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Regularization Methods for Inclusions and Variational Inequalities

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WELLINGTON

2004

The presentation is based on the following papers:

Alber Y., Butnariu D. and Ryazantseva, I., Regularization methods for ill-posed inclusions and variational inequalities with domain perturbations, *Journal of Nonlinear and Convex Analysis*, **2** (2001), 53-79.

Alber Y., Butnariu D. and Kassay G., Convergence and Stability of a Regularization Method for Maximal Monotone Inclusions and Its Applications to Convex Optimization, *preprint,* 2002.

Alber Y., Butnariu D. and Ryazantseva, I., Regularization of monotone variational inequalities with Mosco approximations of the constraint sets, *paper in progress*, 2003.

THE PROBLEM

Given:

$X =$ a reflexive, smooth, strictly convex Banach space

$X^* =$ the dual of X

$A : X \to 2^{X^*}$ a monotone point-to-set operator

$\Omega =$ a nonempty closed convex subset of $Dom(A)$

$f \in X^*$

Find a solution for the **inclusion**

$$\boxed{f \in Ax, \; x \in \Omega}$$

or, alternatively, of the **variational inequality**

$$\boxed{\langle Ax - f, z - x \rangle \geq 0, \; \forall z \in \Omega.}$$

THE DIFFICULTY

Problems like that may happen to be ill-posed in the sense that they may have no solution, or may have infinitely many solutions and/or small data perturbations may lead to significant distortions of the solution set.

More often than not the problem data A, f and Ω are given and/or computable by approximations A_k, f^k and Ω_k, respectively.

Ill-posedness makes solving the perturbed inclusion or variational inequality of little use since there is no guarantee that the solution we find will be close to a solution of the original problem we are supposed to solve.

A REGULARIZATION TECHNIQUE

[Tikhonov, A.N., Regularization of incorrectly posed problems, *Soviet Mathematics Doklady*, 4, 1963, 1035-1038.]

Consider the **perturbed inclusion**

$$f^k \in (A_k + \alpha_k J^\mu)x,\ x \in \Omega_k$$

and, respectively, the **perturbed variational inequality**

$$\left\langle (A_k + \alpha_k J^\mu)x - f^k, y - x \right\rangle \geq 0, \forall y \in \Omega_k,$$

where

$J^\mu : X \rightarrow X^*$ is the duality mapping of gauge μ

α_k is a positive real number.

FACTS

* (Rockafellar) If A_k is maximal monotone, then the perturbed inclusion as well as the perturbed variational inequality have unique solution x^k, no matter how $\alpha_k > 0$ is chosen.

* (A-B-R) If A_k is monotone and s-w-demiclosed, then the perturbed inclusion as well as the perturbed variational inequality have unique solution x^k, no matter how $\alpha_k > 0$ is chosen.

THE BILLION $ QUESTION

Suppose that A is monotone and s-w-demiclosed. We would like to know whether, and under which conditions concerning the problem data, the solution x^k of the perturbed inclusion/variational inequality approximates a solution of the original inclusion/variational inequality. Obviously, the quality of the approximation x^k will depend on the quality of the approximative data which impacts upon the type of stability we can ensure for the regularization process.

"We would like to emphasize that this method [of approximating solutions of the original problem], often used to demonstrate regularity of solutions, may sometimes be used to attain their existence when the assumptions of general [existence] theorems do not apply."

Cf. [D. Kinderlehrer and G. Stampacchia: An Introduction to Variational Inequalities and their Applications, 1980, p. 105]

SOME REFERENCES

Vainberg, M.M., Variational Method...*John Willey & Sons,* New York, 1973.

(*) **Mosco, U.** *Advances in Mathematics*, **3** (1969), 510-585.

(*) **Mosco U.,** Proc. Symp. in Pure Math., 18, Part 1, pp. 182-194, *A. M.S.*, 1970.

Bakushinskii, A.B., *USSR Comput. Math. and Math. Phys.*, 19 (1979), 256-260.

Liskovets, O. A., Variational Methods...(Russian), *Nauka i Tekhnika,* Minsk, 1981.

Alber, Y. and Ryazantseva, I., *Soviet Mathematics Doklady*, 25, 1, 1982, pp. 206-210.

Liskovets, O. A., *Soviet Mathematics Doklady*, 28, 1983, pp. 324-327.

Ryazantseva, I., *USSR Comp. Math and Math. Phys.*, 24, 1984, pp. 194-197

(*) **Liskovets, O. A.**, *Soviet Mathematics Doklady*, 31, 1985, pp. 202-206.

(*) **Liskovets, O. A.**, *Soviet Mathematics Doklady*, 36, 1988, pp. 220-224.

Alber, Y., *Applied Mathematical Letters*, 6, 1993, pp. 63-68.

Alber, Y. and Notik, A.I., *Set-Valued Analysis*, 1, 1993, pp. 393-402.

(*) **Kaplan A. and Tichatschke R.**, Stable Methods..., *Akademie Verlag*, 1994.

Liu F. and Nashed M.Z., *Set Valued Analysis* **6** (1998), 313-344.

Alexandre P., *Serdica Math. Journal*, **25** (1999), 91-102.

Kaplan A. and Tichatschke R., *J. Nonlin. Conv. Anal.* **2** (2001), 305-332.

Alber Y., Butnariu D. and Ryazantseva, I., *J. Nonlin. Conv. Anal.* **2** (2001), 53-79.

(*) **Alber Y., Butnariu D. and Kassay, G.** preprint 2003.

NICE APPLICATIONS

Engl, H.W., Hanke, M. and Neubauer, A., Regularization of Inverse Problems, *Kluwer Academic Publishers*, Dordrecht, 1996.

Lavrent'ev, M.M., Zerkal, S.M. and Trofimov, O.E., Computer Modelling in Tomography, *VSP BV*, Utrecht, 2001.

V.I.: THE MAX MONOTONE CASE

Assumption (A):

There exist sequences of positive real numbers h_k, δ_k, σ_k such that

$$\lim_{k\to\infty} \frac{\delta_k + \sigma_k + \beta(h_k, \sigma_k)}{\alpha_k} = 0$$

and there exist 3 bounded on bounded sets functions $a_1, a_2, a_3 : X \to \mathbb{R}$ which satisfy the following conditions:

(I) $\left\| f - f^k \right\|_* \leq (const.)\delta_k$;

(II) $\forall x \in \Omega$, $\text{dist}_*(x, \Omega_k) \leq a_1(x)\sigma_k$;

(III) $\forall w \in \Omega_k$, $\text{dist}_*(w, \Omega) \leq a_2(w)\sigma_k$;

(IV) $\forall x \in \Omega, \forall u \in Ax$, $\exists z_k \in \Omega_k$:

$$\text{dist}_*(u, A_k z_k)) < a_3(u)\beta(h_k, \sigma_k).$$

THEOREM *(A-B-R 2001): If A is max monotone and if there is a sequence of approximative data (A_k, f^k, Ω_k) with A_k max monotone such that assumption (A) holds, then the variational inequality*

$$\langle Ax - f, y - x\rangle \geq 0, \forall y \in \Omega,$$

has at least one solution and the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ of solutions of the corresponding variational inequalities

$$\left\langle (A_k + \alpha_k J^\mu)x - f^k, y - x\right\rangle \geq 0, \forall y \in \Omega_k,$$

converges strongly to the minimal norm solution of it.

VI: THE MONOTONE CASE

Assumption (B)

(I) *The sets* $\{\Omega_k\}_{k\in\mathbb{N}}$ *have nonempty intersection and converge in Mosco's sense to the set* Ω *and*

$$\Omega_k \subseteq \mathsf{Int}\,(\mathsf{Dom}\,A) \cap \mathsf{Int}\,(\mathsf{Dom}\,A_k)\,;$$

(II) *There exists a sequence of positive real numbers* $\{\alpha_k\}_{k\in\mathbb{N}}$ *which converges to zero and has the next two properties:*

(i) $\forall k \in \mathbb{N}$: $\left\|f^k - f\right\|_* \leq (const)\alpha_k$;

(ii) *There exist two functions* p, $q : X \to \mathbb{R}_+$ *which are bounded on bounded sets and such that, for each* $k \in \mathbb{N}$, *and for any* $x \in \Omega_k$ *we have that*

$(ii-1)$ *If* $\zeta \in Ax$, *then* $\mathsf{dist}_*(\zeta, A_k x) < q(x)\alpha_k$;

$(ii-2)$ *If* $\xi^k \in A_k x$, *then* $\mathsf{dist}_*(\xi^k, Ax) \leq p(x)$.

THEOREM (A-B-R 2003): If A is monotone and s-w-demiclosed, if there exists a sequence of approximative data (A_k, f^k, Ω_k) with A_k monotone and s-w-demiclosed such that assumption (B) holds, and if [we pay the following price]:

For any real number $\beta > 0$, *the set*

$$L_\beta(A) := \{x \in X : \|\xi\| \leq \beta \|x\|, \ \forall \xi \in Ax\}$$

is bounded,

then the variational inequality

$$\langle Ax - f, y - x \rangle \geq 0, \forall y \in \Omega$$

has solutions and the sequence $\left\{x^k\right\}_{k \in \mathbb{N}}$ of solutions of the corresponding variational inequalities

$$\left\langle (A_k + \alpha_k J)x - f^k, y - x \right\rangle \geq 0, \forall y \in \Omega_k,$$

converges to the minimal norm solution of it.

Note: The boundedness condition holds for strongly coercive operators.

INCLUSIONS: THE MAX MONOTONE CASE WITH UNIFORM CONVERGENCE ON BOUNDED SETS

Assumption (C)

There exist the sequences of positive real numbers α_k δ_k and h_k and the continuous functions $a, g, \zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\alpha_k \to 0$, ζ is nondecreasing with $\zeta(0) = 0$,

$$\lim_{k\to\infty} \frac{\delta_k + \zeta(h_k)}{\alpha_k} = 0$$

and such that

(I) $\left\|f - f^k\right\| \leq (const)\delta_k$

(II) $\forall x \in \Omega$, $\exists x' \in \Omega_k$:

$$\left\|x - x'\right\| \leq a(\|x\|)h_k;$$
$$and$$
$$\forall y \in Ax : \mathrm{dist}_*(y, A_k x') \leq g(\|y\|_*)\zeta(h_k).$$

THEOREM (A-B-R 2001): *If* A *is max monotone and if there exists a sequence of approximants* (A_k, f^k, Ω_k) *such that assumption (C) holds, then the inclusion*

$$f \in Ax, \ x \in \Omega$$

has solutions, the sequence $\left\{x^k\right\}_{k\in\mathbf{N}} \subset X$ *given by*

$$x^k = (A_k + \alpha_k J^\mu)^{-1} f^k,$$

is well defined and it converges strongly to the minimal norm solution of it.

INCLUSIONS: THE MAX MONOTONE CASE WITHOUT UNIFORM CONVERGENCE ON BOUNDED SETS

Theorem (A-B-K 2002). *Suppose that there exists a sequence of positive real numbers* $\{\alpha_k\}_{k\in\mathbb{N}}$ *which converges to zero and a sequence of approximants* (A_k, f^k) *with* A_k *maximal monotone such that*

$$w - \overline{\lim}\,\mathrm{Graph}\,(A_k) \subseteq \mathrm{Graph}\,(A)$$

and $f^k \xrightarrow{w} f$. *If for each* $v \in A^{-1}f$, *there exists a sequence* $\left\{v^k\right\}_{k\in\mathbb{N}}$ *which converges strongly to* v *in* X *and such that*

$$0 \in s - \underline{\lim}\,\frac{1}{\alpha_k}\left[A_k v^k - f^k\right],$$

then the inclusion has solutions, the sequence

$$x^k = (A_k + \alpha_k J^{\mu})^{-1} f^k$$

is well defined and converges weakly to the minimal norm solution of it.

AN EXAMPLE

This example illustrates the max monotone case with

UNIFORM CONVERGENCE ON BOUNDED SETS

It shows the degree of stability of the regularization method

Take the inclusion $f \in Ax$ in $X = \mathbf{R}^2$ with $\Omega \subset \mathbf{R}^2$ being the closed convex cone determined by the lines of equations $x_2 = q_1 x_1$ and $x_2 = q_2 x_1$, where $0 < q_1 < q_2$ are constants and the operator $A : \Omega \to \mathbf{R}^2$ is given by

$$Ax = \begin{cases} Bx & \text{if } x \in Int(\Omega), \\ \{Bx + \lambda(q_1, -1) : \lambda \geq 0\} & \text{if } x_2 = q_1 x_1, \\ \{Bx + \lambda(q_2, -1) : \lambda \geq 0\} & \text{if } x_2 = q_2 x_1. \end{cases}$$

where $B : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear positive semidefinite operator

$$Bx = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x.$$

A **is maximal monotone.**

For each $h \in (0,1]$ consider a closed convex cone $\Omega_h \subset \mathbf{R}^2$ determined by the lines $x_2 = q_1(h)x_1$ and $x_2 = q_2(h)x_2$, where $q_1 < q_1(h) < q_2 < q_2(h) < \infty$. Define the operator $B_h : \Omega_h \to \mathbf{R}^2$ by

$$B_h x = \begin{pmatrix} (1+h)^2 & 2+h \\ 2+h & 4+h \end{pmatrix} x,$$

B_h **is positive semidefinite.**

The operator $A_h : \Omega_h \to \mathbf{R}^2$ given by

$$A_h x =$$

$$\begin{cases} B_h x & x \in Int(\Omega_h) \\ \{B_h x + \lambda(q_1(h), -1) : \lambda \geq 0\} & x_2 = q_1(h)x_1 \\ \{B_h x + \lambda(q_2(h), -1) : \lambda \geq 0\} & x_2 = q_2(h)x_1 \end{cases}$$

is maximal monotone too.

If there exists a constant $c_0 > 0$ such that for any $h \in (0,1]$ we have

$$\max\left(|q_1 - q_1(h)|, |q_2 - q_2(h)|\right) \leq c_0 h,$$

then the operators A and $A_k := A_{1/k}$ defined above satisfy the assumtion (C)

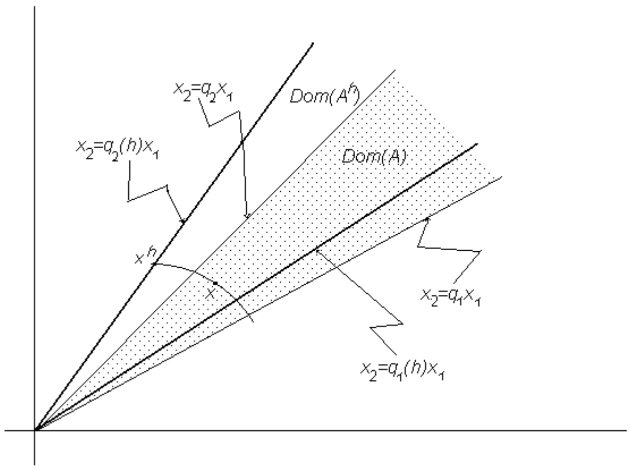

Figure 1: The domains of the given and of the perturbed operators

Computational experiments

DATA

$\mu(t) = t$

$q_1 = 0.25$

$q_2 = 0.5$

$f = (3, 6)^T$

The minimal norm solution is

$x^* = \left(\frac{9}{5}, \frac{3}{5}\right)^T$ with $\|x\| \approx 1.89\,4$.

APPROXIMANTS

$f^k = \left(3 + 0.5k^{-2}, 6 - 0.5k^{-2}\right)^T$

$\Omega_k = B_{1/k}(\Omega)$

The procedure

$$x^k = (A_{1/k} + k^{-1} J^{\mu})^{-1} \begin{pmatrix} 3 + 0.5k^{-2} \\ 6 - 0.5k^{-2} \end{pmatrix}$$

gives

k	x^k	dist. to x*
1	$(.21426, .80954)$	1.5995
10	$(.49803, 1.217)$	1.4408
100	$(.59592, 1.199)$	1.3448
1000	$(1.8003, 0.59983)$	3.4482×10^{-4}

An application

Consider the following optimization problem:

$$(P) \quad \text{Minimize } F(x) \quad \text{subject to} \quad x \in \Omega.$$

represented in the inclusion form

$$(P') \quad \text{Find } x \in X \text{ s.t. } 0 \in Ax, \text{ with } A = \partial F + N_{\Omega}.$$

Presume that the function F and the set Ω can not be exactly determined and that, instead, we have sequences of **approximations of them described as follows:**

★ $F_k : X \to (-\infty, +\infty]$, $(k \in \mathbb{N})$, are convex, l.s.c. functions such that

$$\text{Dom}\, F \subseteq \text{Dom}\, F_k, \quad \forall k \in \mathbb{N},$$

and which approximates F in the following sense:

Condition (A). *There exists a continuous function* $c : [0, +\infty) \to [0, +\infty)$ *and a sequence of positive real numbers* $\{\delta_k\}_{k\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} \delta_k = 0$ *and*

$$|F_k(x) - F(x)| \leq c(\|x\|)\delta_k,$$

whenever $x \in \text{Dom}\, F$ *and* $k \in \mathbb{N}$.

★ $\Omega_k,\ k \in \mathbb{N}$, are closed convex nonempty subsets of Int (Dom F) , which approximate the set Ω in the following sense:

Condition (B). *The next two requirements are satisfied:*

(i) *For any* $y \in \Omega$ *there exists a sequence* $\left\{y^k\right\}_{k\in\mathbb{N}}$ *which converges strongly to* y *in* X *and such that* $y^k \in \Omega_k$ *for all* $k \in \mathbb{N}$*;*

(ii) *If* $\left\{z^k\right\}_{k\in\mathbb{N}}$ *is a sequence in* X *which is weakly convergent and such that for some subsequence* $\left\{\Omega_{i_k}\right\}_{k\in\mathbb{N}}$ *of* $\{\Omega_k\}_{k\in\mathbb{N}}$ *we have* $z^k \in \Omega_{i_k}$ *for all* $k \in \mathbb{N}$*, then there exists a sequence* $\left\{w^k\right\}_{k\in\mathbb{N}}$ *contained in* Ω *with the property that*

$$\lim_{k\to\infty} \left\|z^k - w^k\right\| = 0.$$

For each $k \in \mathbb{N}$, we associate to problem (P) the problem

$$(P_k) \quad \text{Minimize } F_k(x) \quad \text{s.t. } x \in \Omega_k,$$

which can be solved by finding solutions of the inclusion

$$(P_k') \quad 0 \in A_k x^k, \text{ where } A_k := \partial F_k + N_{\Omega_k}.$$

Let

$$x^k := (A^k + \alpha_k J^\mu)^{-1}(0), \text{ where } \alpha_k \to 0^+.$$

x^k is solution to the **regularized problem**

$$(Q_k) \quad \text{Minimize } F_k(x) + \alpha_k \phi\left(\|x\|\right) \quad \text{s.t. } x \in \Omega_k. \tag{1}$$

NOTE: By contrast to problem (P_k) which may have infinitely many solutions, the problem (Q_k) always has unique solution. Moreover, by choosing $\mu(t) = t$ one ensures that the objective function of (Q_k) is strongly convex and, therefore, the problem (Q_k) may be better posed and easier to solve than (P_k).

A consequence of Thm. A-B-K-2002 above:

Theorem (A-B-K). *Suppose that conditions* (A) *and* (B) *are satisfied. If there exists a sequence* $\{\alpha_k\}_{k\in\mathbb{N}}$ *of positive real numbers converging to zero such that for each optimal solution* v *of* (P), *there exists a sequence* $\{v^k\}_{k\in\mathbb{N}}$ *with the properties that* $v^k \in \Omega_k$ *for all* $k \in \mathbb{N}$ *and*

$$\lim_{k\to\infty} \left\| v^k - v \right\| = 0 \tag{2}$$
$$= \lim_{k\to\infty} \alpha_k^{-1} \| \mathrm{Pr}_{\partial F_k(v^k)+N_{\Omega_k}(v^k)}(0) \|_*,$$

then the sequence $\{x^k\}_{k\in\mathbb{N}}$ *converges strongly to the minimal norm solution of the optimization problem* (P).

Condition (c) in itself is difficult to verify but there are simpler conditions which imply it. For instance,

...we have the following surogates for condition (2):

♣...*there exists* $\{v^k\}_{k\in\mathbb{N}}$ *s.t.* $v^k \in \Omega_k$, $(\forall k \in \mathbb{N})$ *and*

$$\lim_{k\to\infty} \left\|v^k - v\right\| = 0 = \lim_{k\to\infty} \alpha_k^{-1} \|\mathrm{Pr}_{\partial F_k(v^k)}(0)\|_*.$$

♣...*there exists* $\{v^k\}_{k\in\mathbb{N}}$ *s.t.* $v^k \in \Omega_k$, $(\forall k \in \mathbb{N})$ *and*

$$\lim_{k\to\infty} \left\|v^k - v\right\| = 0$$
$$= \lim_{k\to\infty} \alpha_k^{-1} \|\mathrm{Pr}_{T_{\Omega_k}(v^k)}\left(-\nabla F_k\left(v^k\right)\right)\|_*.$$

♣...*there exists* $\{v^k\}_{k\in\mathbb{N}}$ *s.t.* $v^k \in \Omega_k$, $(\forall k \in \mathbb{N})$, *and*

$$\lim_{k\to\infty} \alpha_k^{-1} \|\nabla F_k(v^k) - \nabla F(v)\|_* = 0.$$