella proved the following attractive result.

Theorem 1.3 (Terwijn and Zambella [18])**.** *A is low for Schnorr tests iff A is computably traceable.*

We remark that whilst all K-trivials are Δ_2^0 by a result of Chaitin [1], the computably traceable reals are all hyperimmune-free, and there are $2^{\aleph_0}$-many of them.

Subsequently, Kjos-Hanssen, Stephan, and Nies [7] proved that A is low for Schnorr randomness iff A is low for Schnorr tests.

The reader might wonder about analogs of the other results for K. The other members of the coincidence involve K-triviality and lowness for K. What about the Schnorr situation? we want some analog for the characterization of Martin-Löf randomness in terms of prefix-free complexity (A is Martin-Löf random iff for all n, $K(A \upharpoonright n) \geq n - O(1)$.) Such a characterization was discovered by Downey and Griffiths ([3]). They define a prefix-free Turing machine M to be *computable* if the domain of M has computable measure, that is, $\sum_{\{\sigma : M(\sigma)\downarrow\}} 2^{-|\sigma|}$ is a computable real. They then establish the following:

Theorem 1.4 (Downey and Griffiths [3])**.** *A is Schnorr random iff for all computable machines M, for all n, $K_M(A \upharpoonright n) \geq n - O(1)$.*[2]

The quanitification over machines is necessary because (as is the situation for Schnorr tests), there is no universal computable machine. With this result we are in a position to define a real A to be *Schnorr trivial* if for every computable machine N there is a computable machine M such that for all n, $K_M(A \upharpoonright n) \leq K_N(n) + O(1)$. This notion was initially explored by Downey and Griffiths [3], and Downey, Griffiths and LaForte [4], who showed that this class does not coincide with the reals that are low for Schnorr randomness. For instance, there are Turing complete Schnorr trivial reals. Johanna Franklin [6] established the following.

Theorem 1.5 (Franklin [6])**.**

(i) *There is a perfect set of Schnorr trivials.*
(ii) *Every degree above $\mathbf{0}'$ contains a Schnorr trivial.*
(iii) *Every real that is low for Schnorr randomness is also Schnorr trivial.* [3]

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[2] Note that since the range of M need not be all of $2^{<\omega}$, we need to let $K_M(x) = \infty$ for all strings x not in the range of M.

[3] Interestingly, Franklin also showed that the reals that are low for Schnorr randomness are not closed under join.

Thus the relationship between lowness for Schnorr randomness and Schnorr triviality is quite different from the analogous situation for Martin-Löf randomness.

The last piece of the puzzle is the analog for lowness for K. Armed with the machine characterization for Schnorr randomness, we give the following definition.

Definition 1.6. *A real A is* low for computable machines *iff for all A-computable machines M there is a computable machine N such that for all x,*

$$K_M^A(x) \geq K_N(x) - O(1).$$

Note that like "low for K", lowness for computable machines is the strongest notion of all; it is immediate that every real that is low for computable machines is low for Schnorr tests. In this paper we show that unlike the situation for triviality, the coincidence of the reals low for ML randomness and the low for K ones carries over to the Schnorr case:

Theorem 1.7. *A real A is low for computable machines iff A is computably traceable.*

We remark that part (iii) of Theorem 1.5 above is a consequence of Theorem 1.7, since every real A that is low for computable machines is Schnorr trivial. For let N be a computable machine. Let L be an A-computable machine such that for all n, $K_L^A(A \restriction n) = K_N(n)$ (for all x, if $N(x) = n$ then let $L(x) = A \restriction n$.) Then there is some computable machine M such that for all x, $K_M(x) \leq K_L^A(x) + O(1)$; M is as required to witness that A is trivial.

2. The proof

We note that if we enumerate a Kraft-Chaitin set with a computable sum then the machine produced is computable:

Lemma 2.1 (Kraft-Chaitin)**.** *Let $\langle d_0, \tau_0\rangle, \langle d_1, \tau_1\rangle, \dots$ be a computable list of pairs consisting of a natural number and a string. Suppose that $\sum_{i<\omega} 2^{-d_i}$ is a computable real (in particular, is finite). Then there is a* computable *machine N such that for all i, $K_N(\tau_i) \leq d_i + O(1)$.*

(See Downey and Hirschfeldt [5] for a proof of the Kraft-Chaiting theorem; the fact that we get a computable machine is immediate from the proof.)

To prove Theorem 1.7 we need to show that every computably traceable set A is low for computable machines. So let A be a computably traceable set and let M be an oracle machine such that M^A is A-computable. The idea (somewhat following Terwijn and Zambella) is to "break up" the machine M^A into small and finite pieces which we trace. We view M^A as a function from strings to strings. We will partition M^A into finite pieces $f^*, f_0, f_1, f_2, \dots$ where for $n < \omega$, the measure of the domain of f_n is smaller than some small rational ε_n. We then trace the sequence $\langle f_n\rangle$; so for every n, we get $h(n)$ many candidates for f_n, each with domain with measure smaller than ε_n. If we keep $\sum_n h(n)\varepsilon_n$ finite, the union of all of the candidates can be translated into a Kraft-Chaitin set that produces the machine we want.

Let h be the computable function given by Definition 1.2 (again we remark that we can pick any reasonable function; it doesn't matter for this proof.) Fix a computable, decreasing sequence of positive rationals $\varepsilon_0, \varepsilon_1, \dots$ such that $\sum_{n<\omega} h(n)\varepsilon_n$

is finite; moreover, we want the convergence to be quick, say for every $m < \omega$,

$$\sum_{n \geq m} h(n)\varepsilon_n \; < \; 2^{-m}.$$

Let $\langle(\sigma_i, \tau_i)\rangle_{i<\omega}$ be an A-computable enumeration of M^A. We let M_s^A, the machine M^A at stage s, be $\{(\sigma_i, \tau_i) : i < s\}$, and similarly let $M_{\geq s}^A = M^A \setminus M_s^A = \{(\sigma_i, \tau_i) : i \geq s\}$, and for $s < t$, $M_{[s,t)}^A = M_t^A \setminus M_s^A$.

Let t_n be the least stage t such that $\mu(\operatorname{dom} M_{\geq t}^A) < \varepsilon_n$. We let $f^* = M_{t_0}^A$; for $n < \omega$, we let $f_n = M_{[t_n, t_{n+1})}^A$. The point is that the sequence $\langle t_n \rangle$, and so the sequence $\langle f_n \rangle$, are A-computable, as $\mu(\operatorname{dom} M_{\geq t}^A) = \mu(\operatorname{dom} M^A) - \mu(\operatorname{dom} M_t^A)$; the first number is A-computable by assumption, and the latter a rational, computable from the sequence $\langle(\sigma_i, \tau_i)\rangle$ and so from A. For all $n < \omega$, $\mu(\operatorname{dom} f_n) < \varepsilon_n$.

Each f_n is a finite function (and so has a natural number code.) We can thus computably trace the sequence $\langle f_n \rangle$; there is a computable sequence of finite sets $\langle X_n \rangle_{n<\omega}$ (i.e. $X_n = D_{r(n)}$ where r is computable) such that for each n, $|X_n| \leq h(n)$, and for each n, (the code for) $f_n \in X_n$. By weeding out elements, we may assume that for each $n < \omega$, every element of X_n is a code for a finite function f from strings to strings whose domain is prefix-free and has measure at most ε_n.

Enumerate a Kraft-Chaitin set L as follows. Let $\langle d, \tau \rangle \in L$ if there is some σ such that $|\sigma| = d$, and one of the following holds:

- $(\sigma, \tau) \in f^*$;
- For some n and for some $f \in X_n$, $(\sigma, \tau) \in f$.

The set L is computably enumerable. Further, the total of the requests $s = \sum_{(d,\tau)\in L} 2^{-d}$ is a finite, computable real, as we know that for any m,

$$\sum \{2^{-|\sigma|} \colon (\exists n \geq m)(\exists f \in X_n)[\sigma \in \operatorname{dom} f]\} \; \leq \; \sum_{n \geq m} h(n)\varepsilon_n \; \leq \; 2^{-m}.$$

From the "computable" Kraft-Chaitin theorem we get a computable machine N such that for some constant c, if $(d, \tau) \in L$, then $K_N(\tau) \leq d + c$. On the other hand, we know that if τ is in the range of M^A then $(K_{M^A}(\tau), \tau) \in L$ because $f_n \in X_n$ for all n. Thus N is as required.

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