# Effective geometry in the Casimir vacuum

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Abstract:

In 1990 Scharnhorst noticed that photons exhibit anomalous propagation in the Casimir vacuum (the quantum vacuum between perfectly conducting parallel plates).

This Scharnhorst effect can be phrased in the language of nonlinear electrodynamics provided one is careful to calculate the quantum expectation value of the "effective metric".

Based on general symmetry arguments one can uniquely deduce the form of the quantum-averaged effective metric and demonstrate the complete absence of birefringence in the Casimir vacuum.

## Basic Idea:

Consider the Casimir vacuum:

A region of empty space delimited by two perfectly conducting parallel plates.

Orthogonal to the $z$ axis at positions $z = z_0$ and $z = z_0 + a$.

The presence of the plates distorts the quantum vacuum.

$$\langle C|F_{\mu\nu}|C\rangle = 0.$$

$$\langle C|F_{\mu\sigma}\, F^{\sigma}{}_{\nu}|C\rangle \neq 0.$$

This leads to both the Casimir effect (Casimir energy) *and* anomalous propagation of photons.

Details: see quant-ph/0010055;
Liberati, Sonego, Visser.

## Warning:

These effects are *tiny* — probably too small to ever be seen.

We had *hoped* to find birefringence; that would give us some hope for a measurable effect.

Unfortunately, no, there is no birefringence.

Experimentally this seems a lost cause...

In terms of basic physics however, this is still very interesting.

In particular for almost all directions:

$$c_{Casimir\ vacuum} > c_{Minkowski\ vacuum}.$$

And this really is the signal speed I'm talking about here — not the old group velocity versus phase velocity shell game...

## Nonlinear electrodynamics:

From the presentations by Mario Novello, Santiago Bergliaffa, and Jose Salim, we know that nonlinear electrodynamics generically leads to an effective metric description.

Integrate out the fermions to one loop:

$$\mathcal{L}_{\text{Schwinger}} = \mathcal{L}(\mathsf{x}, \mathsf{y}).$$

$$\mathsf{x} \equiv \frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} = \frac{1}{2} \left( \vec{B}^2 - \vec{E}^2 \right),$$

$$\mathsf{y} \equiv \frac{1}{4} F_{\mu\nu}\, {}^{\star}F^{\mu\nu} = \vec{E} \cdot \vec{B}.$$

Specific form of $\mathcal{L}_{\text{Schwinger}}$ not needed yet.

## Nonlinear electrodynamics:

Complete equations of motion for nonlinear electrodynamics:

Bianchi identity

$$F_{[\mu\nu,\lambda]} = 0,$$

Dynamical equation

$$(\partial_{\mathsf{x}}\mathcal{L})\ \partial_\nu F^{\mu\nu} + \frac{1}{2}\, M^{\mu\alpha}{}_{\nu\beta}\ \partial_\alpha F^{\nu\beta} = 0.$$

The tensor

$$\begin{aligned} M^{\mu\alpha}{}_{\nu\beta} \ &\equiv\ \left(\partial^2_{\mathsf{x}}\mathcal{L}\right)\ F^{\mu\alpha}\ F_{\nu\beta} + \left(\partial^2_{\mathsf{y}}\mathcal{L}\right)\ {}^\star F^{\mu\alpha}\ {}^\star F_{\nu\beta} \\ &\quad + \partial_{\mathsf{xy}}\mathcal{L}\ (F^{\mu\alpha}\ {}^\star F_{\nu\beta} + {}^\star F^{\mu\alpha}\ F_{\nu\beta}) \end{aligned}$$

is antisymmetric in both contravariant and co-variant indices.

Define

$$\Omega_{\mu\alpha\nu\beta} = \partial_{\mathsf{x}}\mathcal{L}\,(\eta_{\mu\nu}\,\eta_{\alpha\beta} - \eta_{\mu\beta}\,\eta_{\alpha\nu}) + M_{\mu\alpha\nu\beta}.$$

## Nonlinear electrodynamics:

Then

$$\Omega_{\mu\alpha\nu\beta}\, \partial^\alpha F^{\nu\beta} = 0.$$

Split the electromagnetic field into a background plus a propagating photon.

Apply the eikonal approximation: Introduce a slowly varying amplitude $f^{\mu\nu}$ and a rapidly varying phase $\phi$:

$$F^{\mu\nu} = F^{\mu\nu}_{\mathsf{bkg}} + f^{\mu\nu} \mathrm{e}^{\mathrm{i}\phi}.$$

The wave vector is $k_\mu = \partial_\mu \phi$.

Retain terms linear in the propagating photon:

$$\left(\Omega_{\mu\alpha\nu\beta}\right)_{\mathsf{bkg}}\; k^\alpha\; f^{\nu\beta} = 0.$$

But the background field is itself subject to quantum fluctuations.

## Nonlinear electrodynamics: EM fluctuations

To take quantum fluctuations into account the coefficients of the propagation equation are identified with the expectation value of the corresponding quantum operators in the background state $|\psi\rangle$:

$$\langle\psi|\Omega_{\mu\alpha\nu\beta}|\psi\rangle\, k^{\alpha}\, f^{\nu\beta} = 0.$$

The Bianchi identity constrains $f^{\mu\nu}$ to be of the form

$$f^{\mu\nu} = k^{\mu}\, a^{\nu} - k^{\nu}\, a^{\mu},$$

Then

$$\langle\psi|\Omega_{\mu\alpha\nu\beta}|\psi\rangle\, k^{\alpha}\, k^{\beta}\, a^{\nu} = 0.$$

In general, $\Omega$ can be decomposed into an isotropic part plus anisotropic contributions:

$$\langle\psi|M_{\mu\alpha\nu\beta}|\psi\rangle = d_1\,(\eta_{\mu\nu}\,\eta_{\alpha\beta} - \eta_{\mu\beta}\,\eta_{\alpha\nu}) + \Delta_{\mu\alpha\nu\beta},$$

# Nonlinear electrodynamics: Casimir vacuum

Here $d_1$ is a function that can be computed from the Lagrangian.

Define the function $d_0 \equiv \langle\psi|\partial_{\mathsf{x}}\mathcal{L}|\psi\rangle$:

$$\langle\psi|\Omega_{\mu\alpha\nu\beta}|\psi\rangle = (d_0 + d_1)\,(\eta_{\mu\nu}\,\eta_{\alpha\beta} - \eta_{\mu\beta}\,\eta_{\alpha\nu}) + \Delta_{\mu\alpha\nu\beta}.$$

In the Casimir vacuum, considerable information can be extracted by using only symmetry considerations.

(Similar to what Bryce DeWitt did for the stress-energy-momentum tensor in the Einstein Centenary Survey.)

The presence of a preferred direction, and the symmetry of the configuration, allow us to claim that the functions $d_0$ and $d_1$ can only depend on the $z$ coordinate.

# Nonlinear electrodynamics: Casimir vacuum

Furthermore

$$\Delta^{\mu\alpha}{}_{\nu\beta} = d_2(z)\,(\delta^{\mu}{}_{\nu}\, n^{\alpha}\, n_{\beta} - \delta^{\mu}{}_{\beta}\, n^{\alpha}\, n_{\nu} + \delta^{\alpha}{}_{\beta}\, n^{\mu}\, n_{\nu} - \delta^{\alpha}{}_{\nu}\, n^{\mu}\, n_{\beta}),$$

where $n^{\mu} \equiv (0,0,0,1)$ is a unit vector orthogonal to the plates, while $d_2(z)$ is a function of $z$.

Define the function

$$\xi(z) \equiv \frac{d_2(z)}{d_0(z) + d_1(z)},$$

and the tensor

$$\mathsf{g}_{\mu\nu} = \eta_{\mu\nu} + \xi\, n_{\mu} n_{\nu}.$$

Then

$$\langle C|\Omega_{\mu\alpha\nu\beta}|C\rangle = [d_0(z)+d_1(z)]\,(\mathsf{g}_{\mu\nu}\,\mathsf{g}_{\alpha\beta} - \mathsf{g}_{\mu\alpha}\,\mathsf{g}_{\nu\beta})$$

## Nonlinear electrodynamics: Casimir vacuum

Insert this into the photon EOM:

$$\langle C|\Omega_{\mu\alpha\nu\beta}|C\rangle \; k^\alpha \; k^\beta \; a^\nu = 0.$$

Then

$$[\mathsf{g}_{\mu\nu} \; a^\nu] \; \left[\mathsf{g}_{\alpha\beta} \; k^\alpha \; k^\beta\right] - [\mathsf{g}_{\mu\alpha} \; k^\alpha] \; \left[\mathsf{g}_{\nu\beta} \; a^\nu \; k^\beta\right] = 0.$$

This implies:

$$\left[\mathsf{g}_{\alpha\beta} \; k^\alpha \; k^\beta\right] = 0.$$

$$\left[\mathsf{g}_{\nu\beta} \; a^\nu \; k^\beta\right] = 0.$$

That is: photons follow null curves of $\mathsf{g}_{\alpha\beta}$ and so we are justified in calling this the effective metric.

## Key points:

There is an effective metric that governs photon propagation.

Polarization vector is orthogonal to 4-momentum — orthogonal with respect to this effective metric.

As usual, there are two polarizations. They travel at the same speed.

Therefore no birefringence in the Casimir vacuum.

Different from the usual situation in nonlinear electrodynamics; the key here is that:

$$\langle C|F_{\mu\nu}|C\rangle = 0.$$

## Phase and group velocities:

Dispersion relation:

$$\omega^2 = |\vec{k}|^2 + \xi \; |\vec{k}|^2 \; \cos^2\theta.$$

Phase velocity:

$$v_{\text{phase}}(\theta) = \frac{\omega}{|\vec{k}|} = \left(1 + \xi \; \cos^2\theta\right)^{1/2}.$$

Group velocity:

$$v^i_{\text{group}} \equiv \frac{\partial\omega}{\partial k_i} = \frac{1}{\omega}\left[k^i + \xi \; (k \cdot n) \; n^i\right],$$

so

$$\vec{v}_{\text{group}}(\theta) = \frac{1}{v_{\text{phase}}(\theta)}(\sin\theta, 0, (1+\xi)\cos\theta).$$

Indeed

$$v_{\text{group}}(\theta) = \sqrt{\frac{1 + 2\,\xi \; \cos^2\theta + \xi^2\cos^2\theta}{1 + \xi \; \cos^2\theta}}.$$

## Size of the effect:

So far we have argued solely on the basis of symmetry.

Our results apply equally well to Euler–Heisenberg and Born–Infeld.

For real world estimates use Euler–Heisenberg.

$$\mathcal{L}_{\mathsf{EH}} = -\frac{1}{4\pi}\mathsf{x} + c_1\mathsf{x}^2 + c_2\mathsf{y}^2,$$

$$c_1 = \frac{\alpha^2}{90\pi^2 m_{\mathsf{e}}^4}, \qquad c_2 = \frac{7\alpha^2}{360\pi^2 m_{\mathsf{e}}^4}.$$

In this case

$$M^{\mu\nu}{}_{\alpha\beta} = 2c_1\, F^{\mu\nu}\, F_{\alpha\beta} + 2c_2\, {}^{\star}F^{\mu\nu}\, {}^{\star}F_{\alpha\beta}.$$

## Size of the effect:

Then $d_1(z)$ and $d_2(z)$ are of order $\alpha^2$.

In the Casimir vacuum

$$d_0(z) = -\frac{1}{4\pi} + 2c_1 \langle C|\mathsf{x}|C\rangle.$$

To first order in $\alpha^2$

$$\xi(z) = -4\pi\, d_2(z),$$

and

$$v_{\text{phase}}(\theta) = v_{\text{group}}(\theta) = 1 - 2\pi d_2(z)\ \cos^2\theta.$$

In principle the coefficient $d_2(z)$ could depend on $z$, the position relative to the two plates. In the specific case of the Casimir vacuum it is simply a position-independent number.

## Size of the effect:

Performing a brief calculation

$$d_2(z) = \frac{16\pi}{3}(c_1 + c_2)\ \langle C\,|T_{zz}|\,C\rangle\,.$$

The symmetries of the Casimir vacuum stress-energy then imply

$$d_2(z) = 16\pi(c_1 + c_2)\ \langle C\,|T_{00}|\,C\rangle\,.$$

Finally the well-known result

$$\langle C|T_{00}|C\rangle = -\pi^2/(720a^4)$$

allows us to write

$$d_2 = -\frac{11\pi\alpha^2}{16200\,a^4\,m_\text{e}^4}.$$

In particular, this implies $d_2(z)$ is position independent. Thus, at first order in $\alpha^2$,

$$v_\text{phase}(\theta) = v_\text{group}(\theta) = 1 + \frac{11\pi^2\alpha^2}{8100\,a^4\,m_\text{e}^4}\cos^2\theta.$$

## Conclusions:

The Scharnhorst effect is interesting fundamental physics.

Unfortunately it is well outside the realm of experimental detectability.

Nevertheless it is good for "proof of principle" calculations:

— it shows how the effective metric is a useful tool.

— it shows how quantum vacuum polarization modifies photon propagation.

— it forces you to sharpen your notions of what birefringence means.