The HB theorem and maximal monotonicity

by

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Abstract

We introduce a generalized form of the Hahn–Banach theorem, which we will use to prove various results on the existence of linear functionals in functional analysis, convex analysis and optimization, and also to prove a minimax theorem. We also deduce a sharp version of the Fenchel duality theorem, which we will apply to the Fitzpatrick function to obtain criteria for a monotone multifunction, T, on a reflexive Banach space to be maximal monotone, with various sharp lower bounds on the solutions, x of the equation $(T + J)x \ni 0$. We do not use any renorming theorems, any fixed–point theorems, or any result that depends on Baire's theorem.

Downloads

You can download files containing these slides and several related papers from
<www.math.ucsb.edu/~simons/preprints/Wellington.html>.

Sublinear functionals

Let E be a nonzero real vector space[†]. A **sublinear functional** on E is a map $S\colon\ E \mapsto \mathbb{R}$ such that

$$x, y \in E \quad\Longrightarrow\quad S(x+y) \le S(x) + S(y)$$

and

$$x \in E \text{ and } \lambda > 0 \quad\Longrightarrow\quad S(\lambda x) = \lambda S(x).$$

- Norms and linear functionals are sublinear.

Affine functions

Let D be a nonempty convex subset of a vector space, E be a vector space and $a\colon D \mapsto E$. a is **affine** if

$$x, y \in D \text{ and } \lambda \in (0,1) \quad\Longrightarrow\quad a(\lambda x + (1-\lambda)y) = \lambda a(x) + (1-\lambda)a(y).$$

- Note that an affine function can map into a vector space.

Convex functions

Let C be a nonempty convex subset of a vector space, and $f\colon C \mapsto (-\infty, \infty]$. f is **convex** if

$$x, y \in C \text{ and } \lambda \in (0,1) \quad \Longrightarrow \quad f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),$$

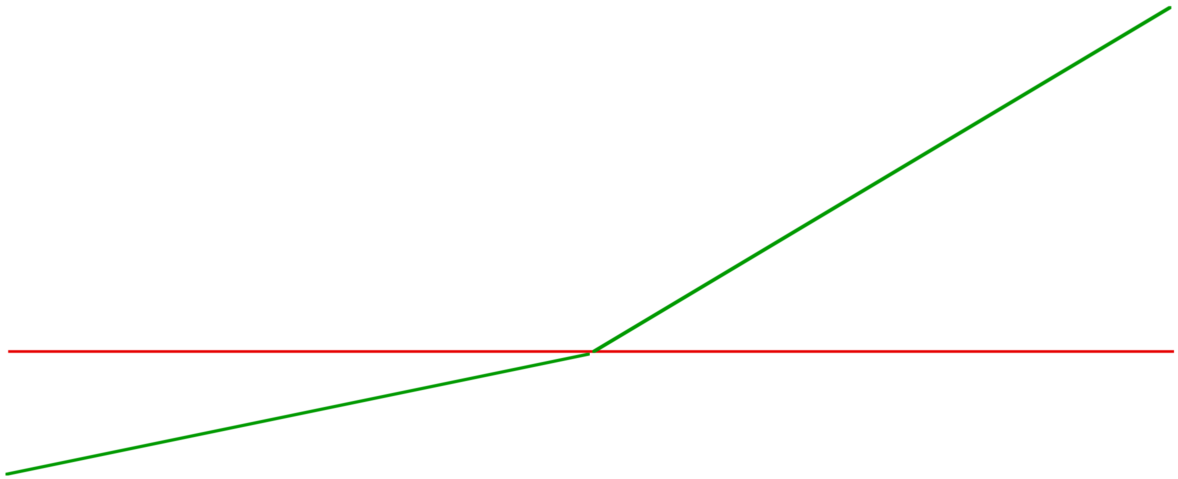
provided $\infty + \infty := \infty$, and $\lambda \times \infty := \infty$ for $\lambda > 0$. f is **proper** if

$$\exists\, x \in C \text{ such that } f(x) \in \mathbb{R}.$$

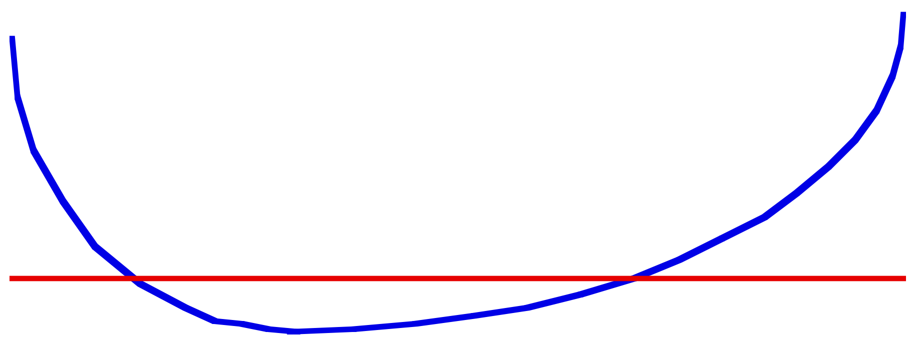
- Sublinear functionals are convex.

— The Hahn–Banach theorem and maximal monotonicity —

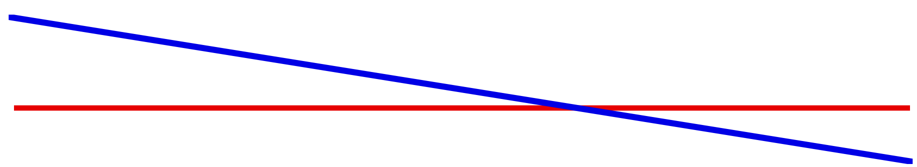
Sublinear functional



Convex function



Affine function



Sublinear functionals

Let E be a nonzero real vector space. A *sublinear functional on* E is a map $S\colon\ E \mapsto \mathbb{R}$ such that

$$x, y \in E \quad \Longrightarrow \quad S(x+y) \le S(x) + S(y)$$

and

$$x \in E \text{ and } \lambda > 0 \quad \Longrightarrow \quad S(\lambda x) = \lambda S(x).$$

The Hahn-Banach theorem

Let S be a sublinear functional on E. Then $\exists$ a linear functional L on E such that[†]

$$L \le S \text{ on } E.$$

A generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let D be a nonempty convex subset of a (possibly different) vector space, and $a\colon\ D \mapsto E$ be affine. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf\nolimits_D L \circ a = \inf\nolimits_D S \circ a.$$

A generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let D be a nonempty convex subset of a (possibly different) vector space, and $a\colon\ D \mapsto E$ be affine. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_D L \circ a = \inf_D S \circ a.$$

Proof Let $\beta := \inf_D S \circ a$. If $\beta = -\infty$, the result is immediate from the Hahn-Banach theorem (take any linear functional L on E such that $L \le S$ on E). So we can suppose that $\beta \in \mathbb{R}$. The result follows by applying the Hahn–Banach theorem to the function $T\colon\ E \mapsto \mathbb{R} \cup \{-\infty\}$ defined by

$$T(x) := \inf_{d \in D,\ \lambda > 0} \big[S\big(x + \lambda a(d)\big) - \lambda\beta\big],$$

which is, in fact, real and sublinear. ∎

• The technique used above is called the technique of the "auxiliary sublinear functional".

A generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let D be a nonempty convex subset of a (possibly different) vector space, and $a\colon D \mapsto E$ be affine. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_D L \circ a = \inf_D S \circ a.$$

- If E is a normed space, E^* stands for the norm–dual of E.

A separation theorem ("bipolar theorem")

Let D be a nonempty convex subset of a normed space E and $x \in E \setminus \overline{D}$. Then $\exists\ z^* \in E^*$ such that

$$\sup_D z^* < \langle x, z^* \rangle.$$

Proof Let $S := \|\cdot\|$ and $a(y) := x - y$ and apply the gHBt. ∎

A generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let D be a nonempty convex subset of a (possibly different) vector space, and $a\colon D \mapsto E$ be affine. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_D L \circ a = \inf_D S \circ a.$$

We will prove:

A more generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let C be a nonempty convex subset of a (possibly different) vector space, $k\colon C \mapsto (-\infty, \infty]$ be proper and convex and $j\colon C \mapsto E$ be S–convex. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_C \big[L \circ j + k\big] = \inf_C \big[S \circ j + k\big].$$

- "j is S–convex" means that

$x_1, x_2 \in C,\ \mu_1, \mu_2 > 0$ and $\mu_1 + \mu_2 = 1 \quad \Longrightarrow \quad j(\mu_1 x_1 + \mu_2 x_2) \le_S \mu_1 j(x_1) + \mu_2 j(x_2)$,
where the ordering "$\le_S$" on E is defined by

$$y \le_S z \iff S(y - z) \le 0.$$

A more generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let C be a nonempty convex subset of a (possibly different) vector space, $k\colon C \mapsto (-\infty,\infty]$ be proper and convex and $j\colon C \mapsto E$ be S–convex. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_C \big[L\circ j + k\big] = \inf_C \big[S\circ j + k\big].$$

Picture :

$$\begin{array}{ccccc} & & C & & \\ & \overset{j}{\swarrow} & & \overset{k}{\searrow} & \\ E & & \overset{S,L}{\longrightarrow} & & \mathbb{R}. \end{array}$$

Proof This follows from the gHBt with E replaced by $E\times\mathbb{R}$, the sublinear functional defined on $E\times\mathbb{R}$ by $(y,\lambda)\mapsto S(y)+\lambda$, the convex set D defined by

$$D := \big\{(x,y,\lambda)\in C\times E\times\mathbb{R}\colon\ S\big(j(x)-y\big)\le 0,\ k(x)\le\lambda\big\},$$

and the affine function $a\colon D\mapsto \widetilde{E}$ defined by

$$a(x,y,\lambda) := (y,\lambda). \blacksquare$$

- Affine functions are S–convex, so the mgHBt generalizes the gHBt.

A more generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let C be a nonempty convex subset of a (possibly different) vector space, $k\colon C \mapsto (-\infty, \infty]$ be proper and convex and $j\colon C \mapsto E$ be S–convex. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_C \left[L \circ j + k\right] = \inf_C \left[S \circ j + k\right].$$

Sandwich theorem

Let S be a sublinear functional on E and $k\colon E \mapsto (-\infty, \infty]$ be proper and convex and $-k \le S$ on E. Then $\exists$ a linear functional L on E such that

$$-k \le L \le S \text{ on } E.$$

Proof Let $C := E$, $j(x) := x$ and apply the mgHBt. ▌

The extension form of the Hahn-Banach theorem

Let E be a normed space, F be a subspace of E and $y^* \in F^*$. Then $\exists$ $x^* \in E^*$ such that

$$x^*|_F = y^* \quad \text{and} \quad \|x^*\|_E \le \|y^*\|_F.$$

Proof Let $S := \|y^*\|_F \|\cdot\|$, $C := F$, $j(y) := y$ and $k(y) := -\langle y, y^*\rangle$, and apply the mgHBt. ▌

Lagrange multipliers for constrained convex problems

Let E be a normed space and $\preceq$ be a vector ordering on E. Let C be a nonempty convex subset of a vector space, $k\colon C \mapsto (-\infty, \infty]$ be proper and convex, $j\colon C \mapsto E$ be $\preceq$–convex and $\inf\{k(x)\colon x \in C, j(x) \preceq 0\} = \mu_0 \in \mathbb{R}$. When can we assert that

$$\exists \preceq\text{–positive } z^* \in E^* \text{ such that } \inf\{\langle j(x), z^*\rangle + k(x)\colon x \in C\} = \mu_0? \qquad (👓)$$

Let $N := \{y \in E\colon y \preceq 0\}$ and $A := \{x \in C\colon k(x) < \mu_0\} \neq \emptyset$.

Classical result: Let $B := \{x \in C\colon j(x) \in \operatorname{int} N\} \neq \emptyset$ then (👓).

Necessary condition with a bound on the norm

Suppose that $B \neq \emptyset$. Then (👓) with

$$\|z^*\| \leq \inf_{v \in B} \frac{k(v) - \mu_0}{\operatorname{dist}\big(j(v), E \setminus N\big)}.$$

Necessary and sufficient condition with sharp bound on the norm

$$(👓) \iff \sup_{x \in A} \frac{\mu_0 - k(x)}{\operatorname{dist}\big(j(x), N\big)} < \infty.$$

Further,
$$\sup_{x \in A} \frac{\mu_0 - k(x)}{\operatorname{dist}\big(j(x), N\big)} = \min\{\|z^*\|\colon z^* \text{ satisfies } (👓)\}.$$

A more generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let C be a nonempty convex subset of a (possibly different) vector space, k: $C \mapsto (-\infty, \infty]$ be proper and convex and j: $C \mapsto E$ be S–convex. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_C \big[L \circ j + k\big] = \inf_C \big[S \circ j + k\big].$$

Lemma on m convex functions

Let C be a nonempty convex subset of a vector space and $f_1, \ldots, f_m$ be convex real functions on C. Then: $\exists\ \lambda_1, \ldots, \lambda_m \ge 0$ such that

$$\lambda_1 + \cdots + \lambda_m = 1 \quad \text{and} \quad \inf_C [f_1 \vee \cdots \vee f_m] = \inf_C [\lambda_1 f_1 + \cdots + \lambda_m f_m].$$

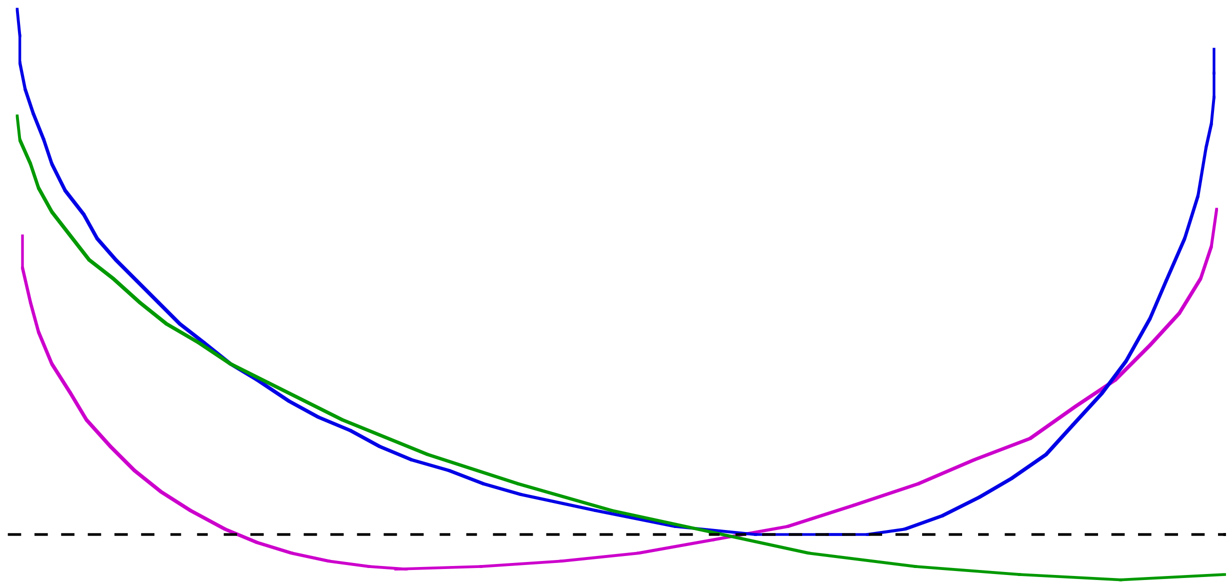
Proof This follows from the mgHBt with† $E := \mathbb{R}^m$, $k := 0$, and S and j defined by

$$S(\mu_1, \ldots, \mu_m) := \mu_1 \vee \cdots \vee \mu_m \quad \text{and} \quad j(c) := \big(f_1(c), \ldots, f_m(c)\big). \blacksquare$$

Lemma on m convex functions

Let B be a nonempty convex subset of a vector space and $f_1, \ldots, f_m$ be convex real functions on B. Then: $\exists\ \lambda_1, \ldots, \lambda_m \geq 0$ *such that*

$$\lambda_1 + \cdots + \lambda_m = 1 \quad \textit{and} \quad \inf_C [f_1 \vee \cdots \vee f_m] = \inf_C [\lambda_1 f_1 + \cdots + \lambda_m f_m].$$



Let A, B be nonempty sets, and $h\colon\ A \times B \mapsto \mathbb{R}$.

- It is easily seen that

$$\sup_{a\in A} \inf_{b\in B} h(a,b) \le \inf_{b\in B} \sup_{a\in A} h(a,b).$$

- This inequality can be strict, take for instance $A = B = \{0,1\}$ and $h(a,b) = 0$ if $a \neq b$ and $h(a,b) = 1$ if $a = b$.

The minimax theorem

Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact space. Let $h\colon\ A \times B \mapsto \mathbb{R}$ be concave on A, and convex and lower semicontinuous on B. Then

$$\sup_{a\in A} \min_{b\in B} h(a,b) = \min_{b\in B} \sup_{a\in A} h(a,b).$$

- h is "concave on A" means that

$$\forall b \in B, \quad -h(\cdot, b) \text{ is convex on } A.$$

h is "convex and lower semicontinuous on B" mean that

$$\forall a \in A, \quad h(a, \cdot) \text{ is convex and lower semicontinuous on } B.$$

- Note that the set A has no topological structure.

- We can write "min" instead of "inf" because h is lower semicontinuous on B and B is compact.

The minimax theorem

Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact space. Let $h\colon A \times B \mapsto \mathbb{R}$ be concave on A, and convex and lower semicontinuous on B. Then

$$\sup_{a\in A} \min_{b\in B} h(a,b) = \min_{b\in B} \sup_{a\in A} h(a,b).$$

Proof Let $\beta := \sup_{a\in A} \min_{b\in B} h(a,b)$. If we had $\beta < \min_{b\in B} \sup_{a\in A} h(a,b)$ then

$$\bigcup_{a\in A}\{b \in B\colon\ h(a,b) > \beta\} = B.$$

Since h is lower semicontinuous on B, the sets $\{b \in B\colon\ h(a,b) > \beta\}$ are open and B is compact, there would exist $a_1, \dots, a_m \in A$ such that

$$\{b \in B\colon\ h(a_1,b) > \beta\} \cup \cdots \cup \{b \in B\colon\ h(a_m,b) > \beta\} = B$$

and so $\min_{b\in B}[h(a_1,b) \vee \cdots \vee h(a_m,b)] > \beta$. From the Lemma on m convex functions with $f_i := h(a_i,\cdot)$, there would exist $\lambda_1, \dots \lambda_m \geq 0$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and

$$\min_{b\in B}[\lambda_1 h(a_1,b) + \cdots + \lambda_m h(a_m,b)] > \beta.$$

Since h is concave on A, it would follow from this that

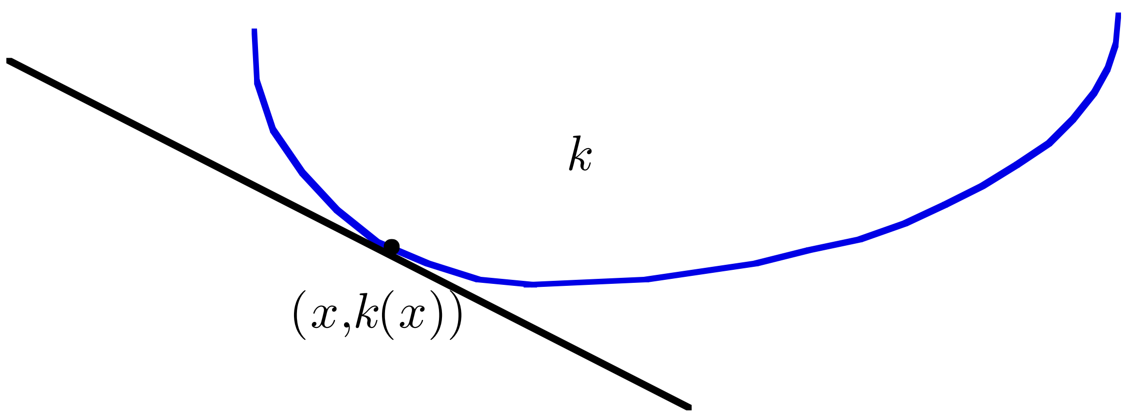
$$\min_{b\in B} h(\lambda_1 a_1 + \cdots + \lambda_m a_m, b) > \beta,$$

which would contradict the definition of β. So $\beta \geq \min_{b\in B} \sup_{a\in A} h(a,b)$. ∎

On the existence of subgradients

Let E be a normed space, $k\colon E \mapsto (\infty, \infty]$ be convex, $x \in E$ and $k(x) \in \mathbb{R}$. Does there exist $x^* \in E^*$ such that

$$y \in E \quad \Longrightarrow \quad k(x) + \langle y - x, x^* \rangle \leq k(y)?$$



On the existence of subgradients

Let E be a normed space, $k\colon E \mapsto (\infty, \infty]$ be convex, $x \in E$ and $k(x) \in \mathbb{R}$. Does there exist $x^* \in E^*$ such that $\quad y \in E \quad \Longrightarrow \quad k(x) + \langle y - x, x^* \rangle \le k(y)$?

$$\Longleftrightarrow$$

Do there exist $M \ge 0$ and a linear functional L on E such that $L \le M\|\cdot\|$ on E and

$$y \in E \quad \Longrightarrow \quad k(y) + L(x - y) \ge k(x)?$$

A more generalized Hahn–Banach theorem

Let S be a sublinear functional on E. Let C be a nonempty convex subset of a (possibly different) vector space, $k\colon\ C \mapsto (-\infty, \infty]$ be proper and convex and $j\colon\ C \mapsto E$ be S–convex. Then $\exists$ a linear functional L on E such that

$$L \le S \text{ on } E \quad \text{and} \quad \inf_C \big[L \circ j + k\big] = \inf_C \big[S \circ j + k\big].$$

From the mgHBt with $S := M\|\cdot\|$, $C := E$ and $j(x) := x - y$, this $\Longleftrightarrow$

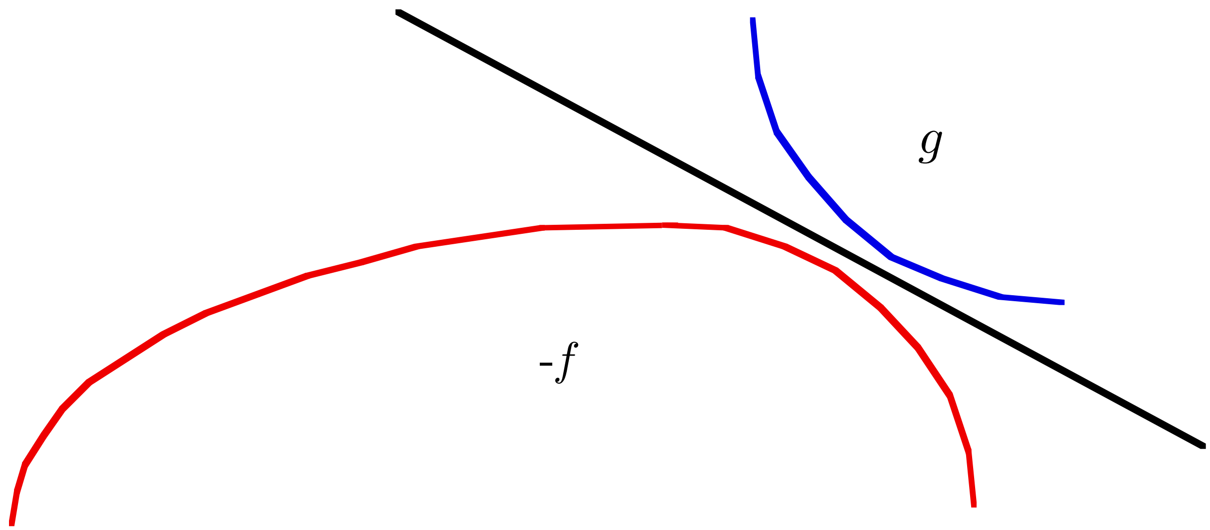
Does there exist $M \ge 0$ such that, $y \in E \quad \Longrightarrow \quad k(y) + M\|x - y\| \ge k(x)$?

Thus we have transformed the original problem on the existence of continuous linear functionals into the (much simpler) problem of finding a real constant M. This is an example of the "discovery method".

Separating a convex and a concave function

Let E be a normed space and $f, g\colon\ E \mapsto (\infty, \infty]$ be proper and convex. Do there exist $z^* \in E^*$ and $\beta \in \mathbb{R}$ such that

$$-f \leq z^* + \beta \leq g \quad \text{on} \quad E?$$



Using the same technique as before, with $C := E \times E$, $j(x, y) := x - y$ and $k(x, y) := f(x) + g(y)$, the above problem reduces to:

Does there exist $M \geq 0$ such that

$$\forall\ x,\ y \in E, \quad f(x) + g(y) + M\|x - y\| \geq 0?$$

Separating a convex and a concave function

Let E be a normed space and $f, g\colon\ E \mapsto (\infty, \infty]$ be proper and convex. Do there exist $z^* \in E^*$ and $\beta \in \mathbb{R}$ such that

$$-f \le z^* + \beta \le g \quad \text{on} \quad E? \qquad (🍦)$$

- The **Fenchel conjugate** f^* is defined by $f^*(x^*) := \sup_E(x^* - f)$.

- $$\begin{aligned} (🍦) &\iff -z^* - f \le \beta \text{ on } E \text{ and } z^* - g \le -\beta \text{ on } E \\ &\iff f^*(-z^*) \le \beta \text{ and } g^*(z^*) \le -\beta, \end{aligned}$$

- So our question $\iff$ is it true that

$$\exists\ z^* \in E^* \quad \text{such that} \quad f^*(-z^*) + g^*(z^*) \le 0? \qquad (🐕)$$

When (🐕) holds, we say that the Fenchel duality theorem is true.

- Rockafellar and Attouch–Brezis have given sufficient conditions for the Fenchel duality theorem to be true. The condition on the previous slide is both necessary and sufficient.

- We will use the following special case of Rockafellar's version, that (🐕) is true if $f + g \ge 0$ on E and g is continuous.

The following result is very useful in the theory of monotone multifunctions.

A sharp case of Fenchel duality

Let F be a normed space, $f\colon F \mapsto (\infty, \infty]$ be proper and convex and

$$y \in F \implies f(y) + \tfrac{1}{2}\|y\|^2 \geq 0. \qquad (☆)$$

Let

$$M := \sup_{y \in F} \left[\|y\| - \sqrt{2f(y) + \|y\|^2}\right] \vee 0.$$

Then there exists $y^* \in F^*$ such that $\|y^*\| \leq M$ and

$$f^*(y^*) + \tfrac{1}{2}\|y^*\|^2 \leq 0. \qquad (☾)$$

- *In fact* $\min\left\{\|y^*\|\colon\ y^* \text{ is as in } (☾)\right\} = M.$

Outline of proof. One can prove using (☆) and Dedekind section that

$$y \in F \implies \big|\|y\| - M\big| \leq \sqrt{2f(y) + \|y\|^2} \implies f(y) + M\|y\| \geq \tfrac{1}{2}M^2.$$

Rockafellar's version of the Fenchel duality theorem now gives $y^* \in F^*$ such that

$$f^*(y^*) + (M\|\cdot\|)^*(-y^*) \leq -\tfrac{1}{2}M^2,$$

thus $\|y^*\| \leq M$ and $f^*(y^*) \leq -\frac{1}{2}M^2$, from which (☾) is immediate. Finally, it is not hard to show that

$$\text{if } y^* \in F^* \text{ satisfies } (☾) \text{ then } \|y^*\| \geq M. \ \blacksquare$$

- E is a reflexive Banach space and E^* is its topological dual space.

Maximal monotone multifunctions

$T\colon\ E \rightrightarrows E^*$ means that $\forall\ x \in E$, Tx is a (possibly empty) subset of E^*. Then

$$G(T) := \big\{(x, x^*)\colon\ x \in E,\ x^* \in Tx\big\}.$$

Let $G(T) \neq \emptyset$. T is **monotone** if

$$(x, x^*) \text{ and } (y, y^*) \in G(T) \quad \Longrightarrow \quad \langle x - y, x^* - y^* \rangle \geq 0.$$

T is **maximal monotone** if T is monotone and

$$(w, w^*) \in E \times E^* \quad \text{and} \quad \big((t, t^*) \in G(T) \quad \Longrightarrow \quad \langle w - t, w^* - t^* \rangle \geq 0\big)$$

$$\Downarrow$$

$$(w, w^*) \in G(T).$$

J and $-J$ and $T + J$

The duality multifunction $J\colon\ E \rightrightarrows E^*$ is defined by:

$$x^* \in Jx \iff \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle.$$

J is maximal monotone. $-J\colon\ E \rightrightarrows E^*$ is defined by: $(-J)x := -Jx \quad (x \in E)$. Then:

$$x^* \in -Jx \iff \langle x, x^* \rangle + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 = 0.$$

If $T\colon\ E \rightrightarrows E^*$ then, $\forall x \in E$, $(T + J)x := \big\{x^* + y^*\colon\ x^* \in Tx,\ y^* \in Jx\big\}$.

- If $(x,x^*) \in E \times E^*$ then $\big\|(x,x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}$.
- The topological dual of $E \times E^*$ is $E^* \times E$, under the pairing

$$\big\langle (x,x^*),(u^*,u)\big\rangle := \langle x,u^*\rangle + \langle u,x^*\rangle.$$

- We have $\big\|(u^*,u)\big\| = \sqrt{\|u\|^2 + \|u^*\|^2}$.

The Fitzpatrick function of T

Let $T\colon E \rightrightarrows E^*$ be maximal monotone. We define its *Fitzpatrick function*, φ_T, by

$$\varphi_T(x,x^*) := \sup\nolimits_{(t,t^*)\in G(T)} \big[\langle t,x^*\rangle + \langle x,t^*\rangle - \langle t,t^*\rangle\big].$$

φ_T is a proper, convex and lower semicontinuous function from $E \times E^*$ into $(\infty,\infty]$,

$$(x,x^*) \in E \times E^* \quad\Longrightarrow\quad \varphi_T^*(x^*,x) \ge \varphi_T(x,x^*) \ge \langle x,x^*\rangle, \qquad (🌵)$$

and

$$\varphi_T^*(x^*,x) = \langle x,x^*\rangle \iff (x,x^*) \in G(T). \qquad (🔑)$$

A new property of φ_T

$$y \in E \times E^* \quad\Longrightarrow\quad \varphi_T(y) + \tfrac12\|y\|^2 \ge 0.$$

Proof. If $y = (x,x^*)$ then, from (🌵),

$$\varphi_T(y) + \tfrac12\|y\|^2 \ge \langle x,x^*\rangle + \tfrac12\|x\|^2 + \tfrac12\|x^*\|^2 \ge \tfrac12\|x\|^2 + \tfrac12\|x^*\|^2 - \|x\|\|x^*\| \ge 0.$$

A new property of φ_T

$$y \in E \times E^* \quad \Longrightarrow \quad \varphi_T(y) + \tfrac{1}{2}\|y\|^2 \geq 0.$$

A sharp case of Fenchel duality

Let F be a normed space, $f\colon\ F \mapsto (\infty, \infty]$ be proper and convex and

$$y \in F \quad \Longrightarrow \quad f(y) + \tfrac{1}{2}\|y\|^2 \geq 0. \qquad (\star)$$

Let

$$M := \sup_{y \in F} \left[\|y\| - \sqrt{2f(y) + \|y\|^2}\right] \vee 0.$$

Then there exists $y^* \in F^*$ such that $\|y^*\| \leq M$ and

$$f^*(y^*) + \tfrac{1}{2}\|y^*\|^2 \leq 0. \qquad (☾)$$

- *In fact* $\min\left\{\|y^*\|\colon\ y^* \text{ is as in } (☾)\right\} = M.$

Now let $N := \frac{1}{\sqrt{2}} \sup_{y \in E \times E^*} \left[\|y\| - \sqrt{2\varphi_T(y) + \|y\|^2}\right] \vee 0.$

Combination result

$\exists\ (z, z^*) \in E \times E^*$ *such that* $\quad \|z\|^2 + \|z^*\|^2 \leq 2N^2 \quad$ *and*

$$\left[\varphi_T^*(z^*, z) - \langle z, z^* \rangle\right] + \left[\langle z, z^* \rangle + \tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}\|z^*\|^2\right] = \varphi_T^*(z^*, z) + \tfrac{1}{2}\|(z^*, z)\|^2 \leq 0.$$

Proof. We have $y^* \in E^* \times E$ such that $\|y^*\| \leq \sqrt{2}N$ and $\varphi_T^*(y^*) + \frac{1}{2}\|y^*\|^2 \leq 0$. Let $(z, z^*) \in E \times E^*$ be such that $y^* = (z^*, z)$. ▌

- Let E be reflexive, $T\colon E \rightrightarrows E^*$ be maximal monotone and

$$N := \tfrac{1}{\sqrt{2}} \sup_{y \in E \times E^*} \left[\|y\| - \sqrt{2\varphi_T(y) + \|y\|^2}\right] \vee 0.$$

Combination result

$\exists\ (z, z^*) \in E \times E^*$ *such that* $\quad \|z\|^2 + \|z^*\|^2 \le 2N^2 \quad$ *and*

$$\left[\varphi_T^*(z^*, z) - \langle z, z^*\rangle\right] + \left[\langle z, z^*\rangle + \tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}\|z^*\|^2\right] \le 0.$$

Now $\langle z, z^*\rangle + \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z^*\|^2 \ge 0$, and (🌵) gives $\varphi_T^*(z^*, z) - \langle z, z^*\rangle \ge 0$, thus

$$\varphi_T^*(z^*, z) = \langle z, z^*\rangle \quad \text{and} \quad \langle z, z^*\rangle + \tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}\|z^*\|^2 = 0.$$

From (🔑), $(z, z^*) \in G(T)$. Also $(z, -z^*) \in G(J)$, from which $\|z^*\| = \|z\|$ and so $\|z\| \le N$. Since $0 = z^* + (-z^*)$, we also have $0 \in (T + J)z$. This proves the "existence" part of:

Reflexivity with maximality theorem

$$\exists\ x \in E \text{ such that } \|x\| \le N \text{ and } (T + J)x \ni 0.$$

In fact,

$$\min\left\{\|x\|\colon\ (T + J)x \ni 0\right\} = N.$$

— The Hahn–Banach theorem and maximal monotonicity —

- Let E be reflexive, $T\colon E \rightrightarrows E^*$ be maximal monotone and

$$N := \tfrac{1}{\sqrt{2}} \sup_{y \in E\times E^*} \left[\|y\| - \sqrt{2\varphi_T(y) + \|y\|^2}\right] \vee 0.$$

Reflexivity with maximality theorem

$$\exists\, x \in E \textit{ such that } \|x\| \le N \textit{ and } (T+J)x \ni 0.$$

In fact,

$$\min\left\{\|x\|\colon\ (T+J)x \ni 0\right\} = N.$$

Rest of Proof. Now we must show that

$$x \in E \text{ and } (T+J)x \ni 0 \quad \Longrightarrow \quad \|x\| \ge N.$$

So suppose that $x \in E$ and $(T+J)x \ni 0$. Then there exists $x^* \in Tx$ such that $-x^* \in Jx$. From (🔑) again,

$$\varphi_T^*(x^*,x) + \tfrac{1}{2}\left\|(x^*,x)\right\|^2 = \left[\varphi_T^*(x^*,x) - \langle x, x^*\rangle\right] + \left[\langle x, x^*\rangle + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] = 0.$$

The sharp case of Fenchel duality now gives

$$\left\|(x^*,x)\right\| \ge \sqrt{2}N.$$

But

$$\|x\| = \tfrac{1}{\sqrt{2}}\left\|(x^*,x)\right\|. \blacksquare$$

Reflexivity with maximality theorem

Let E be reflexive, $T\colon E \rightrightarrows E^*$ be maximal monotone and … . Then

$$\exists\, x \in E \text{ such that } (T + J)x \ni 0 \ldots .$$

The $-J$ criterion for maximality

Let E be reflexive and $T\colon E \rightrightarrows E^*$ be monotone. Then

$$T \text{ is maximal monotone} \iff G(T) + G(-J) = E \times E^*.$$

Proof ($\Longrightarrow$) Let $(w, w^*) \in E \times E^*$ and apply the reflexivity with maximality theorem, with T replaced by the multifunction with graph $G(T) - (w, w^*) \subset E \times E^*$, which is also maximal monotone. We obtain $(t, t^*) \in G(T)$ such that $(t - w, t^* - w^*) \in G(-J)$. But then $(w - t, w^* - t^*) \in G(-J)$ and so $(w, w^*) = (t, t^*) + (w - t, w^* - t^*) \in G(T) + G(-J)$.

($\Longleftarrow$) Let $(w, w^*) \in E \times E^*$ and

$$(t, t^*) \in G(T) \quad \Longrightarrow \quad \langle w - t, w^* - t^* \rangle \geq 0.$$

Choose $(t, t^*) \in G(T)$ so that $(w - t, w^* - t^*) \in G(-J)$. Then

$$\tfrac{1}{2}\|w - t\|^2 + \tfrac{1}{2}\|w^* - t^*\|^2 = -\langle w - t, w^* - t^* \rangle \leq 0.$$

So $(w, w^*) = (t, t^*) \in G(T)$. ▌

The $-J$ criterion for maximality

Let E be reflexive and $T\colon E \rightrightarrows E^*$ be monotone. Then

$$T \text{ is maximal monotone} \iff G(T) + G(-J) = E \times E^*.$$

The range of a multifunction

If $T\colon E \rightrightarrows E^*$,

$$R(T) := \bigcup_{x \in E} Tx.$$

Rockafellar's surjectivity theorem

Let E be reflexive, $T\colon E \rightrightarrows E^*$ be maximal monotone and, $\forall x \in E$,

$$(T + J)x := \{x^* + y^*\colon\ x^* \in Tx,\ y^* \in Jx\}.$$

Then

$$R(T + J) = E^*.$$

Proof Let $w^* \in E^*$. From the $-J$ criterion for maximality,

$$(0, w^*) \in G(T) + G(-J).$$

Thus $\exists\, x \in E$, $x^* \in Tx$ and $y^* \in (-J)(-x)$ such that $x^* + y^* = w^*$. But then $y^* \in Jx$, hence

$$w^* = x^* + y^* \in (T + J)x \subset R(T + J). \blacksquare$$

Minty's Theorem

If E is a **Hilbert space** *and $T\colon E \rightrightarrows E^*$ is monotone then*

$$T \text{ is maximal monotone} \iff R(T+J) = E^*.$$

It was proved by Rockafellar that this also holds if E is a reflexive Banach space such that the norm on E and the dual norm on E^* are strictly convex. Further, it was proved by Asplund that any reflexive Banach space can be renormed so that this property holds.

This result does **not hold** in a reflexive space where J or J^{-1} is not single–valued.

Various formulas for the minimum norm of solutions of $(T+J)x \ni 0$

If E is reflexive and $T\colon E \rightrightarrows E^*$ is maximal monotone then

$$\min\left\{\|x\|\colon x \in E,\ (T+J)x \ni 0\right\}$$
$$= \tfrac{1}{\sqrt{2}} \sup\nolimits_{y \in E \times E^*} \left[\|y\| - \sqrt{2\varphi_T(y) + \|y\|^2}\right] \vee 0$$
$$= \tfrac{1}{2} \sup\nolimits_{(x,x^*) \in E \times E^*} \left[\|x\| + \|x^*\| - \sqrt{4\varphi_T(x,x^*) + \big(\|x\| + \|x^*\|\big)^2}\right] \vee 0$$
$$= \sup_{(x,x^*) \in E \times E^*} \left[\|x\| \vee \|x^*\| - \sqrt{\varphi_T(x,x^*) + \|x\|^2 \vee \|x^*\|^2}\right] \vee 0.$$