VICTORIA UNIVERSITY OF WELLINGTON

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School of Mathematical and Computing Sciences

Computer Science

Meaning and structural rules in natural deduction

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Abstract

Classical and constructive natural deduction can be formulated using exactly the same introduction and elimination rules, with the difference between them expressed using a structural rule.

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Author Information

Neil's main interest is in exploiting the observation that propositions and types are the same thing. He also often works on his favourite hobby, the rejection of the law of the excluded middle.

1 Introduction

Lets assume that the meaning of a logical constant is determined by the rules that govern its use. Lets also agree that intuitionistic[1] and classical logic give different meanings to the logical constants.

Think of your favourite natural deduction system for intuitionistic logic. Now extend it to handle classical logic. Where does the difference lie? If you added this rule:

$$\dfrac{\begin{matrix}\vdots \\ \neg\neg A\end{matrix}}{A}\;\text{DN}$$

then you might think that the difference between classical and constructive logic lies in the meaning of repeated negations.

On the other hand, if you added this rule:

$$\dfrac{}{A \vee \neg A}\;\text{TND}$$

you might think that the meanings of disjunction and negation both have to change.

But if you added this rule:

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\neg A] \\ \vdots \\ C\end{matrix}}{C}$$

then you might think that the difference just lies in a change to the meaning of negation.

Then again, you might add this rule:

$$\dfrac{}{((A \supset B) \supset A) \supset A}\;\text{Peirce}$$

In which case you might think that it is implication which has changed its meaning.

But, of course the, first two rules now add this theorem

$$\vdash ((A \supset B) \supset A) \supset A$$

which mentions only $\supset$, and the last rule adds this theorem

$$\vdash (A \mathbin{\&} B) \vee \neg A \vee \neg B$$

which omits only $\supset$. So, for classical logic it seems plainly false to assert that the rules for a connective govern its meaning. It also seems hard to hold only one connective responsible for the shift in meaning between classical and constructive logic. We need to look bit closer.

2 Dummett, Prawitz and intuitionistic logic

Michael Dummett [3, 4, 7, 6] and Dag Prawitz [15] amongst others, have developed a proof-theoretic explanation of the meanings of the logical constants, essentially based on the assertion by Gentzen:[9]:

[1] We use 'intuitionistic' and 'constructive' identically.

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.

We must take care with 'the final analysis', as Prior's `tonk` [16] reminds us. This connective has a perfectly good introduction rules but has a discordant elimination rule. We respond to the challenge of `tonk` by following the observation of Belnap [1]:

> we are not defining our connectives *ab initio*, but rather in terms of an *antecedently given context of deducibility*, concerning which we have some definite notions.

The emphasis is in the original. We need to say something about the *antecedently given context of deducibility*. We use a theory of judgements based on that given by Martin-Löf, and developed by him in the context of intuitionistic type theory [12, 13]. The only judgement form we will use is:

$$A \textit{ has a proof}$$

or, alternatively,

$$A \textit{ is true}.$$

In addition to dealing with categorical judgements we will also have hypothetical judgements, like:

$$\begin{matrix} A \\ \vdots \\ B \end{matrix}$$

This hypothetical judgement tells us that the judgement of B follows from and assumption, the *judgement* of A. It is important to notice that assumptions are judgements, and can themselves be hypothetical judgements. We are, perhaps, not used to seeing hypothetical hypotheses in propositional logic, but they are used in type theory in the rules for Π and W types and for universes [14].

We can combine and create judgements using *inference rules*.

2.1 Structural rules

Inference rules which do not mention any of the logical constants can be called structural rules, by analogy with the structural rules of the sequent calculus. Even at this point in the development of the theory we have a notion of the validity of a rule. For example, we probably want to accept this rule:

$$\frac{\begin{matrix} \vdots \\ A \end{matrix} \qquad \begin{matrix} \vdots \\ B \end{matrix}}{A}$$

Given a judgement of A (resting on some assumptions, which remain implicit), and a judgement of B (again, resting on some implicit assumptions), we can judge A (resting on all the assumptions the judgements which either A or B rest on).

We almost certainly want to reject this rule:

$$\frac{\begin{matrix} \vdots \\ B \end{matrix}}{A}$$

Given a judgement of of B (resting on some assumptions), we can judge A (resting on the assumptions B rests on).

We might have to think a little about this rule:

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}\qquad\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{B}$$

Given a judgement of A (resting on some assumptions), and a judgement of B (resting on some assumptions, perhaps including the judgement of A), we can judge B (resting on all the assumptions the judgement of A rests and all the assumptions, except A, that B rests on).

There is no reason why we cannot have hypothetical judgements in structural rules. It may be worth pondering whether the following rule is valid:

$$\dfrac{\begin{array}{c}[[A]\cdots B]\\ \vdots\\ A\end{array}}{A}\,\dagger$$

2.2 Introduction rules

If we are to treat the introduction rules as definitional we need to be able to formulate some idea of what the form of these rules is, and we need to be able to give (necessarily informal) explanations of the introduction rules. The informal explanations come from the Brouwer-Heyting-Kolmogorov (BHK) explanation [10, 11]. The form required [6] for the introduction rules to be suitable for use in explaining the meaning of the connectives is that each rule:

- is only an introduction rule, and does not also act as an elimination rule;[2]
- introduces only one occurence of one constant;
- the conclusion is the most complex formula in the rule.

The rules that we find in Gentzen all follow this pattern:

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \supset B}\supset \mathrm{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array}\qquad\begin{array}{c}\vdots\\ B\end{array}}{A \,\&\, B}\,\&\,\mathrm{I}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\vee\mathrm{I} \qquad \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B}\vee\mathrm{I}$$

There is no immediate evidence for $\perp$, and hence no introduction rule of the appropriate form.

Since we identify $\neg A$ with $A \supset \perp$, the $\neg$ introduction rule is produced from the $\supset$ introduction rule by identifying B with $\perp$:

[2] A rule which allows us to derive $A \,\&\, B$ from a derivation of $B \,\&\, A$ acts as both an introduction and an elimination rule for $\&$.

$$\cfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\ \neg\mathrm{I}$$

None of the rules that we presented in Section 1 are of the appropriate form, and so cannot be taken as ‘primitive’ rules.

3 More general elimination rules

Given the introduction rule or rules for a logical constant we can construct the elimination rule for it in a systematic way. To do this we follow the arguments familiar from Martin-Löf’s Type Theory [12, 14]. Type theory is an extensible theory, in the sense that we are free to add new types. In the computer science applications of type theory this is a normal activity, as we wish to use lists, trees and so on. Although we don’t normally think of logic as extensible in the same way, thanks to the Curry-Howard ‘propositions-as-types’ analogy [2], we can treat type theory as a logic. So we have a systematic way to present the rules for the logical constants too. The elimination rules that we get are harmonious, and give us a normalisation property for derivations. We generate the elimination rule in the following way. Suppose there are n introduction rules for $\#A_1 \dots A_n$. The conclusion of the # elimination rules will be some arbitrary proposition C. There will be $n+1$ premisses: one is a derivation of $\#A_1 \dots A_n$, and the other n are derivations of C which may use the premisses of each of the introduction rules. For $\vee$ there are two introduction rules, one when we have a derivation of the left disjunct and one when we have a derivation of the right. Hence the $\vee$ elimination rule will have three premisses:

1. a derivation of $A \vee B$
2. a derivation of C from A
3. a derivation of C from B

The conclusion will be C and the conclusion will not rely on the use of A in the second premiss or B in the third. Hence we get this familiar rule:

$$\cfrac{\begin{matrix}\\ \vdots \\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

The elimination rule for & is:

$$\cfrac{\begin{matrix}\\ \vdots \\ A \,\&\, B\end{matrix} \qquad \begin{matrix}[A, B]\\ \vdots \\ C\end{matrix}}{C}$$

If we let C be A or B in this rule, and appeal to a structural rule, we can recover the more familiar rules.

The $\bot$ elimination rule is simply:

$$\dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{C}$$

The $\supset$ elimination rule follows the same pattern as the other elimination rules, and looks rather different from *Modus Ponens*. It will have two premisses:

1. a derivation of $A \supset B$
2. a derivation of C which may use the premiss to the $\supset$ introduction rule, that is a derivation of B from A

$$\dfrac{\begin{array}{c}\\ \vdots\\ A \supset B\end{array} \qquad \begin{array}{c}[[A]\cdots B]\\ \vdots\\ C\end{array}}{C} \supset \mathrm{E}$$

This rule discharges a hypothetical judgement, so it may look unfamiliar. It is however exactly the rule we get by appealing to the Curry-Howard isomorphism and treating $\supset$ as non-dependent Π, using the rules give in [14], so it is not as radical as it seems. Another $\supset$ elimination rule is given by Dyckhoff [8]:

$$\dfrac{\begin{array}{c}\\ \vdots\\ A \supset B\end{array} \qquad \begin{array}{c}\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C} \supset \mathrm{E}'$$

The rule $\supset E'$ can be recovered from $\supset E$. Suppose we have a derivation of A. Now, given a derivation of A and a derivation of B from A, we can construct a derivation of B. So, given a derivation of A the second premiss of $\supset E$ becomes a derivation of C, given a derivation of B. Hence adding a derivation of A as an extra premiss to $\supset E$ allows us to justify Dyckhoff's rule.

Modus Ponens can be justified using $\supset E'$ by identifying B and C:

$$\dfrac{\begin{array}{c}\\ \vdots\\ A \supset B\end{array} \qquad \begin{array}{c}\\ \vdots\\ A\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ B\end{array}}{B}$$

The third premiss is trivial, and we recover the familiar rule.

Negation is defined as before.

The introduction and elimination rules that we have presented are all the rules that need which mention logical constants in order to define *intuitionistic* propositional logic. They are also all the rules that need which mention logical constants in order to define *classical* propositional logic.

4 Peirce's Law as a structural rule

We gave Peirce's Law above. We know that intuitionistic logic augmented with Pierce's Law gives classical logic. Lets take a closer look at what judgement the classical logician accepts in accepting the validity of Pierce's Law that the intuitionist rejects. We know that the system of rules we gave before is normalising, so we know that if there is a proof of Pierce's Law then there is a normal proof. So the final step is $\supset$ introduction:

$$\dfrac{\begin{matrix}[(A\supset B)\supset A]\\ \vdots \\ A\end{matrix}}{((A\supset B)\supset A)\supset A}\supset\mathrm{I}$$

The premiss will be justified using $\supset$ elimination:

$$\dfrac{\dfrac{[(A\supset B)\supset A] \qquad \begin{matrix}[[A\supset B]\cdots A]\\ \vdots\\ A\end{matrix}}{A}\supset\mathrm{E}}{((A\supset B)\supset A)\supset A}\supset\mathrm{I}$$

If we have a derivation of $A \supset B$, then we could have a normal proof.

$$\dfrac{\dfrac{[(A\supset B)\supset A] \qquad \begin{matrix}\left[\left[\begin{matrix}[A]\\ \vdots\\ \dfrac{B}{A\supset B}\end{matrix}\right]\cdots A\right]\\ \vdots\\ A\end{matrix}}{A}\supset\mathrm{E}}{((A\supset B)\supset A)\supset A}\supset\mathrm{I}$$

The second premiss is a derivation of A relying on a derivation of A relying on a derivation of B relying on a derivation of A. In other words, accepting the validity of Pierce's Law is just accepting the validity of the structural rule † which we presented above.

5 Comments and conclusions

We have managed to isolate the difference between classical and constructive logic in a natural deduction rule which does not mention any connective. This reflects the situation in the sequent calculus where the difference between Gentzen's systems **LJ** and **LK** is expressed independently of any of the connectives.

The introduction and elimination rules deal with local properties of proofs: it is generally our experience that classical and constructive logicians really do agree about the local properties of proofs, but that classical logicians make some extra global stipulation, such demanding to be able to judge every proposition either true or false.

It is also pleasant to absolve any individual connective of the blame for classical logic.

We can also present natural deduction systems for some intermediate logics using structural rules. For example, adding:

$$\dfrac{\begin{matrix}[[A]\cdots B]\\ \vdots\\ C\end{matrix}\qquad\qquad \begin{matrix}[[B]\cdots A]\\ \vdots\\ C\end{matrix}}{C}$$

to intuitionistic logic gives us Gödel-Dumett logic.

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