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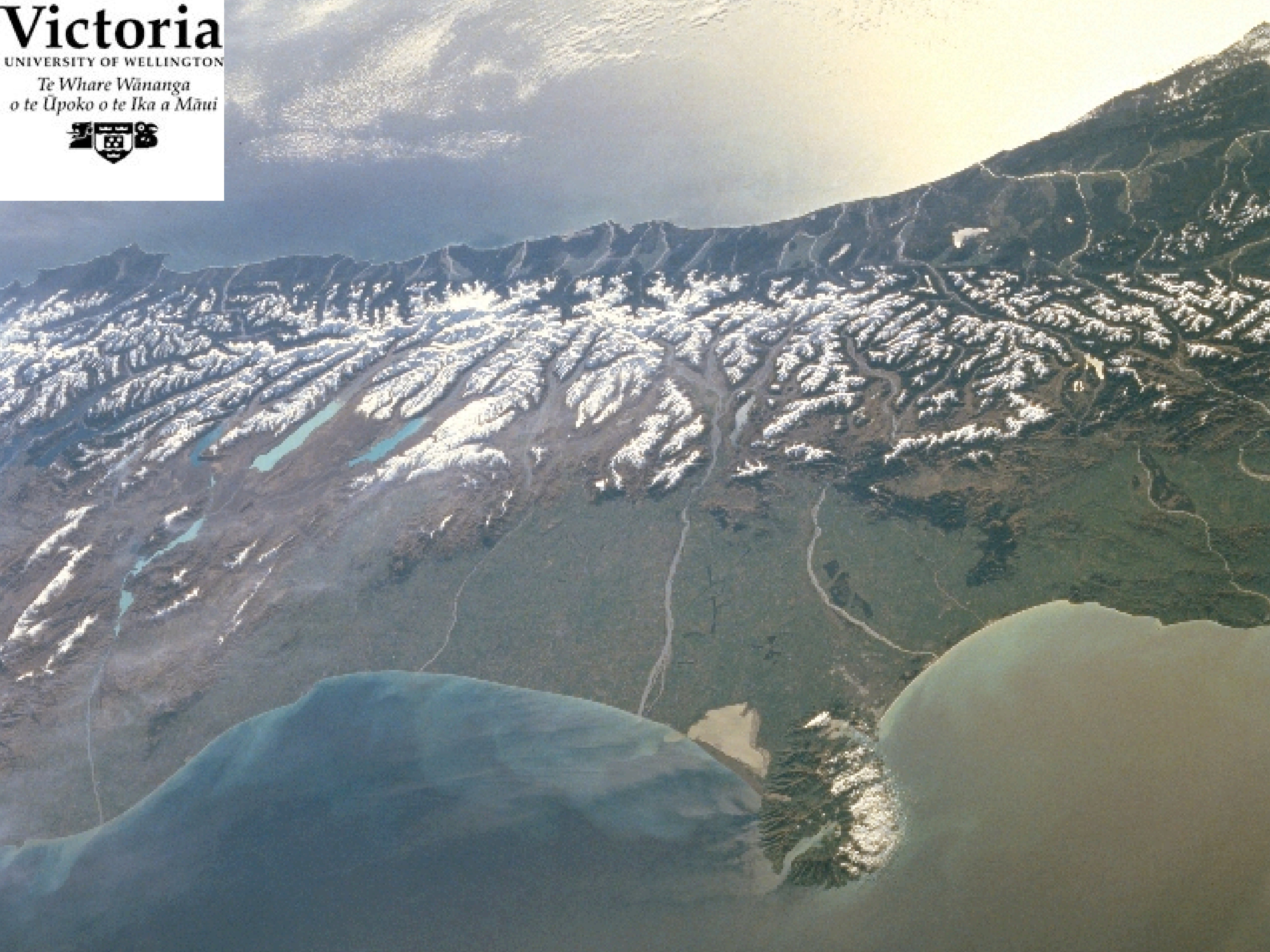
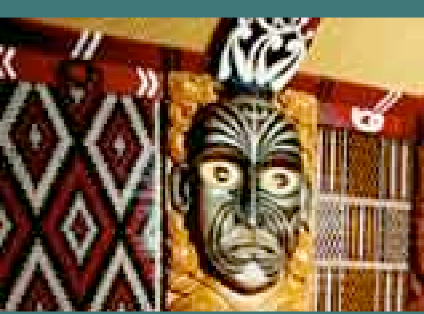
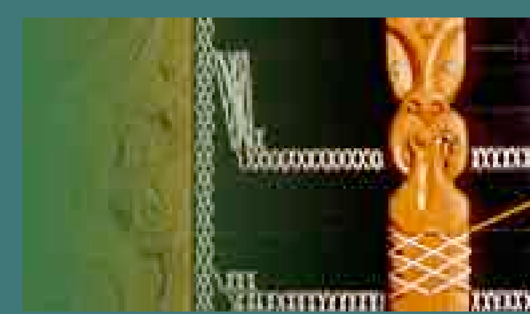


Vorticity in the acoustic analogue of gravity

Matt Visser

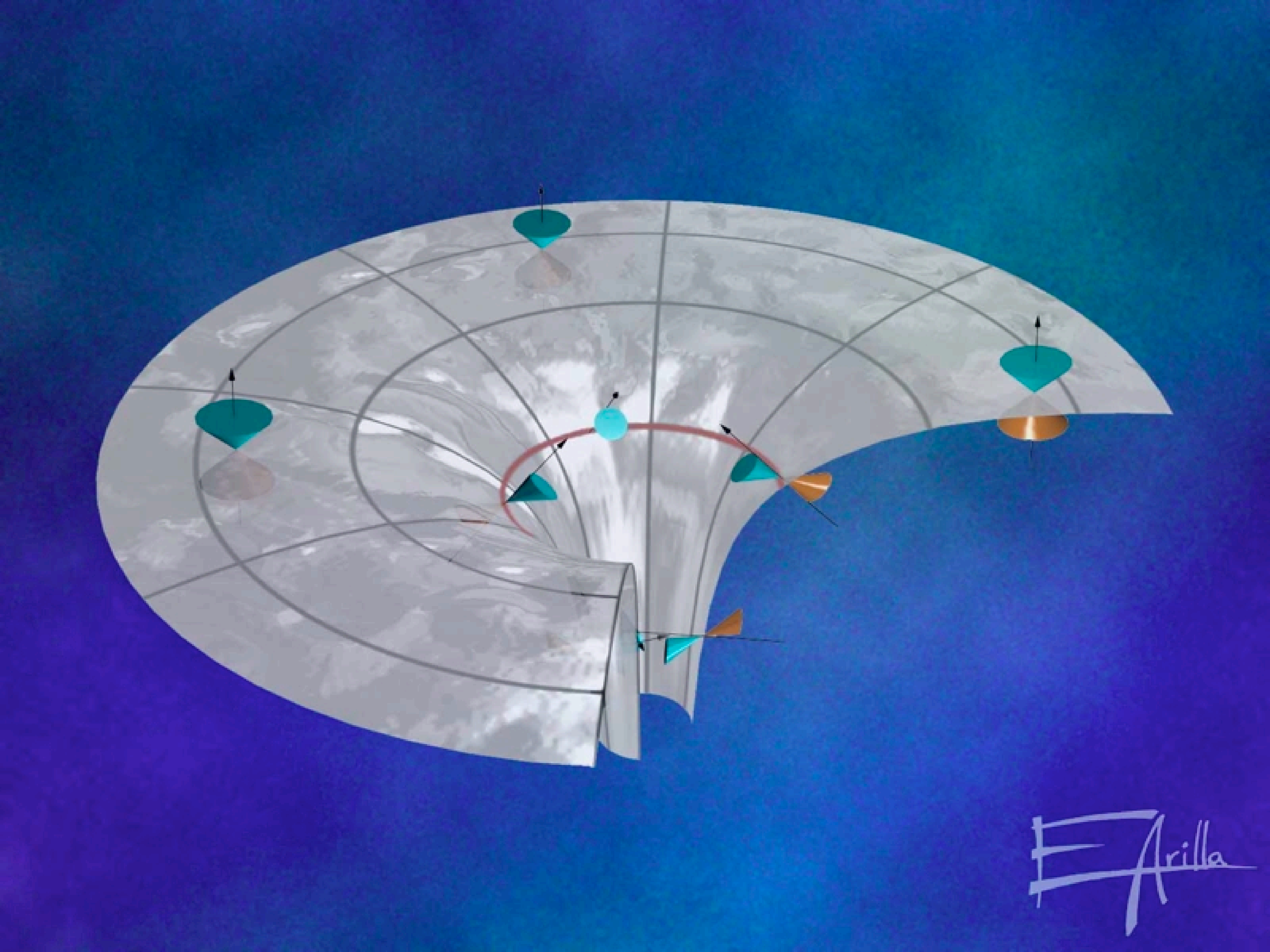




And I cherish more than anything else the Analogies, my most trustworthy masters.

They know all the secrets of Nature, and they ought least to be neglected in Geometry.

--- Johannes Kepler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Abstract:

Sound waves propagating in a flowing fluid can be described by a wave equation on an “acoustic geometry”.

This geometry is described by an “effective metric” that leads to a “curved spacetime” qualitatively similar to that of general relativity.

In particular, it is relatively easy to set up “dumb holes” which trap sound in the same way that the black holes of general relativity trap light.

Abstract:

Vortex flow creates an "acoustic geometry" that is qualitatively similar to the equatorial plane of a rotating Kerr black hole.

There is an acoustic analogy to the general relativity notion of an "ergosphere" as well as to the general relativity notion of "horizon".

I will first qualitatively explain these ideas, and describe why they are useful, and then make a quantitative comparison between the analog acoustic and general relativistic sytems.

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Key idea:

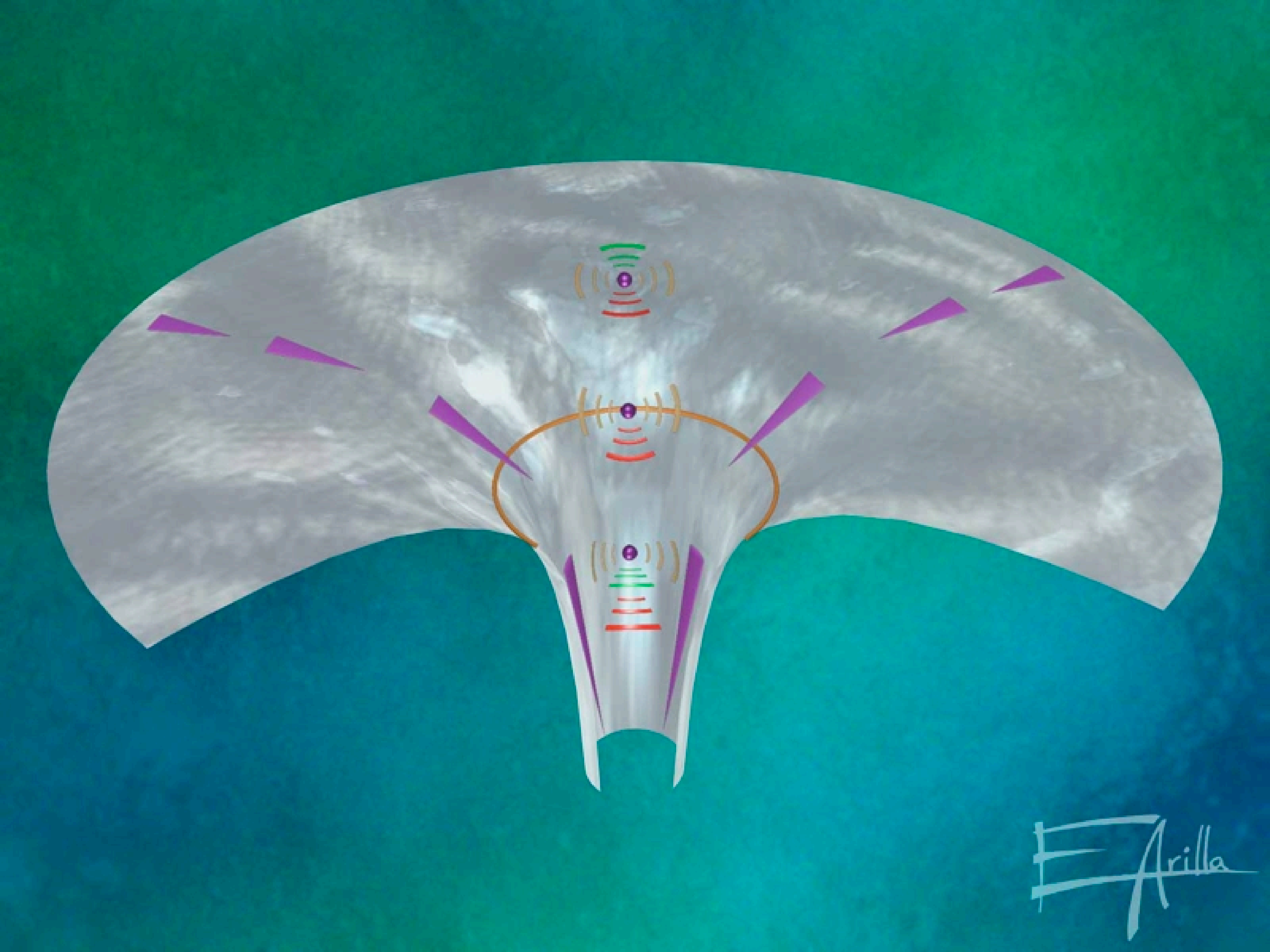
Consider sound waves in a flowing fluid.

If the fluid is moving faster than sound, then the sound waves are swept along with the flow, and cannot escape from that region.

This sounds awfully similar to a black hole in general relativity --- is there any connection?

Yes!

(Of course the devil is in the details...)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Key results:

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Acoustic propagation in fluids can be described in terms of Lorentzian differential geometry.

The acoustic metric depends algebraically on the fluid flow.

Acoustic geometry shares kinematic aspects of general relativity, but not the dynamics.

Einstein equations versus Euler equation.

In particular:
Acoustic black holes divorce kinematic aspects of black hole physics from the specific dynamics due to the Einstein equations.

Advanced features:

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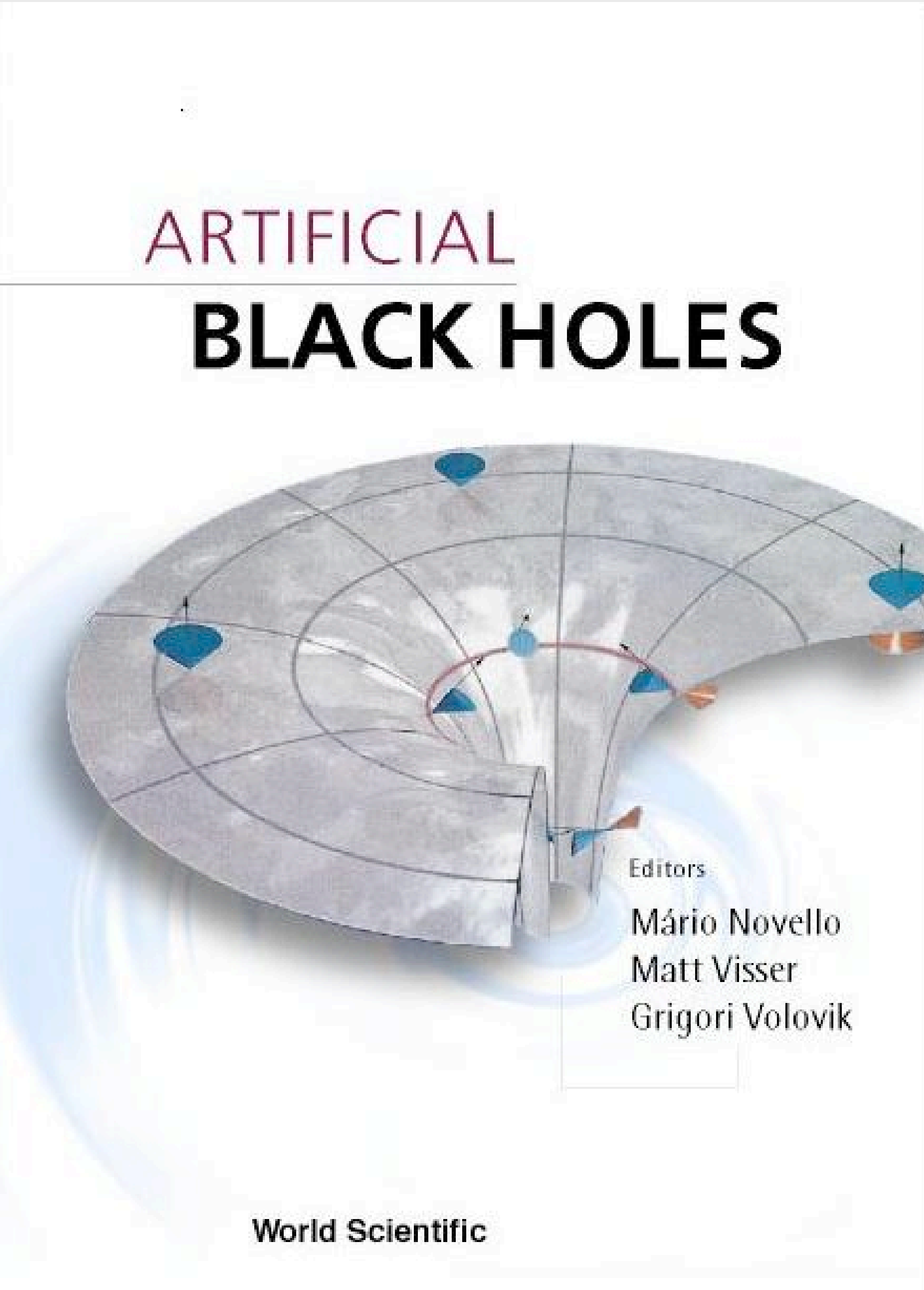
There are also other "analogue models" of general relativity, apart from the acoustic models.

Acoustic black holes have Hawking radiation without black hole entropy.

Hawking radiation is a purely kinematic effect that exists independent of whether or not the Lorentzian geometry obeys any particular geometrodynamics.

You do not need the Einstein equations to get Hawking radiation.



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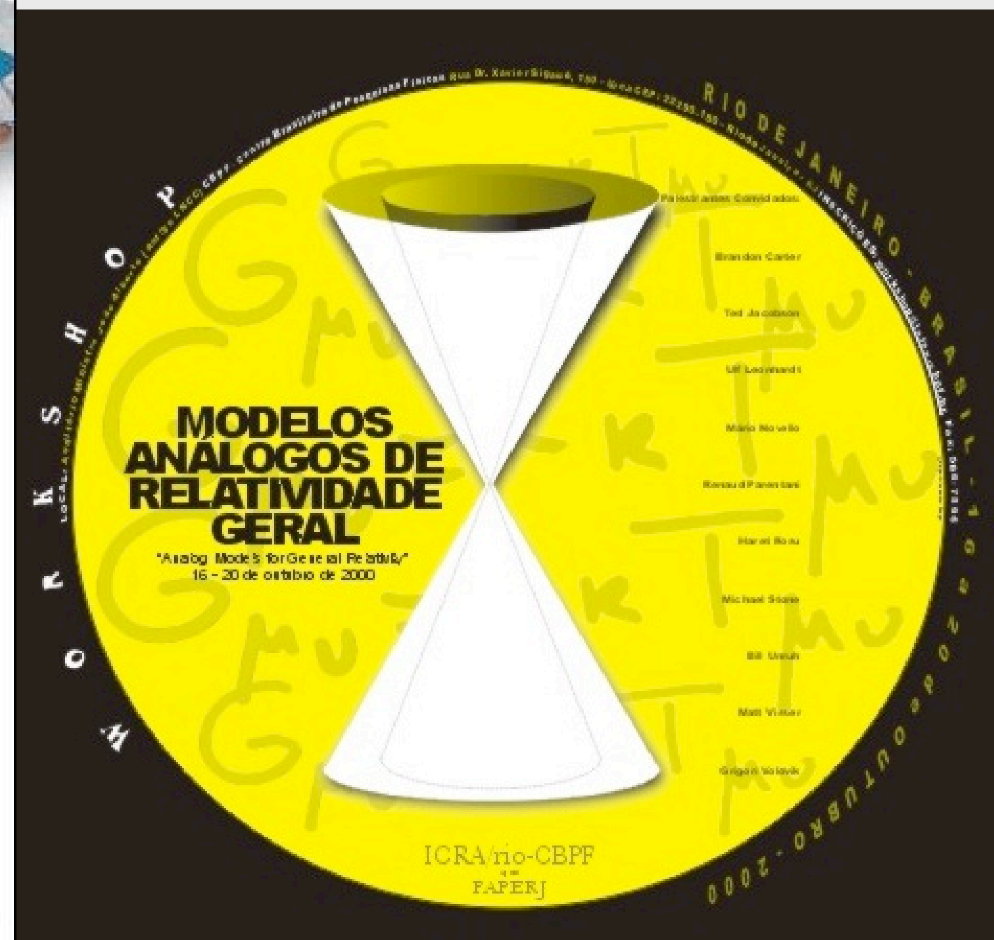

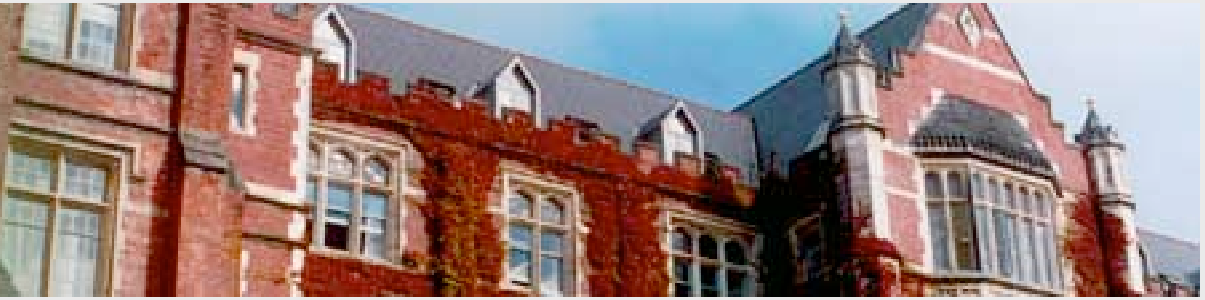

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Photonics Technology

April 2000

Photonics Technology News

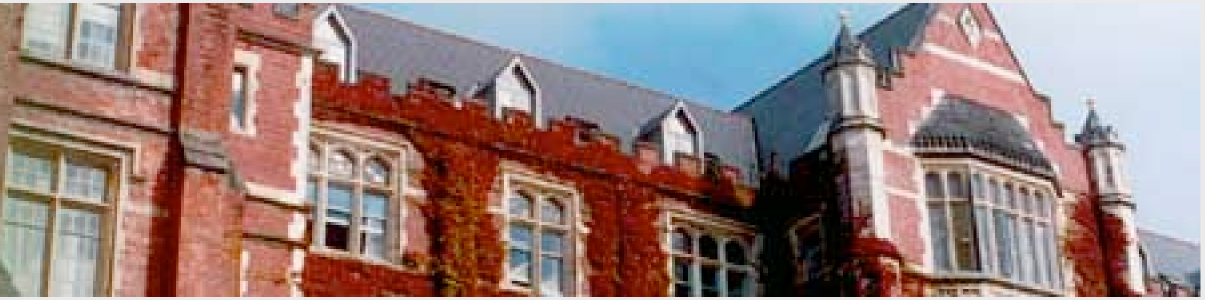
www.Photonics.com

Water down the Drain Leads to Black Holes in the Lab

Archimedes, the famous story goes, cried, "Eureka!" and leapt from his overflowing bathtub, having discovered the principle of buoyancy. But the drain has much to offer, too. Physicists Ulf Leonhardt and Paul Piwnicki of the Royal Institute of Technology suggest in the Jan. 31 issue of *Physical Review Letters* that vortices or other rapid flows analogous to the swirl of water down the drain may enable researchers to model the relativistic effects of black holes.

Leonhardt explained that, as early as 1818, Augustin-Jean Fresnel predicted from theories of the luminiferous ether that a flowing medium will drag light with it. "For quite a long time we were rather puzzled and could not understand what is happening in moving media if one takes them seriously," said Leonhardt. "Then, suddenly, the picture emerged, and we realized the connection to gravity and curved space-time. This has come as a surprise."

The researchers demonstrated that, if a medium swirls more quickly than the speed of light through it, the vortex it forms can decelerate and trap light in much the same way as a black hole. If they could be created in the lab, these optical black holes would enable physicists to test such controversial concepts as Hawking radiation, the emissions by black holes theorized by Stephen W. Hawking. The problem is that no known materials display

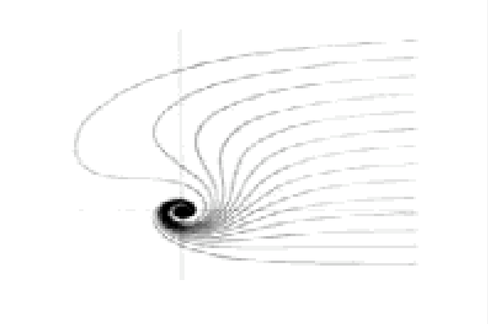


Week of Feb. 5, 2000; Vol. 157, No. 6

Black hole recipe: Slow light, swirl atoms

By P. Weiss

Physicists may soon create artificial black holes in the laboratory, analogous to the ones expected to lurk in distant space. A new study by a pair of theorists in Sweden describes how swirling clouds of atoms could slug down all nearby light, making them as black as their astronomical cousins.



Computer-generated plot shows paths of light rays sucked into optical black hole. (Leonhardt and Piwnicki/*Physical Review A*)

Called optical black holes, these eddies could provide an extraordinary test-bench for the theory of general relativity, which gave rise to the concept of gravitational black holes, the researchers say. Ulf Leonhardt and Paul Piwnicki of the Royal Institute of Technology in Stockholm find that the same mathematics describes both the terrible tug of an astronomical black hole on light and the gentle corralling of rays by an atom vortex.

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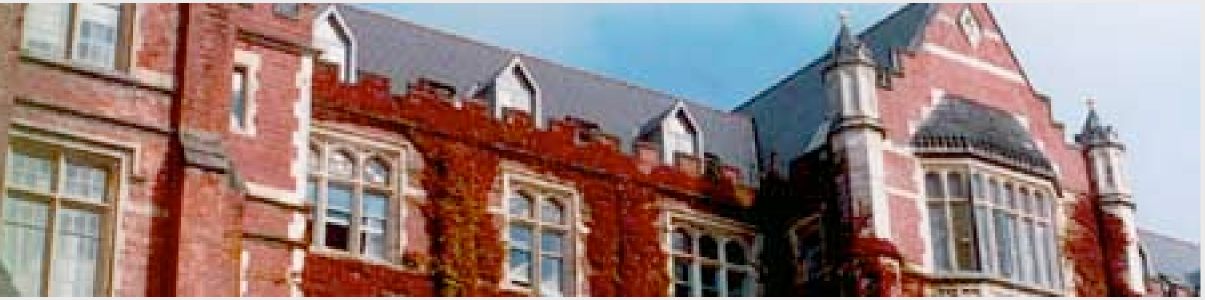
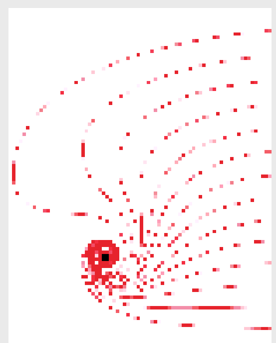


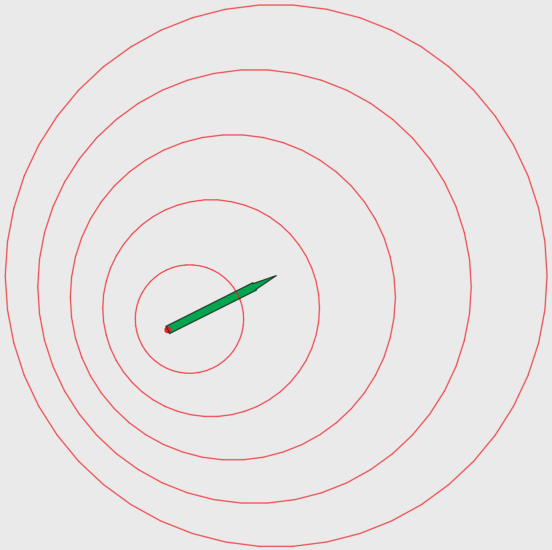
Optical black holes could be made in the lab



Optical black hole

Black holes could be created in the laboratory, claim physicists from Sweden and Scotland. Their calculations show that, thanks to some recent advances in condensed matter physics, it is theoretically possible to create an optical black hole to attract and trap specific colours of light in just the same way as an astronomical black hole attracts and consumes matter.

The researchers unearthed a 1923 paper in which Walter Gordon, starting from Einstein's idea of gravity mediated by changes in the metric of space and time, realized that space-time is effectively a medium and that consequently, any moving dielectric medium acts on light as an effective gravitational field. However, to see effects as dramatic as a black hole, the velocity of light in the medium must be low compared with the velocity of the medium.



Geometrical acoustics:

In a flowing fluid, if sound moves a distance $\mathrm{d}\vec{x}$ in time $\mathrm{d}t$ then

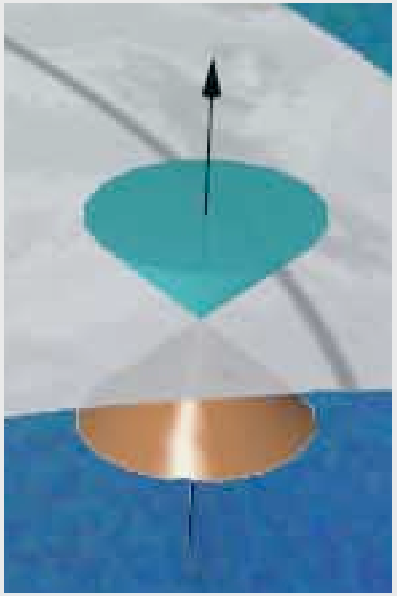
$$||\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t|| = c_s\,\mathrm{d}t.$$

Write this as

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) \cdot (\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) = c_s^2 \mathrm{d}t^2.$$

Now rearrange a little:

$$-(c_s^2 - v^2)\,\mathrm{d}t^2 - 2\,\vec{v} \cdot \mathrm{d}\vec{x}\;\;\mathrm{d}t + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = 0.$$



Geometrical acoustics:

Notation — four-dimensional coordinates:

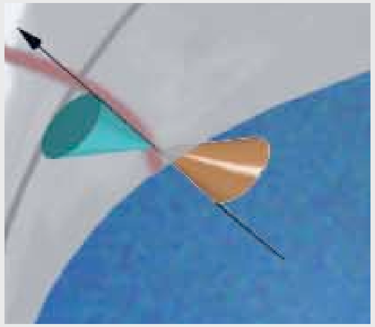
$$x^\mu = (x^0; x^i) = (t; \vec{x}).$$

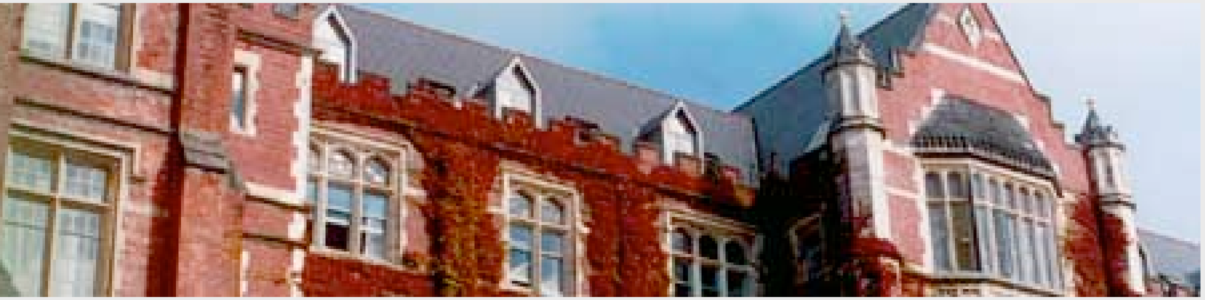
Then you can write this as

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = 0.$$

With an effective acoustic metric

$$g_{\mu\nu}(t, \vec{x}) \propto \left[\begin{array}{ccc} -(c_s^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{array} \right].$$





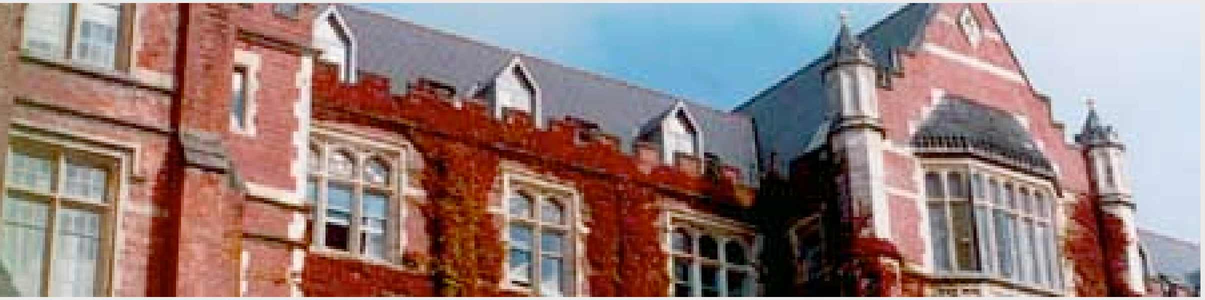
Geometrical acoustics:

Eikonals:

The sound paths of geometrical acoustics are the null geodesics of this effective metric.

Geometrical acoustics, by itself, does not give you enough information to fix an overall multiplicative factor (conformal factor).

Note: This also works for geometrical optics in a flowing fluid, with $c_s \to c/n$; replace the speed of sound by the speed of light in the medium (speed of light divided by refractive index).



Geometrical acoustics:

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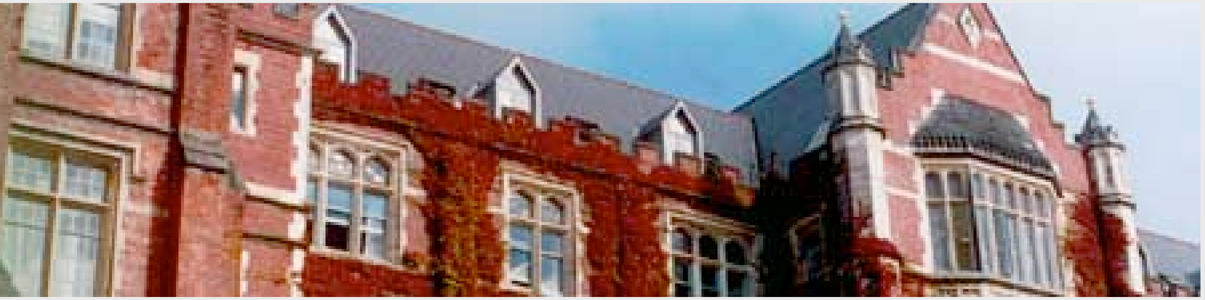
This is already enough to give you some very powerful results:

Fermat's principle is now a special case of geodesic propagation.

Sound focussing can be described by the Riemann tensor of this effective metric.

But there is a lot more hiding in the woodwork.



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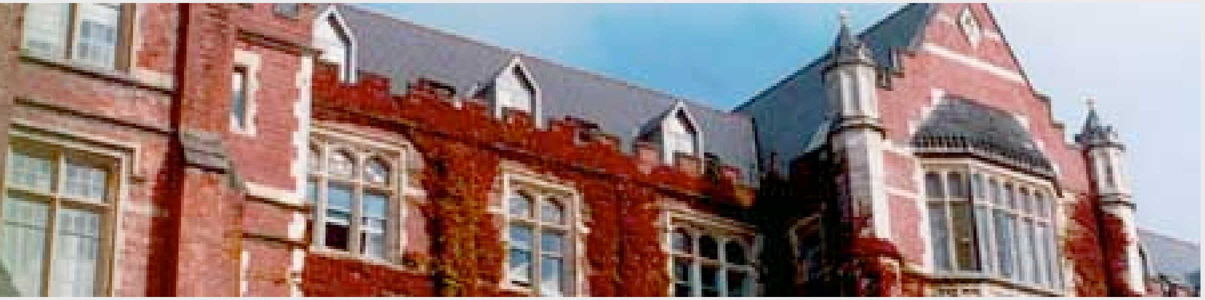
The Acoustic metric:

Suppose you have a non-relativistic flowing fluid, governed by the Euler equation plus the continuity equation.

Suppose the fluid flow is barotropic, irrotational, and inviscid.

Suppose we look at linearized fluctuations.

This then leads to a wave equation...

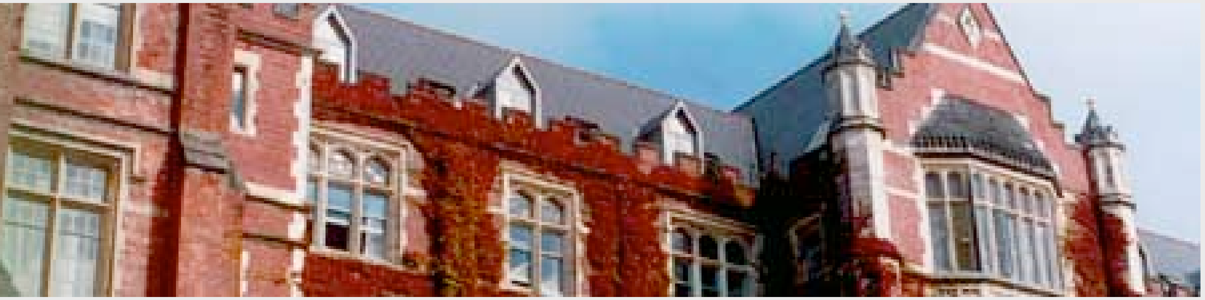


Physical acoustics:

Then the linearized fluctuations (*aka* sound waves, *aka* phonons) are described by a massless minimally coupled scalar field propagating in a (3+1)-dimensional acoustic metric

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho}{c} \begin{bmatrix} -(c^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{bmatrix}.$$

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\; \partial_\nu \psi \right) = 0.$$



Physical acoustics:

Fundamental fluid dynamics:

Continuity —

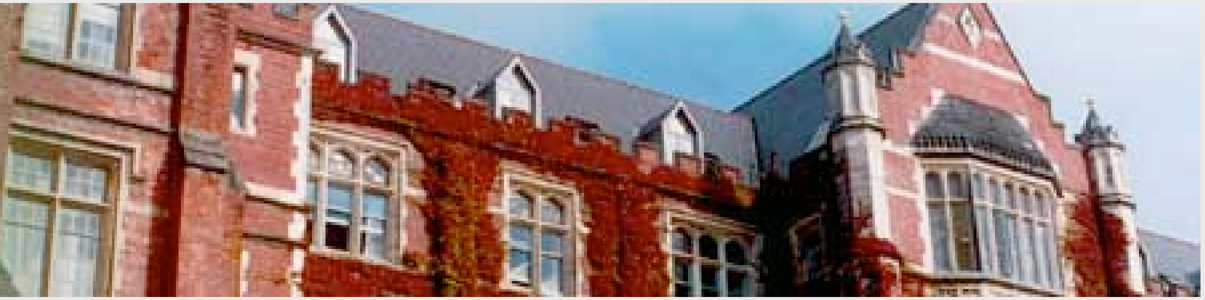
$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0.$$

Euler (inviscid) —

$$\rho \left[\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p - \rho \nabla \phi.$$

Barotropic —

$$\rho = \rho(p); \qquad \zeta(p) = \int_0^p \frac{dp'}{\rho(p')}; \qquad \nabla \zeta = \frac{1}{\rho} \nabla p.$$



Standard Manipulations:

Euler + Barotropic —

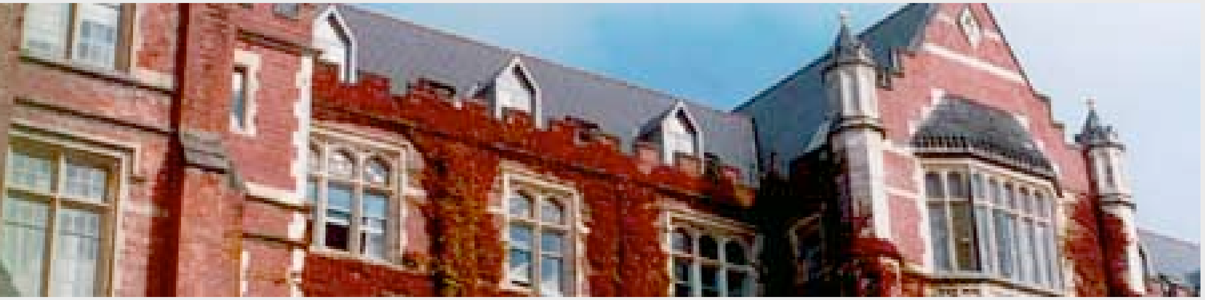
$$\partial_t \vec{v} = \vec{v} \times (\nabla \times \vec{v}) - \nabla(\tfrac{1}{2}v^2 + \zeta + \phi).$$

Irrotational —

$$\nabla \times \vec{v} = 0; \qquad \vec{v} = -\nabla \psi.$$

Euler + Barotropic + Irrotational —

$$-\partial_t \psi + \zeta + \tfrac{1}{2}(\nabla \psi)^2 + \phi = 0.$$



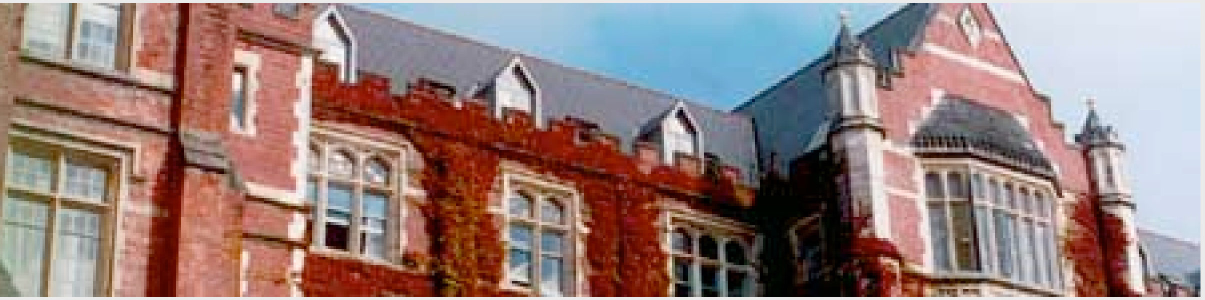
Linearize:

$$\rho \equiv \rho_0 + \epsilon \rho_1; \quad p \equiv p_0 + \epsilon p_1; \quad \psi \equiv \psi_0 + \epsilon \psi_1; \quad \phi \equiv \phi_0;$$

$$\zeta \equiv \zeta_0 + \epsilon \zeta_1 = \zeta_0 + \epsilon (p_1/\rho_0); \quad \rho_1 = \frac{\partial \rho}{\partial p} p_1.$$

- Sound $\equiv$ linearized fluctuations.

- Substitute the linearized Euler equation into the linearized continuity equation. This gives a wave equation for the scalar potential.



Linearized equations:

Continuity —

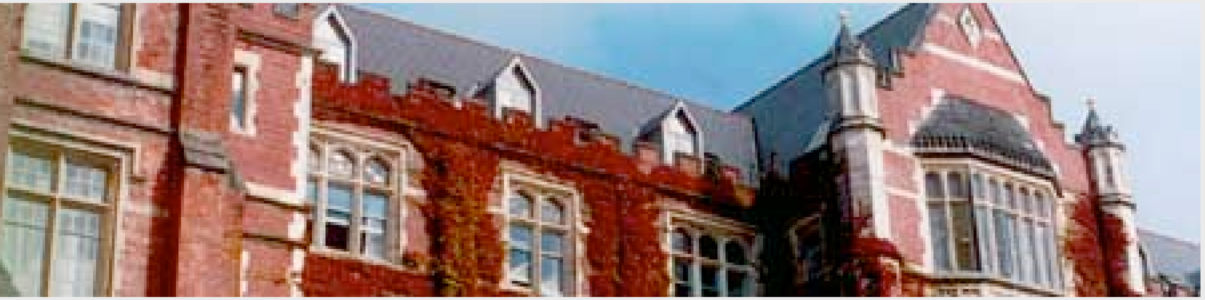
$$\partial_t \rho_1 + \nabla \cdot (\rho_1 \vec{v}_0 + \rho_0 \vec{v}_1) = 0.$$

Euler —

$$-\partial_t \psi_1 + \frac{p_1}{\rho_0} - \vec{v}_0 \cdot \nabla \psi_1 = 0.$$

Rearrange:

$$p_1 = \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1).$$



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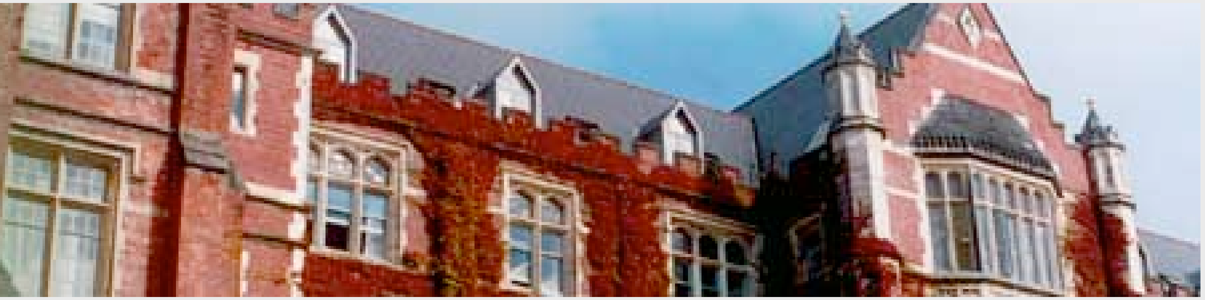
Wave equation:

$$
\begin{aligned}
- \quad & \partial_t \left[\frac{\partial \rho}{\partial p} \rho_0 \left(\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 \right) \right] \\
+ \quad & \nabla \cdot \left[\rho_0 \nabla \psi_1 - \frac{\partial \rho}{\partial p} \rho_0 \vec{v}_0 \left(\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 \right) \right] = 0.
\end{aligned}
$$

Coefficients ρ_0, $\vec{v}_0$, and $1/c^2 \equiv (\partial\rho/\partial p)$ can have arbitrary time and space dependencies.



Physical acoustics:

Define:

$$f^{\mu\nu} \equiv \frac{\rho_0}{c^2} \left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{array} \right].$$

Then:

$$\partial_\mu (f^{\mu\nu} \partial_\nu \psi_1) = 0.$$

That's really it!

Everything else is minor technical fiddling.

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Rewrite the wave equation as:

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\; \partial_\nu \psi\right) = 0.$$

$$f^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}; \qquad g = \frac{1}{\det(g^{\mu\nu})}$$

$f^{\mu\nu}$ is independent of the number of space dimensions.

$g^{\mu\nu}$ depends on the number of space dimensions.

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In (d+1) dimensions:

Inverse metric:

$$g^{\mu\nu} = \left(\frac{\rho}{c}\right)^{-2/(d-1)} \left[\begin{array}{c|c} -1/c^2 & -\vec{v}^T/c^2 \\ \hline -\vec{v}/c^2 & -\left(\mathbf{I}_{d\times d} - \vec{v}\,\vec{v}^T/c^2\right) \end{array}\right]$$

Metric:

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)^{2/(d-1)} \left[\begin{array}{c|c} -\left(c^2 - v^2\right) & \vec{v}^T \\ \hline \vec{v} & \mathbf{I}_{d\times d} \end{array}\right]$$

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To be specific:

The acoustic line-element for three space and one time dimension reads

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right) \left[\begin{array}{c|c} -(c^2 - v^2) & \vec{v}^T \\ \hline \vec{v} & \mathbf{I}_{3\times 3} \end{array} \right].$$

The acoustic line-element for two space and one time dimension reads

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)^2 \left[\begin{array}{c|c} -(c^2 - v^2) & \vec{v}^T \\ \hline \vec{v} & \mathbf{I}_{2\times 2} \end{array} \right].$$

There is a formal difficulty in (1+1) dimensions.

The conformal factor is raised to an infinite power.

Note that this issue only presents a difficulty for physical systems that are intrinsically one-dimensional. A three dimensional system with plane symmetry is perfectly well behaved as in the case $d = 3$ above.

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There are two distinct metrics.

Call it a bi-metric theory.

The physical metric is the flat Minkowski metric.

Fluid particles couple only to the physical metric.

Acoustic perturbations do not "see" the physical metric --- they couple only to the acoustic metric.

The acoustic metric inherits several nice properties from the underlying physical metric.

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Simple topology --- $\Re^4$

(with maybe a few excised regions due to boundaries.)

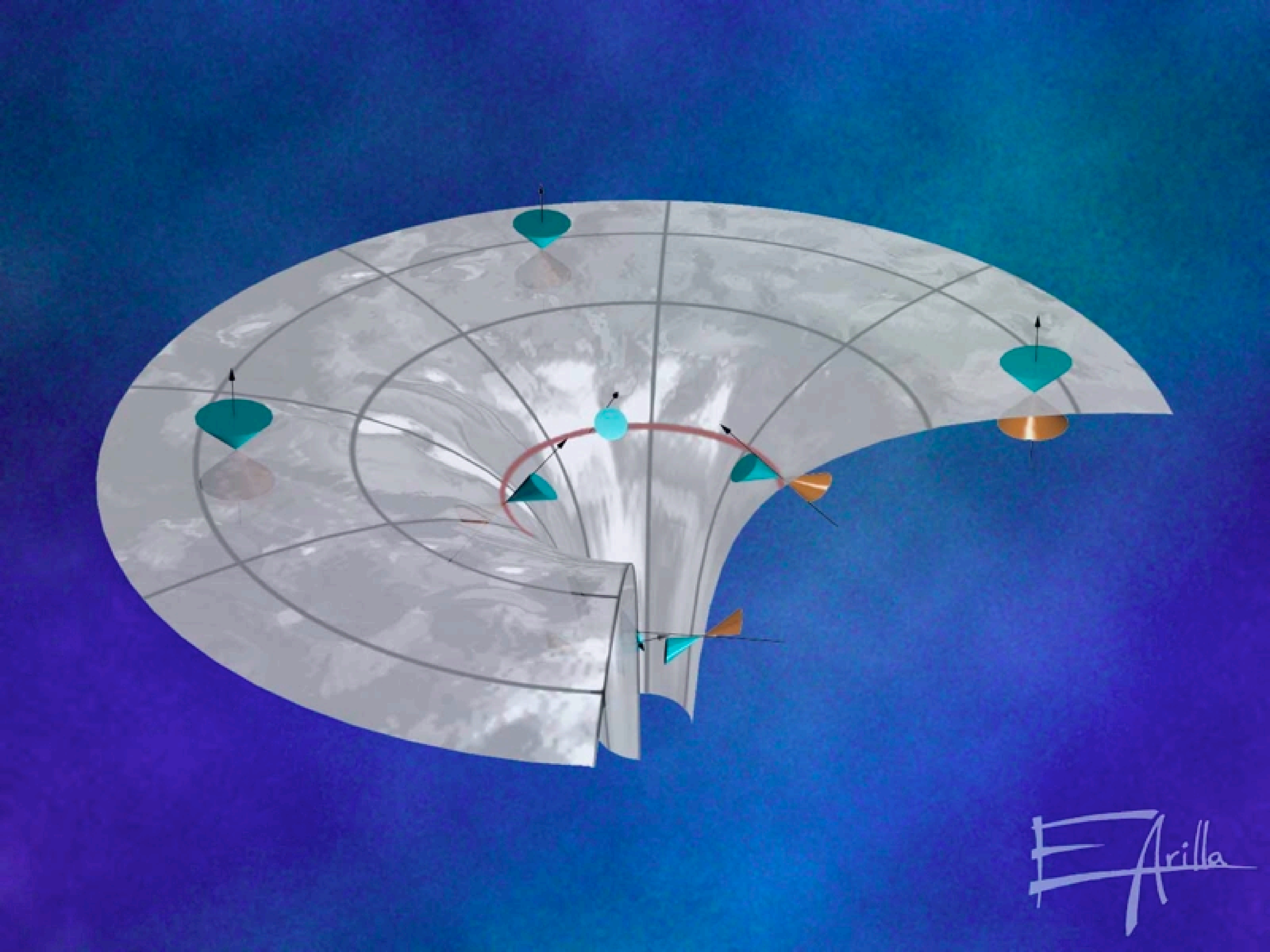
Stable causality: $g^{\mu\nu}(\nabla_\mu t)(\nabla_\nu t) = -1/(\rho_0 c) < 0.$

(means we don't need to worry about time travel.)

Space (not space-time) is conformally flat: $g_{ij} = (\rho/c)\; h_{ij}$

Here h_{ij} is the metric of flat space (possibly in curvilinear coordinates)

Notation: "stationary" $\Longleftrightarrow$ "steady flow"
"static" $\Longleftrightarrow$ "fluid at rest".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History:

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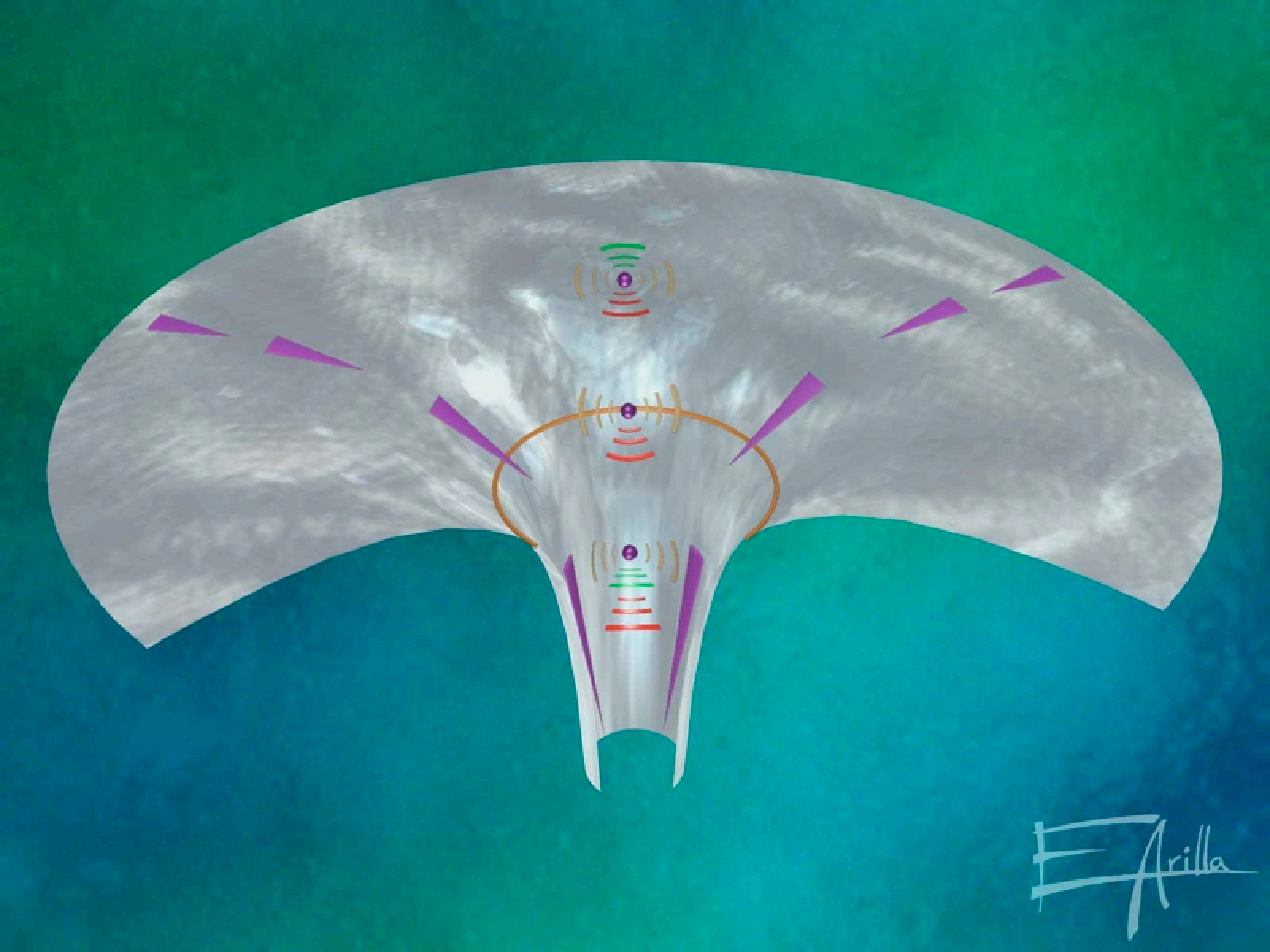
Recent: Unruh 81
Visser 93
Jacobson
Barcelo, Liberati, Sonego
Stone, Perez-Bergliaffa, Hibberd
Fischer, Fedichev
Garay, Cirac, Zoller, Anglin
Laughlin, Santiago, Chapline
Schutzhold
Volovik, Leonhardt, Piwnicki
Novello, Salim, Klippert, Weinfurtner

Older: Gordon, Madelung, ...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Example: Draining bathtub geometry

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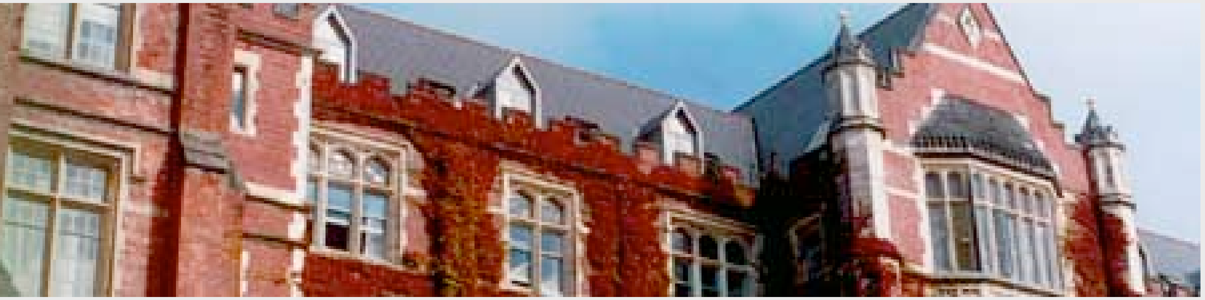
A $(2+1)$ dimensional flow with a sink.

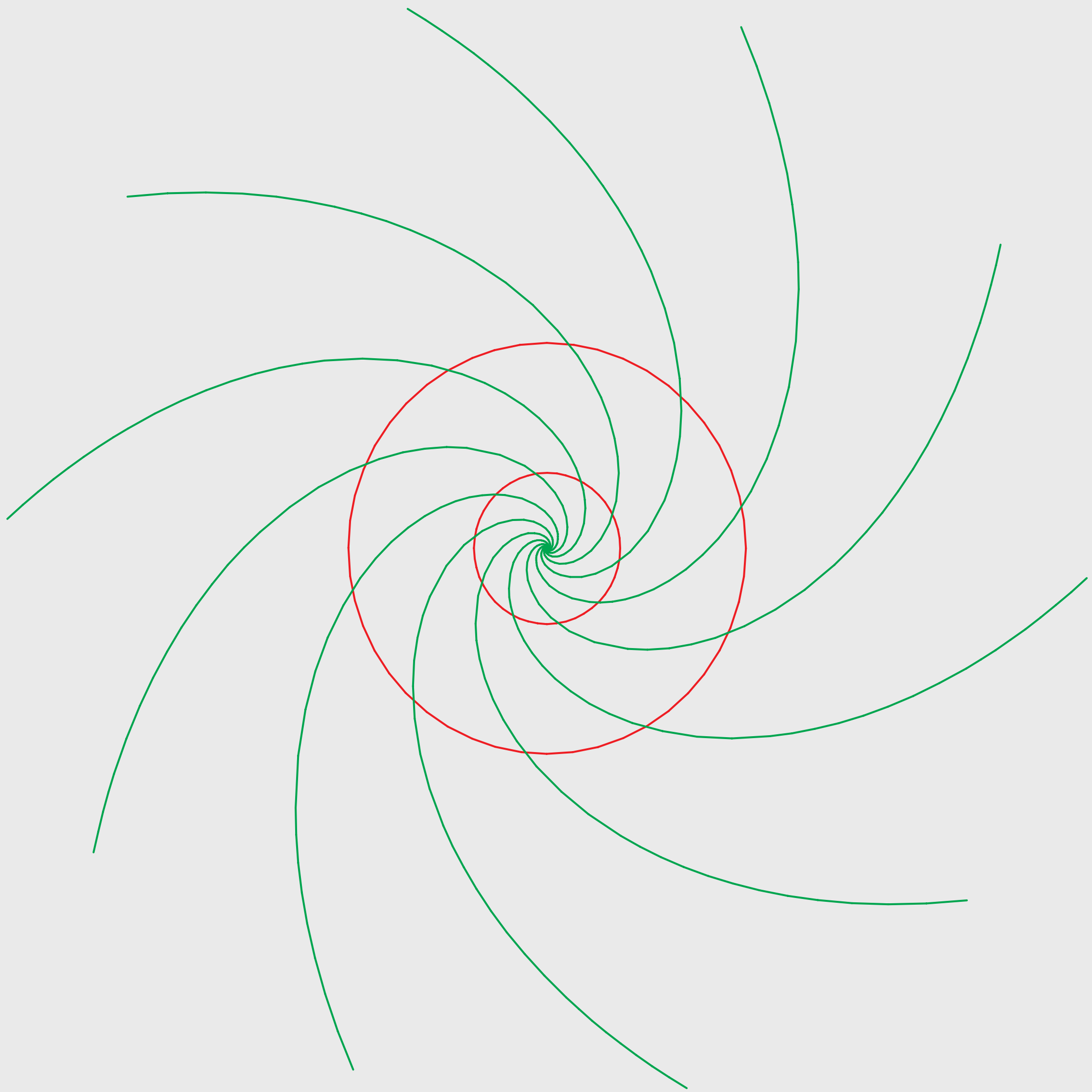
Use: constant density, constant sound speed, zero torque. (You will need an external force.)

The velocity of the fluid flow is

$$\vec{v} = \frac{(A\,\hat{r} + B\,\hat{\theta})}{r}.$$

Streamlines are equiangular spirals.





Most general line vortex:

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The background flow, which determines the "acoustic metric" is governed by the continuity, Euler, and barotropic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \vec{v}) = 0.$$

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] = -\nabla p + \vec{f}.$$

$$p = p(\rho).$$

Engineering perspective:

$$\vec{f} = \rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] + \nabla p,$$

Physical acoustics: Draining bathtub

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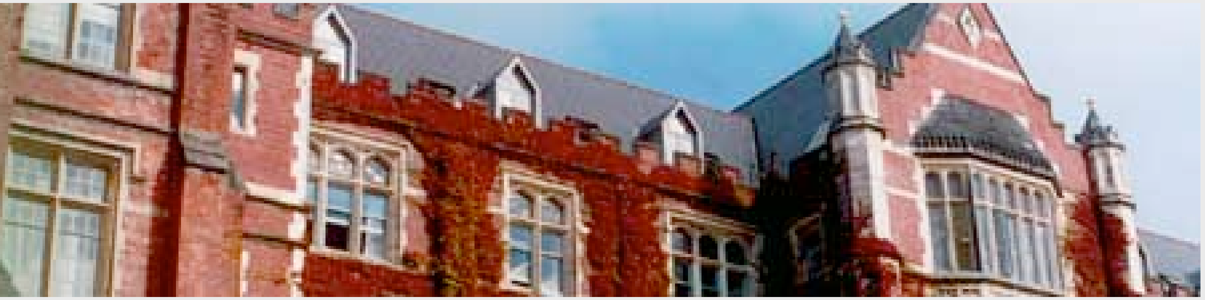
The acoustic metric is

$$\mathrm{d}s^2 = -c_s^2 \mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\mathrm{d}t\right)^2.$$

The acoustic horizon forms once the radial velocity exceeds the speed of sound.

An ergo-surface forms once the speed exceeds the speed of sound.

$$r_{horizon} = \frac{|A|}{c}. \qquad r_{ergo-surface} = \frac{\sqrt{A^2+B^2}}{c}.$$



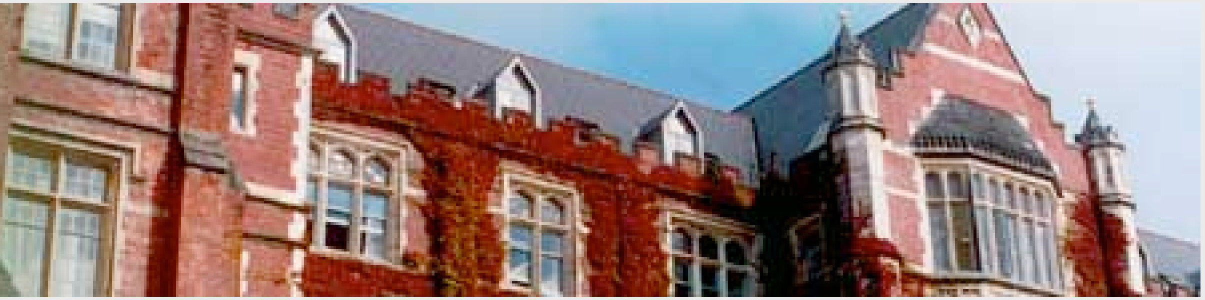
Event horizons and ergo-regions:

Event horizon: the boundary of the region from which null geodesics (phonons; sound rays) cannot escape.

At the event horizon, the inward normal component of fluid velocity equals the speed of sound.

Ergo surface: the boundary of the region of supersonic flow.

In general relativity this is important for spinning black holes.



Surface gravity:

For a static geometry, we can apply all of the standard tricks for calculating the "surface gravity" developed in general relativity.

The surface gravity is a useful characterization of general properties of the event horizon and is given in terms of a normal derivative by

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v_\perp^2)}{\partial n} = c\,\frac{\partial(c - v_\perp)}{\partial n}.$$

Physical acoustics: Surface gravity

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The surface gravity is essentially the acceleration of the fluid as it crosses the horizon.

Even non-static geometries are not too bad.

The second background metric simplifies some of the technical complications encountered in general relativity.

(The second metric gives you an unambiguous background for making comparisons.)

There is a reason that for the importance of surface gravity, patience....

Back to geometric acoustics:

Eikonal limit:

Take the short wavelength/high frequency limit. Sound rays (phonons) follow the *null geodesics* of the acoustic metric. Null geodesics are insensitive to any overall conformal factor in the metric. Simplify life by considering

$$h_{\mu\nu} \equiv \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta^{ij} \end{array} \right].$$

In the geometric acoustics limit, sound propagation is insensitive to the density of the fluid. It depends only on the local speed of sound and the velocity of the fluid. The density of the medium is important only for specifically wave related properties.

Back to geometric acoustics:

Sanity check: Parameterize $X^\mu(t) \equiv (t, \vec{x}(t))$.
— The null condition implies

$$h_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} = 0$$

$$\iff -(c^2 - v_0^2) - 2 v_0^i \frac{dx^i}{dt} + \frac{dx^i}{dt} \frac{dx^i}{dt} = 0$$

$$\iff \left\| \frac{d\vec{x}}{dt} - \vec{v}_0 \right\| = c.$$

The norm is taken in the flat physical metric.
Interpretation: the ray travels at the local speed of sound relative to the moving medium.

From geometric acoustics to Fermat:

Stationary geometry:

Let $X^\mu(s) \equiv (t(s); \vec{x}(s))$ be some null path from $\vec{x}_1$ to $\vec{x}_2$ parameterized in terms of physical arc length (*i.e.* $||d\vec{x}/ds|| \equiv 1$).

The condition for the path to be null (though not yet necessarily a null geodesic) is

$$-(c^2 - v_0^2)\left(\frac{dt}{ds}\right)^2 - 2v_0^i\left(\frac{dx^i}{ds}\right)\left(\frac{dt}{ds}\right) + 1 = 0.$$

— Solve the quadratic —

$$\left(\frac{dt}{ds}\right) = \frac{-v_0^i\left(\frac{dx^i}{ds}\right) + \sqrt{c^2 - v_0^2 + \left(v_0^i\frac{dx^i}{ds}\right)^2}}{c^2 - v_0^2}.$$

From geometric acoustics to Fermat:

The total time taken to traverse the path is

$$T[\gamma] = \int_{\vec{x}_1}^{\vec{x}_2} (dt/ds) ds$$
$$= \int_\gamma \{ \sqrt{(c^2 - v_0^2) ds^2 + (v_0^i dx^i)^2} - v_0^i dx^i \} / (c^2 - v_0^2).$$

• Extremizing the total time taken *is* Fermat's principle. Cf p 262 Landau and Lifshitz.

We have now come full circle:

Geometric acoustics ==> physical acoustics ==> wave equation ==> eikonal ==> geometric acoustics ==>

Why bother?

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If you are a general relativist, this acoustic analogy gives you simple concrete physical models for curved spacetime.

If you are a fluid mechanic (or more generally a condensed matter physicist) the differential geometry of curved spacetimes gives you a whole new way of looking at sound.

Of course these analogue models can be greatly generalized: all you really need are well-defined characteristic speeds.

Simple example: Linearize any Lagrangian field theory.

Lagrangian analysis:

$$\mathcal{L}(\partial_\mu\phi, \phi).$$

Convention:

$$\partial_\mu\phi = (\partial_t\phi\ ;\ \partial_i\phi) = (\partial_t\phi\ ;\ \nabla\phi).$$

Action:

$$S[\phi] = \int \mathrm{d}^{d+1}x\, \mathcal{L}(\partial_\mu\phi, \phi).$$

Euler-Lagrange:

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Lagrangian analysis:

Linearize the field around a solution:

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) + \epsilon\phi_1(t,\vec{x}) + \frac{\epsilon^2}{2}\phi_2(t,\vec{x}) + O(\epsilon^3).$$

Linearized action

$$\begin{aligned} S[\phi] &= S[\phi_0] \\ &+ \frac{\epsilon^2}{2}\int \mathrm{d}^{d+1}x \left[\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)} \right\} \partial_\mu\phi_1\,\partial_\nu\phi_1 \right. \\ &\qquad \left. + \left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu \left\{ \frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi} \right\} \right) \phi_1\,\phi_1 \right] \\ &+ O(\epsilon^3). \end{aligned}$$

Lagrangian analysis:

Linear pieces $[O(\epsilon)]$ vanish by equations of motion.

Quadratic in $\phi_1 \Rightarrow$ field-theory normal modes.

Linearized equations of motion:

$$\partial_\mu \left(\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \partial_\nu \phi_1 \right) - \left(\frac{\partial^2 \mathcal{L}}{\partial \phi\, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial \phi} \right\} \right) \phi_1 = 0.$$

Formally self-adjoint.

Lagrangian analysis:

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Geometrical interpretation:

$$[\Delta(g(\phi_0)) - V(\phi_0)]\,\phi_1 = 0.$$

Metric:

$$\sqrt{-g}\; g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\;\partial(\partial_\nu \phi)} \right\}\Bigg|_{\phi_0}.$$

Potential:

$$V(\phi_0) \;=\; \frac{1}{\sqrt{-g}} \left(\frac{\partial^2 \mathcal{L}}{\partial\phi\;\partial\phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\;\partial\phi} \right\} \right).$$

Lagrangian analysis:

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Key results:

$$\text{linearization} \Rightarrow \text{metric;}$$

$$\text{hyperbolic} \Rightarrow \text{pseudo-Riemannian;}$$

$$\text{parabolic} \Rightarrow \text{degenerate;}$$

$$\text{elliptic} \Rightarrow \text{Riemannian.}$$

This strongly suggests that the "analogue gravity" phenomena is generic to almost any linearization.

Lagrangian analysis: barotropic, irrotational, inviscid fluid:

$$\mathcal{L} = -\rho\, \partial_t \theta - \frac{1}{2}\rho(\nabla\theta)^2 - \int_0^\rho d\rho'\, h(\rho').$$

$$h(\rho) = h[p(\rho)] = \int_0^{p(\rho)} \frac{dp'}{\rho(p')}.$$

Vary $\rho \Rightarrow$ Bernoulli equation (Euler equation).

$$\partial_t \theta + \frac{1}{2}(\nabla\theta)^2 + h(\rho) = 0.$$

Vary $\theta \Rightarrow$ continuity equation.

$$\partial_t \rho + \nabla(\rho\, \nabla\theta) = 0.$$

Lagrangian analysis: barotropic, irrotational, inviscid fluid:

Use the Bernoulli equation to algebraically eliminate ρ:

$$\rho = h^{-1}(z) = h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right).$$

But then:

$$\mathcal{L}(z) = z\,\rho(z) - \int_0^{\rho(z)} d\rho'\, h(\rho').$$

Furthermore:

$$p(\rho) = \rho\, h(\rho) - \int_0^{\rho} d\rho'\, h(\rho').$$

Lagrangian analysis: barotropic, irrotational, inviscid fluid:

Finally:

$$\mathcal{L} = p(\rho(z)) = p\left(h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right)\right).$$

This reduces everything to a Lagrangian depending on only a single scalar field.

Use the preceding single-field Lagrangian analysis.

There is a metric hiding here waiting to be found...

Lagrangian analysis: barotropic, irrotational, inviscid fluid:

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Some boring manipulations:

$$\sqrt{-g}\, g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \Bigg|_{\phi_0}.$$

$$\frac{\partial z}{\partial(\partial_\mu \phi)} = -(1; \nabla\theta)^\mu = -(1; \nabla^i\theta).$$

$$\frac{\partial^2 z}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} = -\delta^{ij}.$$

Therefore:

$$f^{\mu\nu} = \frac{\mathrm{d}^2 p}{\mathrm{d}z^2}\, (1; \nabla\theta)^\mu\, (1; \nabla\theta)^\nu - \frac{\mathrm{d}p}{\mathrm{d}z}\, \delta^{ij}.$$

Lagrangian analysis: barotropic, irrotational, inviscid fluid:

Continued boring manipulations:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \rho; \qquad \frac{\mathrm{d}^2 p}{\mathrm{d}z^2} = \frac{\mathrm{d}\rho}{\mathrm{d}z} = \frac{\mathrm{d}\rho}{\mathrm{d}p}\,\frac{\mathrm{d}p}{\mathrm{d}z} = \rho\, c_s^{-2}.$$

Collecting terms:

$$f^{\mu\nu} = -\rho\, c_s^{-2} \left[\begin{array}{c|c} -1 & -\nabla^i\theta \\ \hline -\nabla^j\theta & c_s^2\, \delta^{ij} - \nabla^i\theta\, \nabla^j\theta \end{array} \right].$$

This should by now be a familiar result...

Where do we go from here?

Four routes to better physics:

1. Adding vorticity (rather non-trivial).

2. Multiple fields (generalized Fresnel equation).

3. More physics examples:
   --- Acoustic horizons (experimental/observational);
   --- BECs (a special type of fluid);
   --- Laval nozzles;
   --- Kerr geometry (a special type of vortex?).

4. Quantum physics in curved spacetimes:
   --- Hawking radiation.

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